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## Maxwell's Equations, Part V

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## Maxwell's Equations. V. $\nabla \times \mathbf{E} = -\delta\mathbf{B}/\delta t$

*This is the fifth installment in a series devoted to explaining Maxwell's equations, the four mathematical statements upon which the classical theory of electromagnetic fields – and light – is based. Previous installments can be found on Spectroscopy's website (whose URL can be found throughout these issues). Maxwell's equations are expressed in the language of vector calculus, so a significant part of some previous columns have been devoted to explaining vector calculus, not spectroscopy. For better or worse, that's par for the course, and it's my job to explain it well – I trust readers will let me know if I fail! The old adage “the truth will set you free” might be better stated, for our purposes, as “math will set you free”. And that's the truth.*

### **Beginnings**

In mid-1820, Danish physicist Hans Oersted discovered that a current in a wire can affect the magnetic needle of a compass. These experiments were quickly confirmed by François Arago and, more exhaustively, André Marie Ampère. Ampère's work demonstrated that the effects generated, which defined a so-called “magnetic field” (labeled  $\mathbf{B}$ ), were centered on the wire, were perpendicular to the wire, and were circularly symmetric about the wire (Figure 31). By convention, the vector component of the field had a direction given by the right hand rule: if the thumb of the right hand were pointing in the direction of the current, the curve of the fingers on the right hand gives the direction of the vector field.

Other careful experiments by Jean-Baptiste Biot and Félix Savart established that the strength of the magnetic field was directly related to the current  $I$  in the wire and inversely related to the radial distance from the wire  $r$ . Thus, we have

$$\mathbf{B} \propto \frac{I}{r}$$

where “ $\propto$ ” means “proportional to”. To make a proportionality an equality, we introduce a proportionality constant. However, because of the axial symmetry of the field, we typically include a factor of  $2\pi$  (the radian angle of a circle) in the denominator of any arbitrary proportionality constant. As such, our new equation is

$$\mathbf{B} = \frac{\mu I}{2\pi r}$$

where the constant  $\mu$  is our proportionality constant and is called the permeability of the medium the magnetic field is in. In a vacuum, the permeability is labeled  $\mu_0$  and, because of how the units of  $\mathbf{B}$  and  $I$  are defined, is equal to exactly  $4\pi \times 10^{-7}$  tesla-meters per ampere (T·m/A).

Not long after the initial demonstrations, Ampère had another idea: curve the wire into a circle. Sure enough, inside the circle, the magnetic field increased in strength as the concentric circles of the magnetic field overlapped on the inside (Figure 32). Biot and Savart found that the magnetic field  $\mathbf{B}$  created by the loop was related to the current  $I$  in the loop and the radius of the loop  $R$ :

$$\mathbf{B} = \frac{\mu I}{2R}$$

Multiple loops can be joined in sequence to increase  $\mathbf{B}$ , and in 1824 – 5 English inventor William Sturgeon wrapped loops around a piece of iron, creating the first electromagnet. Even by then, Ampère had the thought that it was the current – that is, moving charges – that caused the magnetic field.

Joseph Henry was an American scientist who eventually became the first Secretary of the Smithsonian Institution. In 1830, he performed some experiments showing how a magnetic field can induce electricity – and did not publish. Because of this, he lost a larger place in scientific

history when in 1831, Michael Faraday announced that a changing magnetic field could produce an electrical current. (Henry's work has not gone unnoticed, as the SI unit of inductance is named the henry.) Note that Faraday (followed by others) found that a *changing* magnetic field is required; a static, non-changing magnetic field produces no current (Figure 33). This strongly suggests that an electric current  $I$  is related to a varying magnetic field, or

$$I \propto \frac{\delta \mathbf{B}}{\delta t}$$

Actually, this is not far from the truth (which would then be another of Maxwell's equations if it were), but the more complete truth is expressed in a different, more applicable form.

### **Work in an Electrostatic Field**

The simple physical definition of work ( $w$ ) is force ( $F$ ) times displacement ( $\Delta s$ ):

$$w = F \times \Delta s$$

This is fine for straight-line motion, but what about if the motion occurred on a curve (Figure 34 in two dimensions), with perhaps a varying force? Then calculating the work is not as straightforward, especially since force and displacement are both vectors. However, it can be easily justified that the work is the integral, from initial point to final point, of the dot product force vector  $\mathbf{F}$  with the unit vector tangent to the curve, which we will label  $\mathbf{t}$ :

$$w = \int_{\text{initial point}}^{\text{final point}} \mathbf{F} \cdot \mathbf{t} \, ds$$

Because of the dot product, only the force component in the direction of the tangent to the curve contributes to the work.. That makes sense if you remember the definition of the dot product,  $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ : if the force is parallel to the displacement, work is maximized (because the cosine of the angle between the two vectors is  $\cos[0^\circ] = 1$ ) while if the force is perpendicular to the displacement, work is zero (because now the cosine is  $\cos[90^\circ] = 0$ ).

Now consider two random points inside of an electrostatic field  $\mathbf{E}$  (Figure 35). Keep in mind that we have defined  $\mathbf{E}$  as *static*; that is, not moving or changing. Imagine that an electric particle with charge  $q$  were to travel from  $P_1$  to  $P_2$  and back again along the paths  $\mathbf{s}_1$  and  $\mathbf{s}_2$ , as indicated. Since the force  $\mathbf{F}$  on the particle is given by  $q\mathbf{E}$  (this is from Coulomb's law), we have for an imagined two-step process:

$$w = \int_{P_1}^{P_2} q\mathbf{E} \cdot \mathbf{t} \, ds_1 + \int_{P_2}^{P_1} q\mathbf{E} \cdot \mathbf{t} \, ds_2$$

Each integral covers one pathway, but eventually you end up where you started.

This last statement is a crucial one: *eventually you end up where you started*. According to Coulomb's law, the only variable that the force or electric field between the two particles depends on is the radial distance,  $r$ . This further implies that the work,  $w$ , depends only on the radial distance between any two points in the electric field. Even further still, this implies that if you start and end at the *same* point, as we are in our example, the overall work is zero because you are starting and stopping at the same radial point  $r$ . Thus, the equation above must be equal to zero:

$$\int_{P_1}^{P_2} q\mathbf{E} \cdot \mathbf{t} \, ds_1 + \int_{P_2}^{P_1} q\mathbf{E} \cdot \mathbf{t} \, ds_2 = 0$$

Since we are starting and stopping at the same point, the combined paths  $\mathbf{s}_1$  and  $\mathbf{s}_2$  are termed a *closed* path. Notice too that, other than being closed, we have not imposed any requirement on the overall path  $\mathbf{s}$  itself: it can be any path. We say that this integral, which must equal zero, is *path-independent*.

The symbol for an integral over a closed path is  $\oint$ . Thus, we have

$$\oint q\mathbf{E} \cdot \mathbf{t} \, ds = 0$$

We can divide by the constant  $q$  to get something slightly more simple:

$$\oint \mathbf{E} \cdot \mathbf{t} \, ds = 0$$

This is one characteristic of an electrostatic field: the path-independent integral over any closed path in an electrostatic field is exactly zero.

The key word in the above statement is “any”. You can select any random closed path in an electric field, and the integral of  $\mathbf{E} \cdot \mathbf{t}$  over that path is exactly zero. How can we generalize this for any closed path?

Let us start with a closed path in one plane, as shown by Figure 36. The complete closed path has four parts, labeled T, B, L, and R for top, bottom, left, and right, and it surrounds a point at some given coordinates  $(x, y, z)$ . T and B are parallel to the  $x$  axis, while L and R are parallel to the  $y$  axis. The dimensions of the path are  $\Delta x$  by  $\Delta y$  (these will be useful shortly). Right now the area enclosed by the path is arbitrary, but later on we will want to shrink the closed path down so that the area goes to zero. Finally, the path is immersed in a three dimensional field  $\mathbf{F}$  whose components are  $F_x$ ,  $F_y$ , and  $F_z$ . That is,

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y + \mathbf{k}F_z$$

in terms of the three unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  in the  $x$ -,  $y$ -, and  $z$ -dimension, respectively.

Let us evaluate the work of each straight segment of the path separately, starting with path B. The work is

$$w_B = \int_B \mathbf{F} \cdot \mathbf{t} \, ds$$

The tangent vector  $\mathbf{t}$  is simply the unit vector  $\mathbf{i}$ , since path B points along the positive  $x$  axis.

When you take the dot product of  $\mathbf{i}$  with  $\mathbf{F}$  (see expression above), the result is simply  $F_x$ . (Can you verify this?) Finally, since the displacement  $\mathbf{s}$  is along the  $x$  axis,  $d\mathbf{s}$  is simply  $dx$ . Thus, we have

$$w_B = \int_B F_x \, dx$$

Although the value of  $F_x$  can vary as you go across path B – in fact, it is better labeled as  $F_x(x,y,z)$  – let us assume some average value of  $F_x$  as indicated by its value at a  $y$ -axis position of  $y - \Delta y/2$ , which is the  $y$  value  $\frac{1}{2}$  of the height of the box below the point in the center. Thus, we have

$$w_B = \int_B F_x dx \approx F_x(x, y - \frac{\Delta y}{2}, z) \cdot \Delta x$$

where we have replaced the infinitesimal  $dx$  with the finite  $\Delta x$ .

We can do the same for the work at the top of the box, which is path T. There are only two differences: first, the tangent vector is  $-\mathbf{i}$ , because the path is moving in the negative direction, and second, the average value of  $F_x$  is judged at  $y + \Delta y/2$ , which is  $\frac{1}{2}$  of the height of the box *above* the center point. Hence we can simply write

$$w_T \approx -F_x(x, y + \frac{\Delta y}{2}, z) \cdot \Delta x$$

The sum of the work on the top and bottom are thus

$$w_{T+B} \approx F_x(x, y - \frac{\Delta y}{2}, z) \cdot \Delta x - F_x(x, y + \frac{\Delta y}{2}, z) \cdot \Delta x$$

Rearranging this so that it is in the form “top minus bottom” and factoring out the  $\Delta x$ , this becomes

$$w_{T+B} \approx - \left[ F_x(x, y + \frac{\Delta y}{2}, z) - F_x(x, y - \frac{\Delta y}{2}, z) \right] \Delta x$$

Let us multiply this expression by 1, in the form of  $\Delta y/\Delta y$ . We now have

$$w_{T+B} \approx - \frac{[F_x(x, y + \frac{\Delta y}{2}, z) - F_x(x, y - \frac{\Delta y}{2}, z)]}{\Delta y} \Delta x \Delta y$$

Recall that this work is actually a sum of two integrals involving, originally, the integrand  $\mathbf{F} \cdot \mathbf{t}$ .

Reminding ourselves, this last expression can be written as

$$w_{T+B} = \int_{T+B} \mathbf{F} \cdot \mathbf{t} ds \approx - \frac{[F_x(x, y + \frac{\Delta y}{2}, z) - F_x(x, y - \frac{\Delta y}{2}, z)]}{\Delta y} \Delta x \Delta y$$

The term  $\Delta x \Delta y$  is the area of the path,  $A$ . Dividing the area to the other side of the equation, we have

$$\frac{1}{A} \int_{T+B} \mathbf{F} \cdot \mathbf{t} \, ds \approx - \frac{[F_x(x, y + \frac{\Delta y}{2}, z) - F_x(x, y - \frac{\Delta y}{2}, z)]}{\Delta y}$$

Suppose we take the limit of this expression as  $\Delta x = \Delta y = A \rightarrow 0$ . What we would have is the amount of work done over any infinitesimal area defined by any random path – the only restriction is that the path is in the  $(x, y)$  plane. The equation above becomes

$$\lim_{A \rightarrow 0} \left[ \frac{1}{A} \int_{T+B} \mathbf{F} \cdot \mathbf{t} \, ds \right] = \lim_{\Delta y \rightarrow 0} \left[ - \frac{[F_x(x, y + \frac{\Delta y}{2}, z) - F_x(x, y - \frac{\Delta y}{2}, z)]}{\Delta y} \right]$$

Looking at the second limit above and recalling our basic calculus, that limit defines a derivative with respect to  $y$ ! But because  $F_x$  is a function of three variables, this is better defined as the partial derivative with respect to  $y$ . Thus, we have

$$\lim_{A \rightarrow 0} \left[ \frac{1}{A} \int_{T+B} \mathbf{F} \cdot \mathbf{t} \, ds \right] = - \frac{\delta F_x}{\delta y}$$

Note the retention of the minus sign.

We can do the same thing for paths L and R. The analysis is exactly the same, only the variables that are affected change. What we get is (and you are welcome to verify the derivation)

$$\lim_{A \rightarrow 0} \left[ \frac{1}{A} \int_{L+R} \mathbf{F} \cdot \mathbf{t} \, ds \right] = \frac{\delta F_y}{\delta x}$$

Now, combine the two parts: the work done over an infinitesimally small closed path in the  $(x, y)$  plane is given by

$$\lim_{A \rightarrow 0} \frac{w}{A} = \lim_{A \rightarrow 0} \left[ \frac{1}{A} \oint \mathbf{F} \cdot \mathbf{t} \, ds \right] = \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y}$$

Now isn't that a rather simple result?

Let us see an example of this result so we can understand what it means. Consider a two-dimensional sink, in the  $(x, y)$  plane, as diagrammed in Figure 37. A thin film of water is going

down the central drain, and in this case it is spinning in a counter-clockwise direction at some constant angular velocity. The vector field for the velocity of the spinning water is

$$\mathbf{v} = \mathbf{i} \frac{\delta x}{\delta t} + \mathbf{j} \frac{\delta y}{\delta t}$$

In terms of the angular velocity  $\omega$ , this can be written as

$$\mathbf{v} = \omega(-\mathbf{i}y + \mathbf{j}x) = -\mathbf{i}\omega y + \mathbf{j}\omega x$$

(A conversion to polar coordinates was necessary to go to this second expression for  $\mathbf{v}$ , in case you need to do the math yourself.) In this vector field,  $F_x = -\omega y$  and  $F_y = \omega x$ . To determine the limit of the work per unit area, we evaluate the expression

$$\frac{\delta}{\delta x}(\omega x) - \frac{\delta}{\delta y}(-\omega y)$$

This is easy to evaluate:

$$= \omega - (-\omega) = 2\omega$$

Suppose we stand up a piece of cardboard on the sink, centered at the drain. Experience suggests to us that the cardboard piece will start to rotate, with the axis of rotation perpendicular to the flat sink. In this particular case, the axis of rotation will be in the  $z$  dimension, and in order to be consistent with the right hand rule, we submit that in this case the axis points in the positive  $z$  direction. If this axis is considered a vector, then the unit vector in this case is (positive)  $\mathbf{k}$ .

Thus, vectorially speaking, the infinitesimal work per unit area is actually

$$\left[ \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \right] \mathbf{k}$$

Thus, the closed loop in the  $(x,y)$  plane is related to a vector in the  $z$  direction. In the case of a vector field, the integral over the closed path is referred to as the *circulation* of the vector field.

As a counterexample, suppose water in our two-dimensional sink is flowing from left to right at a constant velocity, as shown in Figure 38. In this case, the vector function is

$$\mathbf{v} = K\mathbf{i}$$

where  $K$  is a constant. If we put a piece of cardboard in this sink, centered on the drain, does the cardboard rotate? No, it doesn't. If we evaluate the partial-derivative expression from above (in this case,  $F_x = K$  and  $F_y = 0$ ):

$$\left[ \frac{\delta}{\delta x}(K) - \frac{\delta}{\delta y}(0) \right] \mathbf{k} = 0\mathbf{k} = 0$$

(Recall that the derivative of a constant is zero.) This answer implies that no rotation is induced by the closed loop.

### Introducing the Curl

For a vector function  $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ , I will hereby define the function

$$\left[ \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \right] \mathbf{k}$$

as the *one-dimensional curl* of  $\mathbf{F}$ . Possibly improperly, I designate it “one dimensional” because the result is a vector in one dimension, in this case the  $z$  dimension. The analysis we performed in the earlier section – defining a closed path in a single plane and taking the limit of the path integral – can be performed for the  $(x,z)$  and  $(y,z)$  planes. When we do that, we get the following analogous results:

$$(x,z) \text{ plane: } \left[ \frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x} \right] \mathbf{j}$$

$$(y,z) \text{ plane: } \left[ \frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z} \right] \mathbf{i}$$

The combination of all three expressions gives us a general expression for the *curl of  $\mathbf{F}$* :

$$\text{curl } \mathbf{F} = \left[ \frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z} \right] \mathbf{i} + \left[ \frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x} \right] \mathbf{j} + \left[ \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \right] \mathbf{k}$$

This expression allows us to determine

$$\lim_{A \rightarrow 0} \left[ \frac{1}{A} \oint \mathbf{F} \cdot \mathbf{t} \, ds \right]$$

for any vector function  $\mathbf{F}$  in any plane.

But what does the curl of a vector function mean? One way of thinking about it is that it is a variation in the vector function  $\mathbf{F}$  that causes a rotational effect in about perpendicular axis. (Indeed, “curl  $\mathbf{F}$ ” is sometimes still designated “rot  $\mathbf{F}$ ”, and a vector function whose curl equals zero (see Figure 38) is termed “irrotational”.) Too, a vector function with a non-zero curl can be thought of as curving around a particular axis, with that axis being normal to the plane of the curve. Thus, the rotating water in Figure 37 has a non-zero curl, while the linearly-flowing water in Figure 38 has a zero curl.

A mnemonic (that is, a memory aid) for the general expression for curl  $\mathbf{F}$  takes advantage of the structure of a  $3 \times 3$  determinant:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_y & F_z \end{vmatrix}$$

Understand that curl  $\mathbf{F}$  is NOT a determinant; a determinant is a number that is a characteristic of a square matrix of numerical values. However, the expression for curl  $\mathbf{F}$  can be constructed by performing the same operations on the expressions as one would do with numbers to determine the value of a  $3 \times 3$  determinant: constructing the diagonals and adding the right-downward diagonals and subtracting the left-upwards diagonals. If you have forgotten how, Figure 39 shows how to determine the expression for the curl.

The determinantal form of the curl can be expressed in terms of the del operator,  $\nabla$ .

Recall from Part III of this series [1] that the del operator is

$$\nabla \equiv \mathbf{i} \frac{\delta}{\delta x} + \mathbf{j} \frac{\delta}{\delta y} + \mathbf{k} \frac{\delta}{\delta z}$$

Recall from vector calculus that the cross product of two vectors  $\mathbf{A} \equiv \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$  and  $\mathbf{B}$  defined analogously is written  $\mathbf{A} \times \mathbf{B}$  and is given by the expression

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

By comparing this expression to the determinantal form of the curl, it should be easy to see that the curl of a vector function  $\mathbf{F}$  can be written as

$$\text{curl } \mathbf{F} \equiv \nabla \times \mathbf{F}$$

Like the fact that curl is not technically a determinant, it is technically not a cross product, as  $\nabla$  is an *operator*, not a vector. The parallels, however, make it easy to gloss over this technicality and use the “ $\nabla$  cross  $\mathbf{F}$ ” symbolism to represent the curl of a vector function.

Because the work integral over a closed path through an electrostatic field  $\mathbf{E}$  is zero, it is a short logical step to state that therefore

$$\nabla \times \mathbf{E} = 0$$

This is one more property of an electrostatic field: the field is not rotating about any point in space. Rather, an electrostatic field is a purely radial field, with all field “lines” going from the point in space straight to the electric charge.

## **Faraday’s Law**

The previous section may be all well, magnetic fields relate to *moving*, not static, charges. Also, the electric fields themselves are subtly different.

An electrostatic field caused by a charged particle is thought of as beginning at a positive charge and ending at a negative charge. Since overall, matter is electrically neutral, then every

electric field emanating from a positive charge eventually ends at a negative charge. This is illustrated in Figure 40, top diagram.

However, when a changing magnetic field creates a current in a conductor, this current is the product of an *induced* electric field. In the case of a bar-type magnet, the magnetic field is axially symmetric about the length of the bar, so the induced electric field is axially symmetric as well: that is, it is circular. This is illustrated in Figure 40, bottom diagram.

The induced, circular electric field caused by a moving magnet causes charges in that circle. The circulation of the induced electric field vector can be constructed from our definition of “circulation” above; it is

$$\text{circulation} = \oint_{\mathbf{s}} \mathbf{E} \cdot \mathbf{t} \, ds$$

where  $\mathbf{E}$  is the induced electric field,  $\mathbf{t}$  is the tangent vector along the path, and  $\mathbf{s}$  is the infinitesimal amount of path. In this case, the “circulation” is defined as the “electromotive force”, or emf. What is this force doing? Why, causing charges to move, of course! As such, it is doing work, and our arguments using work in the sections above are all valid here.

What Faraday found experimentally is that a changing magnetic field induced an electric field (which then forced a current). If you imagine that a magnetic field is composed of discrete field lines, what is happening is that as the number of magnetic field lines in a given area changes with time, an electric field is induced. Figure 41 illustrates this. Consider the loop of area outlined by the black line. As the magnet is moved to the right, the number of magnetic field lines intersecting the loop changes. It is this change that induces the electric field.

The number of field lines per area is called the magnetic flux. In Part III of this series [1], we presented how we determine the flux of a vector field. For a changing vector field  $\mathbf{F}$

having a unit vector perpendicular (or normal) to its direction of motion  $\mathbf{n}$  over some surface  $S$ , the flux is defined as

$$\text{flux} = \int_S \mathbf{F} \cdot \mathbf{n} \, dS$$

For our magnetic field  $\mathbf{B}$ , this becomes

$$\text{magnetic flux} = \int_S \mathbf{B} \cdot \mathbf{n} \, dS$$

But the induced electric field is related to the change in magnetic flux with time. Thus, we are actually interested in the time derivative of the magnetic flux:

$$\frac{\delta}{\delta t} \int_S \mathbf{B} \cdot \mathbf{n} \, dS$$

At this stage, we bring everything together by citing the experimental facts as determined by Faraday and others: the electromotive force is equal to the change in the magnetic flux over time. That is,

$$\oint_S \mathbf{E} \cdot \mathbf{t} \, ds = \frac{\delta}{\delta t} \int_S \mathbf{B} \cdot \mathbf{n} \, dS$$

Let us divide each side of this equation by the area  $A$  of the circular path of the induced current. This area also corresponds to the surface  $S$  that the magnetic field flux is measured over, so we divide one side by  $A$  and one side by  $S$ . We get

$$\frac{1}{A} \oint_S \mathbf{E} \cdot \mathbf{t} \, ds = \frac{1}{S} \frac{\delta}{\delta t} \int_S \mathbf{B} \cdot \mathbf{n} \, dS$$

Suppose we want to consider the limit of this expression as the area of the paths shrink to zero size; that is, as  $A \rightarrow 0$ . We would have

$$\lim_{A \rightarrow 0} \oint_S \mathbf{E} \cdot \mathbf{t} \, ds = \lim_{S \rightarrow 0} \frac{\delta}{\delta t} \int_S \mathbf{B} \cdot \mathbf{n} \, dS$$

The left side is, by definition, the curl of  $\mathbf{E}$ . What about the right side? Rather than prove it mathematically, let's consider the following argument. As the surface  $S$  goes to zero, the limit of the magnetic flux ultimately becomes *one* magnetic flux line. This single line will be

perpendicular to the infinitesimal surface – look at the rendering of the magnetic field lines in Figure 41 if you need to convince yourself of this. Thus, the dot product  $\mathbf{B} \cdot \mathbf{n}$  is simply  $\mathbf{B}$ , and the infinite sum of infinitesimal pieces (which is what an integral is) degenerates to a single value of  $\mathbf{B}$ . I therefore argue that

$$\lim_{S \rightarrow 0} \frac{\delta}{\delta t} \int_S \mathbf{B} \cdot \mathbf{n} \, dS = \frac{\delta}{\delta t} \mathbf{B}$$

So what we now have is

$$\nabla \times \mathbf{E} = \frac{\delta}{\delta t} \mathbf{B}$$

We are almost done. The law of conservation of energy must be satisfied. Although it appears that we are getting an induced current from nowhere, understand that this induced current also generates a magnetic field. In order for the law of conservation of energy to be satisfied, the new magnetic flux must *oppose* the original magnetic flux (this concept is known as Lenz's law after Henrich Lenz, a Russian physicist who discovered it). To represent this mathematically, a negative sign must be included in the last equation. By convention, the minus sign is put on the right side, so our final equation is

$$\nabla \times \mathbf{E} = -\frac{\delta}{\delta t} \mathbf{B}$$

This expression is known as Faraday's law of induction, given that Michael Faraday discovered (or rather, first announced) magnetic induction of current. It is considered the third of Maxwell's equations: a changing magnetic field induces an electromotive force, which in a conductor will promote a current.

Not meaning to minimize the importance of Maxwell's other equations, but the impact of what this equation embodies is huge. Electric motors, electrical generators, and transformers are all direct applications of a changing magnetic field being related to an electromotive force.

Given the electrified nature of modern society – and the machines that make it that way – we realize that there is a huge impact of Maxwell’s equations in our everyday lives.

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#### Other References

In writing this series, I have been strongly influenced by the following works:

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- D. Fleisch. “A Student’s Guide to Maxwell’s Equations” Cambridge University Press: Cambridge, 2008. This very approachable book takes the tactic of parsing each equation and explaining what each part means. Very useful in understanding the fundamentals of Maxwell’s equations.

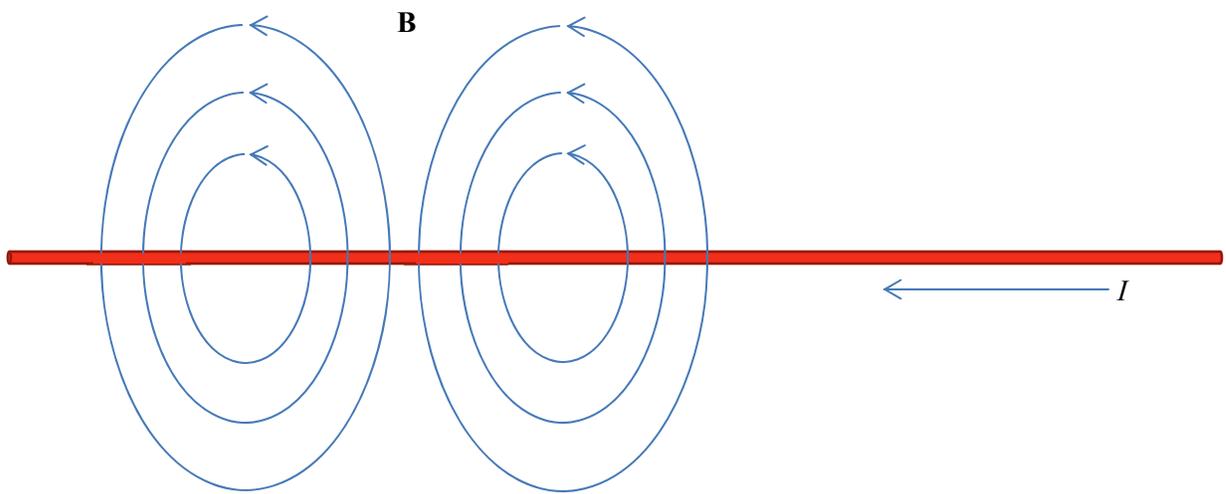


Figure 31. The “shape” of a magnetic field about a wire with a current running through it.

[AU’s note: this may need to be redrawn by an artist.]

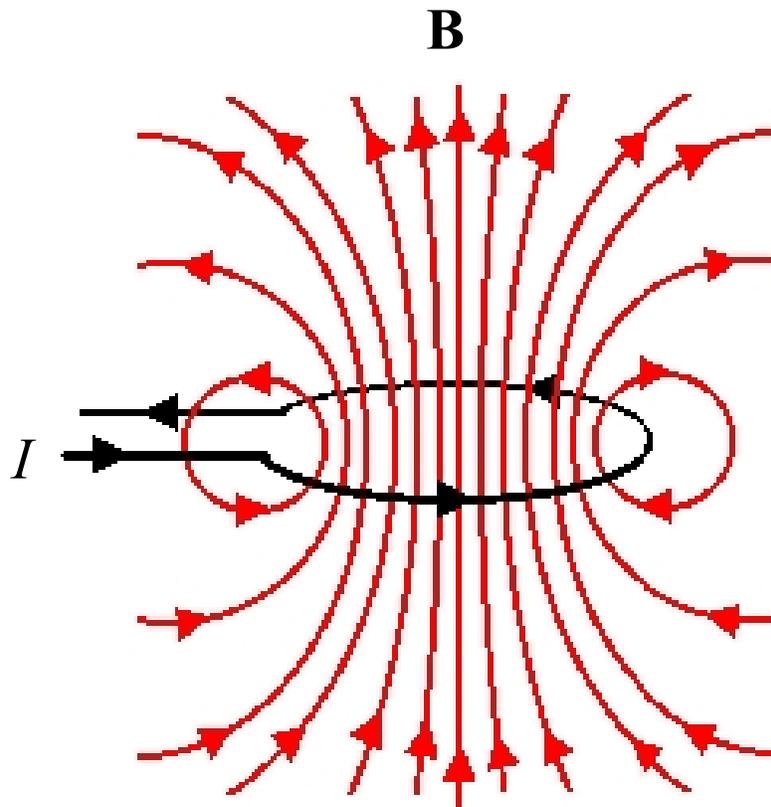


Figure 32. A wire loop generates a magnetic field  $B$  when a current  $I$  runs through the wire. In this case, the magnetic field is an axial field about the central axis of the loop.

[AU's note: Needs to be re-rendered by an artist. Make the black wire of the loop a thicker line and the red lines of the magnetic field thinner.]

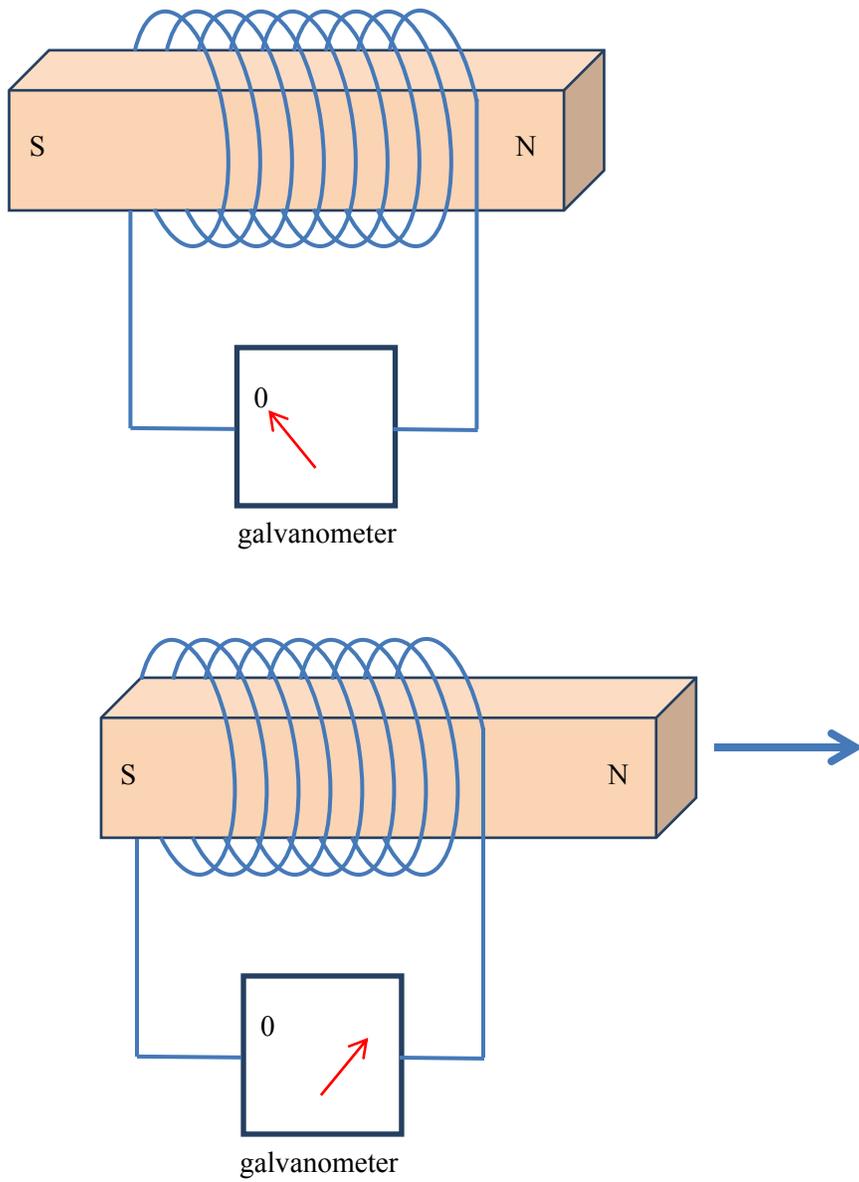


Figure 33. A magnet inside a coil of wire (top) does not generate a current. A magnet moving through a coil of wire (bottom) does generate a current.

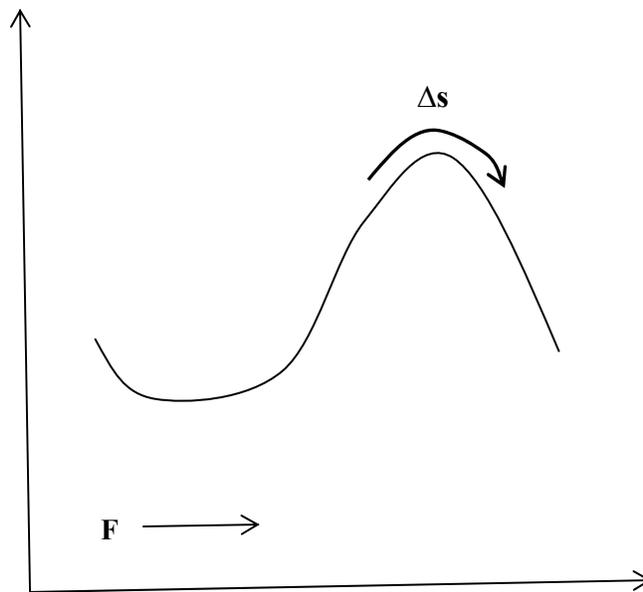


Figure 34. If the force  $\mathbf{F}$  is not parallel to the displacement  $\mathbf{s}$  (shown here as variable, but  $\mathbf{F}$  can be variable too), then the work performed is not as straightforward to calculate.

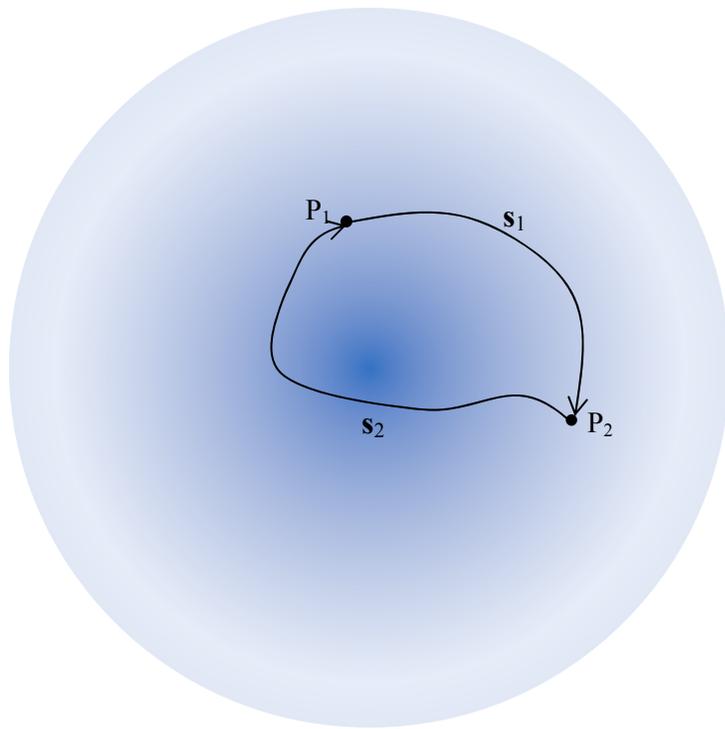


Figure 35. Two arbitrary points in an electric field. The relative strength of the field is indicated by the darkness of the color.

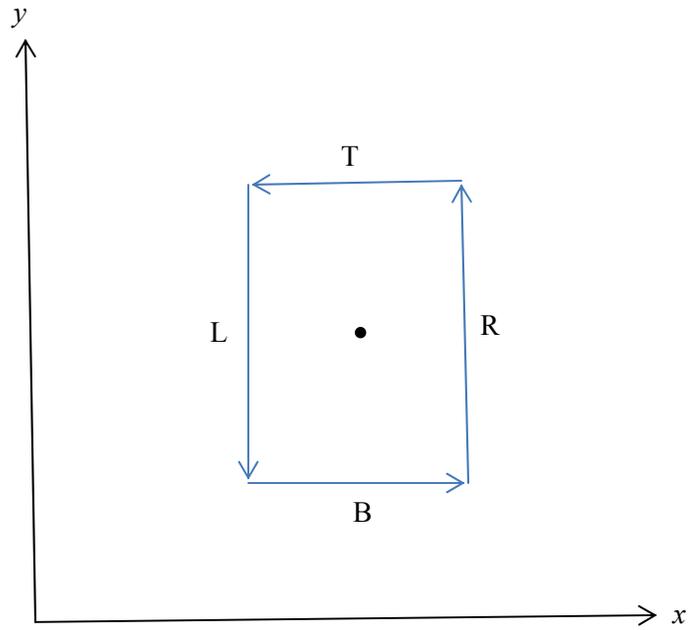


Figure 36. A closed, two-dimensional path around a point.

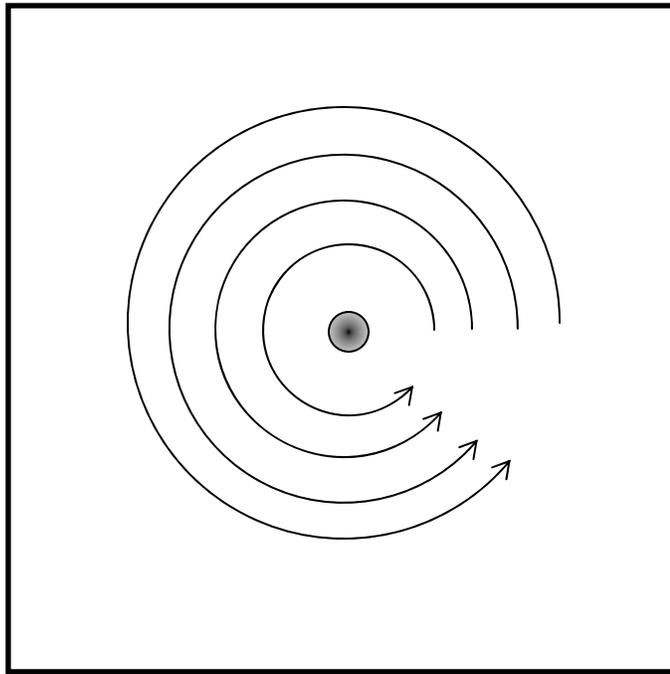


Figure 37. A two-dimensional sink with a film of water rotating counterclockwise as it goes down the drain.

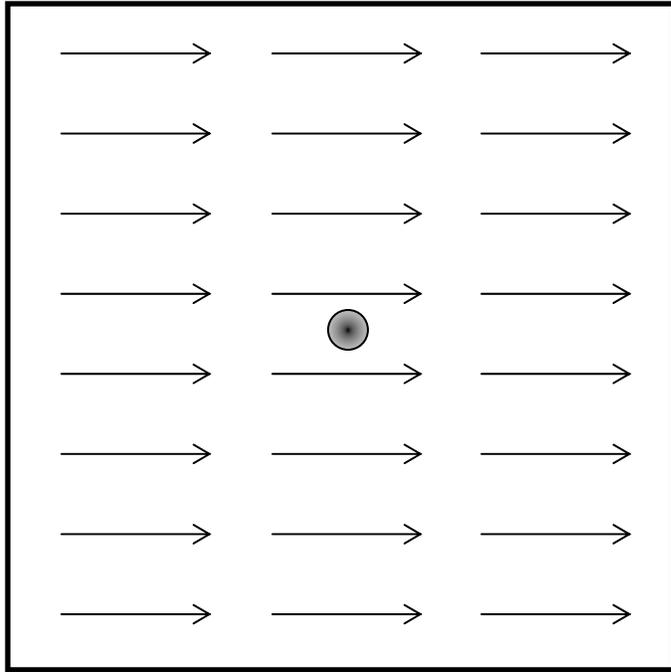


Figure 38. Water flowing in a two-dimensional sink with a constant left-to-right velocity.

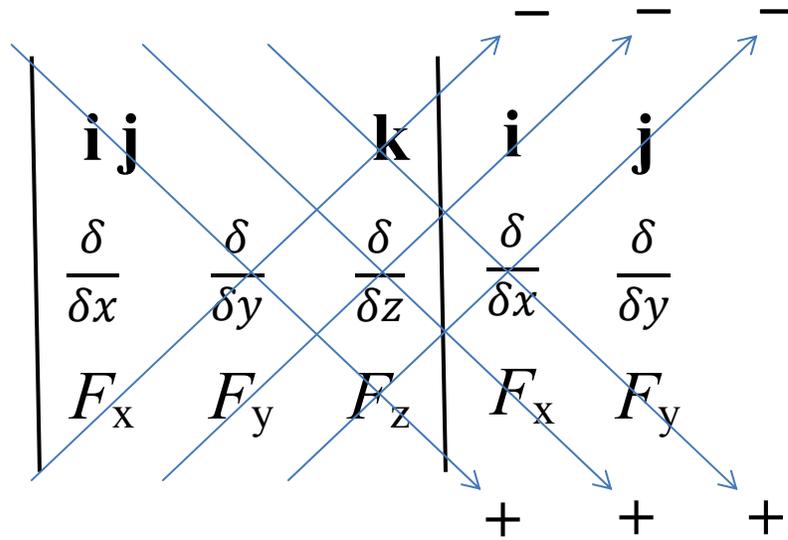


Figure 39. To determine the expression using a determinant, multiply the three terms on each arrow and apply the positive or negative sign to that product, as indicated. Combining all terms yields the proper expression for the curl of a vector function  $\mathbf{F}$  having components  $F_x$ ,  $F_y$ , and  $F_z$ .

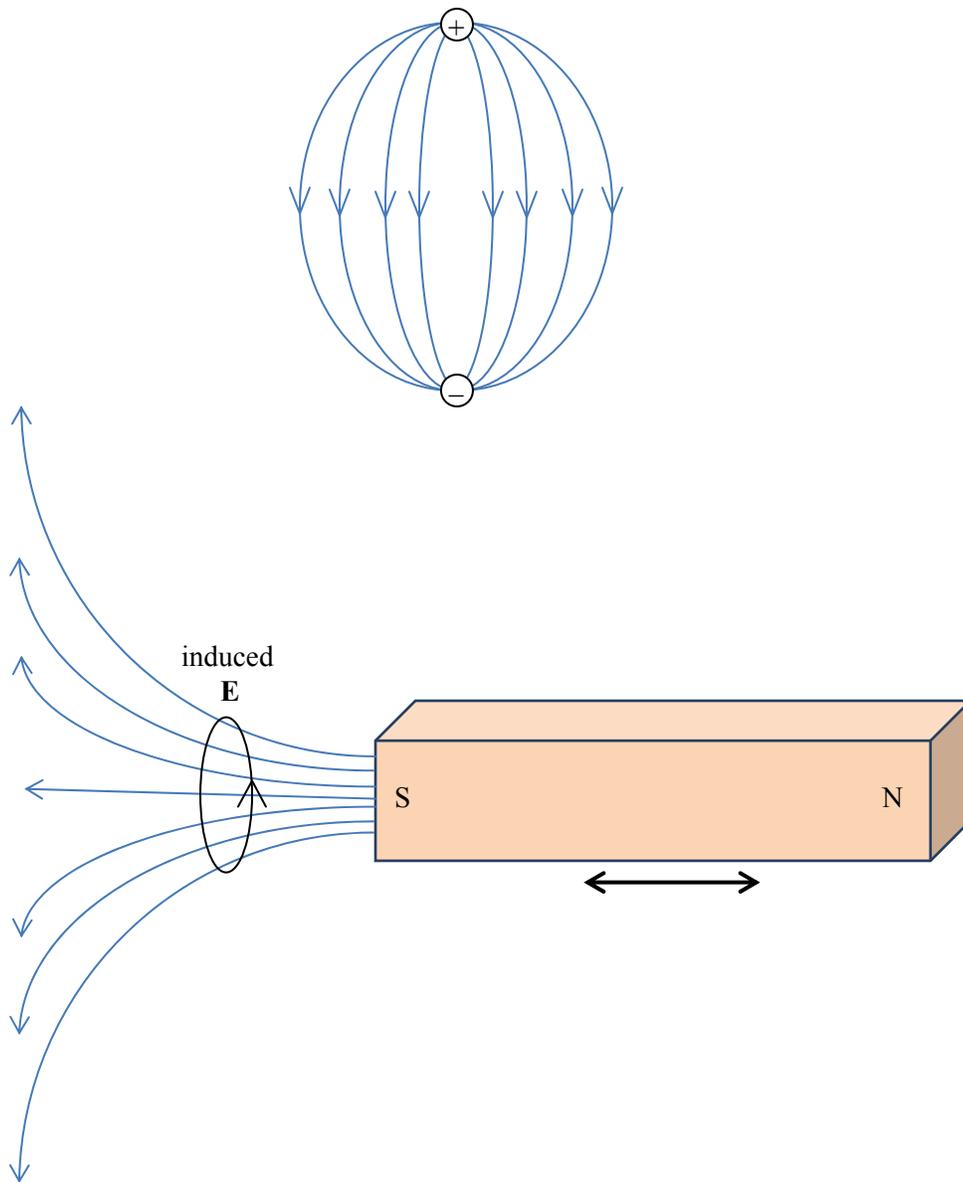


Figure 40. Top: In an electrostatic field, the field lines go from the positive charge to the negative charge. Bottom: a moving magnetic field induces an electric field, but in this case the electric field is in a circle, following the axial nature of the magnetic field lines.

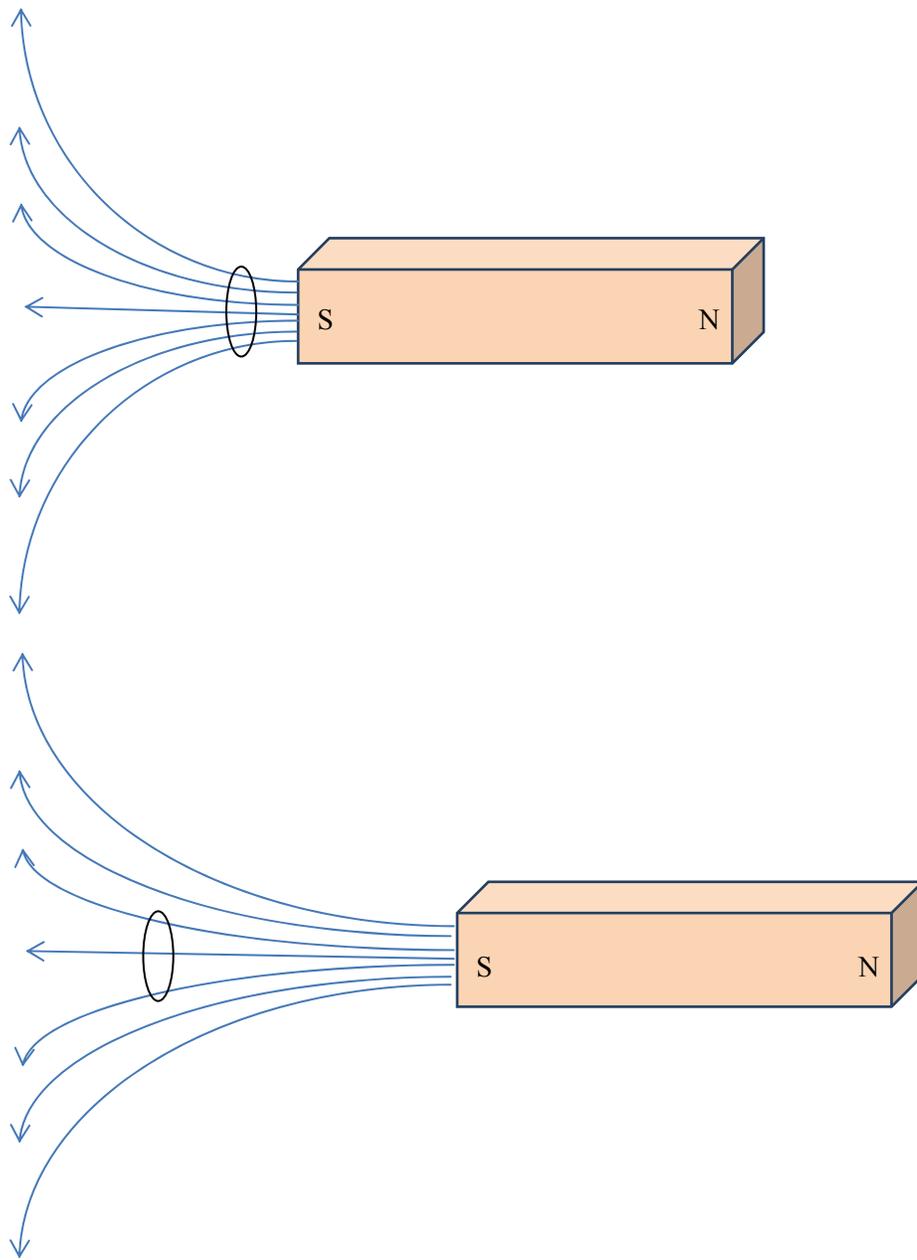


Figure 41. As the magnet is moved farther from the loop, the number of imaginary magnetic field lines intersection the loop changes (here, from seven lines to three lines). It is this change that induces an electric field in the loop.