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Debye series for light scattering by a coated nonspherical particle

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By using the extended boundary condition method, the Debye series is developed for light scattered by a coated nonspherical particle in order to interpret the angular dependence of the scattered intensity in terms of various physical processes. Numerical calculations are performed to study the influence of the coating thickness and the ellipticity of a coated spheroid on the angular position of the α and β primary rainbows, which are produced by partial waves experiencing one internal reflection. The hyperbolic umbilic focal section is demonstrated and is analyzed for both the α and the β rainbows.

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I. INTRODUCTION

Inhomogeneous particles exist widely in nature and industrial processes. Computing their scattering characteristics is significant for elastic-scattering-based particle characterization. As an extension of the Lorenz-Mie theory (LMT), which describes light scattering by a homogeneous sphere [1], the extended boundary condition method [(EBCM), also known as the null-field method] provides a surface-based solution for calculating light scattering by a homogeneous or inhomogeneous nonspherical particle [2–4]. For scattering by a coated particle, it is often of great importance to determine how the properties or the size of either the coating or the core affect the angular dependence of scattering by the entire composite particle. For a coated sphere, the LMT as extended by Aden and Kerker [5] has been widely used to study the effect of the coating or the core. For scattering by a coated nonspherical particle, similar studies have been made using the extension of the EBCM formulation [6–8]. However, the results obtained using these methods give only indirect information on the individual core and coating contributions, since, by nature, the properties of the core and the coating are both considered together when calculating the full scattered intensity.

A more direct way to assess the specific contribution to the scattered intensity of either the core or the coating requires that the coated particle partial-wave-scattering amplitudes be decomposed into the individual contributions of reflection from or transmission through the coating or the core. In this way, one can more clearly determine the importance of either a single physical process or the interference between two or more processes involving the coating or the core to the total scattered intensity. This physically based decomposition of the partial-wave total scattering amplitudes is accomplished by the Debye series. It was first derived for a circular cylinder with normal plane-wave incidence [9]. Using either the variable-separation method (VSM) or the EBCM, the Debye series for a homogeneous sphere and a homogeneous nonspherical particle was derived in Refs. [10] and [11], and Ref. [12], respectively. Concerning various types of inhomogeneous particles, the Debye series for a concentric-coated and radially multilayered sphere was derived in Refs. [13], and Refs. [14] and [15], respectively. By generalizing from plane-wave incidence to more complicated beams, the Debye series, in the context of the generalized Lorenz-Mie theory (GLMT), was derived for a homogeneous sphere [16] and a homogeneous spheroid [17], as well as for a multilayer sphere [18] and a multilayer cylinder [19]. The purpose of this paper is to incorporate the Debye series into the EBCM to interpret scattering by an arbitrarily shaped coated particle.

The body of this study is organized as follows. In order to define the notation we will use for the Debye series and to outline our method of calculation, in Sec. II, we summarize the EBCM formulation of light scattering by a coated particle of arbitrary shape, referring to the formalism of Refs. [6–8]. The verification of the Debye series for scattering of an electromagnetic wave by a coated nonspherical particle entails two separate calculations. The first is the determination of all the partial-wave transmission and reflection amplitudes, which is carried out in Sec. III. We do this in three stages: (i) First, we consider transmission and reflection by a homogeneous particle, (ii) then, we consider the influence of the core, and (iii) finally, we consider scattering by the composite particle. The resulting principal equations for (i)–(iii) of our method are Eqs. (47), (48), (56), and (57); Eqs. (61)–(64); and Eqs. (70)–(73), respectively. The second part is the verification of the Debye series by demonstrating that when all the external and internal reflections are added together, the results exactly match the full partial-wave scattering and interior amplitudes. This is carried out in Sec. IV, using matrix notation. A Taylor series expansion of all the matrix inverses in this section reproduces all scattering orders in Sec. III. We also constructed a computer program based on the principal equations of Sec. III to compute the various Debye-series terms for scattering by a coated nonspherical particle. In Sec. V, in order to test our formalism and computer program, we

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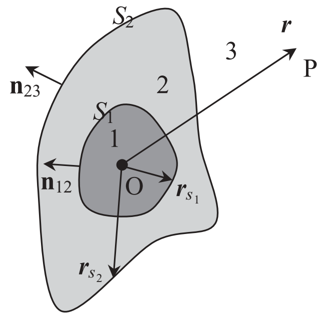


FIG. 1. Cross-sectional geometry of a coated nonspherical particle with the coating bounded by the surface S_2 and the core bounded by the surface S_1. The vector $\boldsymbol{r}$ goes from the origin to any point P in space, and the vectors $\boldsymbol{r}_{S_1}$ and $\boldsymbol{r}_{S_2}$ go from the origin to any point on the surfaces of the core and coating, respectively. The vectors $\mathbf{n}_{12}$ and $\mathbf{n}_{23}$ are the outward directed surface normals of the core and the coating, respectively.

report the results of computations of one-internal-reflection scattering in the vicinity of the twinned first-order rainbow of a coated particle whose core and overall shape were either spherical or spheroidal. In the same section, we also compared our results to the geometric optics rainbow shift by a coated sphere, the Möbius rainbow shift of a homogeneous spheroid, and the evolution of the first-order rainbow into a hyperbolic umbilic caustic in the context of catastrophe optics. Finally, Sec. VI contains a few concluding remarks.

II. LIGHT-SCATTERING THEORY FOR A COATED PARTICLE

Consider a monochromatic and arbitrarily oriented shaped beam incident on a coated nonspherical particle. As indicated in Fig. 1, the coating is bounded by the surface S_2, and the core is bounded by the surface S_1. We designate 3, 2, and 1 to be the regions of the medium, the coating, and the core of the particle, respectively. The particle and the infinite medium are assumed to be isotropic. The medium is nonabsorbing, and, hence, the light in the medium is characterized by a real wave number k_3. The coating and the core can be absorbing, and, hence, the light is characterized by the complex wave numbers k_2 and k_1, respectively. For a nonmagnetic particle and medium, the permeabilities μ_3, μ_2, and μ_1 are equal to the permeability of vacuum μ_0. An incident beam can be described in the particle coordinates either by using standing waves based on the spherical Bessel functions of the first kind or by using incoming and outgoing spherical multipole waves that describe radially propagating waves and are based on Hankel functions of the second and first kinds, respectively. By adopting the first method, the vector spherical wave functions (VSWFs) of the first kind ($\mathbf{M}^{(1)}_{mn}$,$\mathbf{N}^{(1)}_{mn}$) are generated and are used in LMT, GLMT, and the EBCM formulations. By adopting the second method, the VSWFs of the third and fourth kinds ($\mathbf{M}^{(3)}_{mn}$,$\mathbf{N}^{(3)}_{mn}$) and ($\mathbf{M}^{(4)}_{mn}$,$\mathbf{N}^{(4)}_{mn}$) are generated and are used in the Debye-series formulation.

With the time dependence $e^{-i\omega t}$, an incident beam satisfying Maxwell's equations can be represented by an infinite series of partial waves of transverse-electric (TE) and transverse-magnetic (TM) polarizations. These partial waves are weighted by the beam-shape coefficients (BSCs) G^{TE}_{mn} and G^{TM}_{mn} to produce the incident field [12],

$$\mathbf{E}^{(\mathrm{i})}(\boldsymbol{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\left[G^{\mathrm{TE}}_{mn}\mathbf{M}^{(1)}_{mn}(k_3\boldsymbol{r}) + G^{\mathrm{TM}}_{mn}\mathbf{N}^{(1)}_{mn}(k_3\boldsymbol{r})\right], \quad (1)$$

$$\mathbf{H}^{(\mathrm{i})}(\boldsymbol{r}) = -i\frac{k_3}{\omega\mu_3}\sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\left[G^{\mathrm{TM}}_{mn}\mathbf{M}^{(1)}_{mn}(k_3\boldsymbol{r}) + G^{\mathrm{TE}}_{mn}\mathbf{N}^{(1)}_{mn}(k_3\boldsymbol{r})\right], \quad (2)$$

where $\boldsymbol{r}$ is the position vector from the interior origin of coordinates O to an arbitrary field point (see Fig. 1) and the spherical coordinates (r,θ,φ) specifying the position vector $\boldsymbol{r}$ are associated with the particle's Cartesian coordinates (x,y,z). Similarly, the electric field of the scattered light is denoted by the superscript "s" and may be expanded in terms of the VSWFs of the third kind ($\mathbf{M}^{(3)}_{mn}$,$\mathbf{N}^{(3)}_{mn}$):

$$\mathbf{E}^{(\mathrm{s})}(\boldsymbol{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\left[B_{mn}\mathbf{M}^{(3)}_{mn}(k_3\boldsymbol{r}) + A_{mn}\mathbf{N}^{(3)}_{mn}(k_3\boldsymbol{r})\right]. \quad (3)$$

Since our Debye-series decomposition of the partial-wave-scattering amplitudes using the EBCM formulation is built in analogy to the coated particle EBCM formulation, we lay out the major stages of the EBCM formulation for the remainder of this section. By following the development of Ref. [6], we apply Schelkunoff's equivalence theorem [20] to the incident and scattered fields. The scattering object can then be replaced by a set of surface currents radiating the scattered fields ($\mathbf{E}^{(\mathrm{s})}$,$\mathbf{H}^{(\mathrm{s})}$) in region 3 and the negative incident fields in regions 1 and 2 so that

$$\left.\begin{matrix}\mathbf{E}^{(\mathrm{e})}(\boldsymbol{r})\\ 0\end{matrix}\right\} = \mathbf{E}^{(\mathrm{i})}(\boldsymbol{r}) + \int_{S_2}\left\{i\omega\mu_0\left[\mathbf{n}_{23}\times\mathbf{H}_+(\boldsymbol{r}_{S_2})\right]\cdot\overline{\overline{G}}\left(\boldsymbol{r}_{S_2}|\boldsymbol{r}\right) + \left[\mathbf{n}_{23}\times\mathbf{E}_+(\boldsymbol{r}_{S_2})\right]\cdot\left[\nabla\times\overline{\overline{G}}\left(\boldsymbol{r}_{S_2}|\boldsymbol{r}\right)\right]\right\}dS_2 \quad \text{for} \quad \boldsymbol{r}\in\begin{cases}\text{region 3},\\ \text{regions 1 and 2},\end{cases} \quad (4)$$

where $\mathbf{E}^{(\mathrm{e})}$ is the total electric field outside the particle, and $\mathbf{E}^{(\mathrm{e})} = \mathbf{E}^{(\mathrm{i})} + \mathbf{E}^{(\mathrm{s})}$. The integral represents the radiation field of the surface currents, and $\overline{\overline{G}}$ is the free-space Green's dyadic. For $r > r_{S_2}$, substitution of Eq. (4) into Eq. (3) and using the Green's dyadic expansion in terms of the VSWFs of the first and third kinds give [3]

$$\begin{pmatrix}B_{mn}\\ A_{mn}\end{pmatrix} = -(-1)^m k_3\int_{S_2}\left\{\omega\mu_0\left[\mathbf{n}_{23}\times\mathbf{H}_+(\boldsymbol{r}_{S_2})\right]\cdot\begin{pmatrix}\mathbf{M}^{(1)}_{-mn}(k_3 r_{S_2},\theta,\varphi)\\ \mathbf{N}^{(1)}_{-mn}(k_3 r_{S_2},\theta,\varphi)\end{pmatrix} - ik_3\left[\mathbf{n}_{23}\times\mathbf{E}_+(\boldsymbol{r}_{S_2})\right]\cdot\begin{pmatrix}\mathbf{N}^{(1)}_{-mn}(k_3 r_{S_2},\theta,\varphi)\\ \mathbf{M}^{(1)}_{-mn}(k_3 r_{S_2},\theta,\varphi)\end{pmatrix}\right\}dS_2. \quad (5)$$

For $r < r_{S_2}$, substitution of Eq. (4) into Eq. (1) gives

$$\begin{pmatrix} G_{mn}^{\mathrm{TE}} \\ G_{mn}^{\mathrm{TM}} \end{pmatrix} = (-1)^m k_3 \int_{S_2} \left\{ \omega\mu_0 [\mathbf{n}_{23} \times \mathbf{H}_+(\boldsymbol{r}_{S_2})] \cdot \begin{pmatrix} \mathbf{M}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi) \\ \mathbf{N}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi) \end{pmatrix} - i k_3 [\mathbf{n}_{23} \times \mathbf{E}_+(\boldsymbol{r}_{S_2})] \cdot \begin{pmatrix} \mathbf{N}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi) \\ \mathbf{M}_{-mn}^{(3)}(k_2 r_{S_2},\theta,\varphi) \end{pmatrix} \right\} dS_2. \tag{6}$$

The continuity of the tangential components of the fields $(\mathbf{E}_+,\mathbf{H}_+)$ at the coating surface S_2 is ensured by the boundary conditions

$$\mathbf{n}_{23} \times \mathbf{E}_+(\boldsymbol{r}_{S_2}) = \mathbf{n}_{23} \times \mathbf{E}_-(\boldsymbol{r}_{S_2}), \tag{7}$$

$$\mathbf{n}_{23} \times \mathbf{H}_+(\boldsymbol{r}_{S_2}) = \mathbf{n}_{23} \times \mathbf{H}_-(\boldsymbol{r}_{S_2}), \tag{8}$$

where the subscript "ij" of the surface normal vector $\mathbf{n}$ indicates a direction from region i to region j. The quantities $(\mathbf{n}_{23} \times \mathbf{E}_-)$ and $(\mathbf{n}_{23} \times \mathbf{H}_-)$ are obtained from the fields assumed to be expanded in the following form throughout region 2:

$$\mathbf{E}^{(2)}(\boldsymbol{r}) = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{+n'} [\delta_{m'n'} \mathbf{M}_{m'n'}^{(1)}(k_2\boldsymbol{r}) + \gamma_{m'n'} \mathbf{N}_{m'n'}^{(1)}(k_2\boldsymbol{r}) + \beta_{m'n'} \mathbf{M}_{m'n'}^{(3)}(k_2\boldsymbol{r}) + \alpha_{m'n'} \mathbf{N}_{m'n'}^{(3)}(k_2\boldsymbol{r})], \tag{9}$$

$$\mathbf{H}^{(2)}(\boldsymbol{r}) = -i \frac{k_2}{\omega\mu_0} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{+n'} [\gamma_{m'n'} \mathbf{M}_{m'n'}^{(1)}(k_2\boldsymbol{r}) + \delta_{m'n'} \mathbf{N}_{m'n'}^{(1)}(k_2\boldsymbol{r}) + \alpha_{m'n'} \mathbf{M}_{m'n'}^{(3)}(k_2\boldsymbol{r}) + \beta_{m'n'} \mathbf{N}_{m'n'}^{(3)}(k_2\boldsymbol{r})]. \tag{10}$$

Accordingly, we have

$$\mathbf{n}_{23} \times \mathbf{E}_-(\boldsymbol{r}_{S_2}) = \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{+n'} [\delta_{m'n'} \mathbf{n}_{23} \times \mathbf{M}_{m'n'}^{(1)}(k_2 \boldsymbol{r}_{S_2}) + \gamma_{m'n'} \mathbf{n}_{23} \times \mathbf{N}_{m'n'}^{(1)}(k_2 \boldsymbol{r}_{S_2}) + \beta_{m'n'} \mathbf{n}_{23} \times \mathbf{M}_{m'n'}^{(3)}(k_2 \boldsymbol{r}_{S_2}) + \alpha_{m'n'} \mathbf{n}_{23} \times \mathbf{N}_{m'n'}^{(3)}(k_2 \boldsymbol{r}_{S_2})], \tag{11}$$

$$\mathbf{n}_{23} \times \mathbf{H}_-(\boldsymbol{r}_{S_2}) = -i \frac{k_2}{\omega\mu_0} \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{+n'} [\delta_{m'n'} \mathbf{n}_{23} \times \mathbf{N}_{m'n'}^{(1)}(k_2 \boldsymbol{r}_{S_2}) + \gamma_{m'n'} \mathbf{n}_{23} \times \mathbf{M}_{m'n'}^{(1)}(k_2 \boldsymbol{r}_{S_2}) + \beta_{m'n'} \mathbf{n}_{23} \times \mathbf{N}_{m'n'}^{(3)}(k_2 \boldsymbol{r}_{S_2}) + \alpha_{m'n'} \mathbf{n}_{23} \times \mathbf{M}_{m'n'}^{(3)}(k_2 \boldsymbol{r}_{S_2})]. \tag{12}$$

Substitution of Eqs. (7), (8), (11), and (12) into Eq. (6) gives the relation between the field coefficients $(\alpha_{m'n'},\beta_{m'n'},\gamma_{m'n'},\delta_{m'n'})$ in the coating and the BSCs,

$$[Q_{mn,m'n'}^{(1,3),S_2}(k_2,k_3)] \begin{bmatrix} \delta_{m'n'} \\ \cdots \\ \gamma_{m'n'} \end{bmatrix} + [Q_{mn,m'n'}^{(3,3),S_2}(k_2,k_3)] \begin{bmatrix} \beta_{m'n'} \\ \cdots \\ \alpha_{m'n'} \end{bmatrix} = \begin{bmatrix} G_{mn}^{\mathrm{TE}} \\ \cdots \\ G_{mn}^{\mathrm{TM}} \end{bmatrix}, \tag{13}$$

where

$$[Q_{mn,m'n'}^{(1,3),S_2}(k_2,k_3)] = \begin{bmatrix} -ik_2k_3 K_{mn,m'n'}^{(1,3)}(k_2,k_3) - ik_3^2 J_{mn,m'n'}^{(1,3)}(k_2,k_3) & -ik_2k_3 L_{mn,m'n'}^{(1,3)}(k_2,k_3) - ik_3^2 I_{mn,m'n'}^{(1,3)}(k_2,k_3) \\ -ik_2k_3 I_{mn,m'n'}^{(1,3)}(k_2,k_3) - ik_3^2 L_{mn,m'n'}^{(1,3)}(k_2,k_3) & -ik_2k_3 J_{mn,m'n'}^{(1,3)}(k_2,k_3) - ik_3^2 K_{mn,m'n'}^{(1,3)}(k_2,k_3) \end{bmatrix}, \tag{14}$$

$$[Q_{mn,m'n'}^{(3,3),S_2}(k_2,k_3)] = \begin{bmatrix} -ik_2k_3 \dot{K}_{mn,m'n'}^{(3,3)}(k_2,k_3) - ik_3^2 \dot{J}_{mn,m'n'}^{(3,3)}(k_2,k_3) & -ik_2k_3 \dot{L}_{mn,m'n'}^{(3,3)}(k_2,k_3) - ik_3^2 \dot{I}_{mn,m'n'}^{(3,3)}(k_2,k_3) \\ -ik_2k_3 \dot{I}_{mn,m'n'}^{(3,3)}(k_2,k_3) - ik_3^2 \dot{L}_{mn,m'n'}^{(3,3)}(k_2,k_3) & -ik_2k_3 \dot{J}_{mn,m'n'}^{(3,3)}(k_2,k_3) - ik_3^2 \dot{K}_{mn,m'n'}^{(3,3)}(k_2,k_3) \end{bmatrix}. \tag{15}$$

The superscript (i,j) of the Q matrices and their elements indicate the sequence of the two types of VSWFs in the cross product of the kernel of the surface integrals, namely,

$$I_{mn,m'n'}^{(1,3)}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23} \cdot \mathbf{M}_{m'n'}^{(1)}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{M}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi)\, dS_2, \tag{16}$$

$$J_{mn,m'n'}^{(1,3)}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23} \cdot \mathbf{M}_{m'n'}^{(1)}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{N}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi)\, dS_2, \tag{17}$$

$$K_{mn,m'n'}^{(1,3)}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23} \cdot \mathbf{N}_{m'n'}^{(1)}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{M}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi)\, dS_2, \tag{18}$$

$$L_{mn,m'n'}^{(1,3)}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23} \cdot \mathbf{N}_{m'n'}^{(1)}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{N}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi)\, dS_2 \tag{19}$$

and

$$\dot{I}_{mn,m'n'}^{(3,3)}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23} \cdot \mathbf{M}_{m'n'}^{(3)}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{M}_{-mn}^{(3)}(k_3 r_{S_2},\theta,\varphi)\, dS_2, \tag{20}$$

$$\dot{J}^{(3,3)}_{mn,m'n'}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23}\cdot\mathbf{M}^{(3)}_{m'n'}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{N}^{(3)}_{-mn}(k_3 r_{S_2},\theta,\varphi)\, dS_2, \tag{21}$$

$$\dot{K}^{(3,3)}_{mn,m'n'}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23}\cdot\mathbf{N}^{(3)}_{m'n'}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{M}^{(3)}_{-mn}(k_3 r_{S_2},\theta,\varphi)\, dS_2, \tag{22}$$

$$\dot{L}^{(3,3)}_{mn,m'n'}(k_2,k_3) = (-1)^m \int_{S_2} \mathbf{n}_{23}\cdot\mathbf{N}^{(3)}_{m'n'}(k_2 r_{S_2},\theta,\varphi) \times \mathbf{N}^{(3)}_{-mn}(k_3 r_{S_2},\theta,\varphi)\, dS_2. \tag{23}$$

If the coefficients $\alpha_{mn},\beta_{mn},\gamma_{mn},\delta_{mn}$ in the coating were known, substitution of Eqs. (7), (8), (11), and (12) into Eq. (5) gives the relation between the scattered field coefficients (A_{mn},B_{mn}) and the coefficients $(\alpha_{mn},\beta_{mn},\gamma_{mn},\delta_{mn})$,

$$\begin{bmatrix} B_{mn} \\ \cdots \\ A_{mn}\end{bmatrix} = -\left[Q^{(1,1),S_2}_{mn,m'n'}(k_2,k_3)\right]\begin{bmatrix} \delta_{m'n'} \\ \cdots \\ \gamma_{m'n'}\end{bmatrix} - \left[Q^{(3,1),S_2}_{mn,m'n'}(k_2,k_3)\right]\begin{bmatrix} \beta_{m'n'} \\ \cdots \\ \alpha_{m'n'}\end{bmatrix}, \tag{24}$$

where

$$\left[Q^{(1,1),S_2}_{mn,m'n'}(k_2,k_3)\right] = \begin{bmatrix} -ik_2k_3\tilde{K}^{(1,1)}_{mn,m'n'} - ik_3^2\tilde{J}^{(1,1)}_{mn,m'n'} & \vdots & -ik_2k_3\tilde{L}^{(1,1)}_{mn,m'n'} - ik_3^2\tilde{I}^{(1,1)}_{mn,m'n'} \\ \cdots & \vdots & \cdots \\ -ik_2k_3\tilde{I}^{(1,1)}_{mn,m'n'} - ik_3^2\tilde{L}^{(1,1)}_{mn,m'n'} & \vdots & -ik_2k_3\tilde{J}^{(1,1)}_{mn,m'n'} - ik_3^2\tilde{K}^{(1,1)}_{mn,m'n'}\end{bmatrix}, \tag{25}$$

$$\left[Q^{(3,1),S_2}_{mn,m'n'}(k_2,k_3)\right] = \begin{bmatrix} -ik_2k_3\tilde{\dot{K}}^{(3,1)}_{mn,m'n'} - ik_3^2\tilde{\dot{J}}^{(3,1)}_{mn,m'n'} & \vdots & -ik_2k_3\tilde{\dot{L}}^{(3,1)}_{mn,m'n'} - ik_3^2\tilde{\dot{I}}^{(3,1)}_{mn,m'n'} \\ \cdots & \vdots & \cdots \\ -ik_2k_3\tilde{\dot{I}}^{(3,1)}_{mn,m'n'} - ik_3^2\tilde{\dot{L}}^{(3,1)}_{mn,m'n'} & \vdots & -ik_2k_3\tilde{\dot{J}}^{(3,1)}_{mn,m'n'} - ik_3^2\tilde{\dot{K}}^{(3,1)}_{mn,m'n'}\end{bmatrix}. \tag{26}$$

The matrices $Q^{(1,1),S_2}$ and $Q^{(3,1),S_2}$ consist of the elements $(\tilde{I},\tilde{J},\tilde{K},\tilde{L},\tilde{\dot{I}},\tilde{\dot{J}},\tilde{\dot{K}},\tilde{\dot{L}})$ that are expressed similarly as $(I,J,K,L,\dot{I},\dot{J},\dot{K},\dot{L})$ except that $\mathbf{M}^{(3)}_{-mn}$ is replaced by $\mathbf{M}^{(1)}_{-mn}$ and $\mathbf{N}^{(3)}_{-mn}$ is replaced by $\mathbf{N}^{(1)}_{-mn}$.

We then apply the equivalence principle to the electromagnetic fields in region 2, obtaining the surface currents distributed on the two surfaces enclosing region 2, S_1 and S_2. They radiate in space to produce the original fields in region 2 and the null fields outside it so that [6]

$$\left.\begin{matrix}\mathbf{E}^{(2)}(\boldsymbol{r}) \\ 0\end{matrix}\right\} = \int_{S_2}\{i\omega\mu_0[\mathbf{n}_{32}\times\mathbf{H}^{(2)}_{-}(\boldsymbol{r}_{S_2})]\cdot\bar{\bar{G}}(\boldsymbol{r}_{S_2}|\boldsymbol{r}) + [\mathbf{n}_{32}\times\mathbf{E}^{(2)}_{-}(\boldsymbol{r}_{S_2})]\cdot[\nabla\times\bar{\bar{G}}(\boldsymbol{r}_{S_2}|\boldsymbol{r})]\}\,dS_2 + \int_{S_1}\{i\omega\mu_0[\mathbf{n}_{12}\times\mathbf{H}^{(2)}_{+}(\boldsymbol{r}_{S_1})]\cdot\bar{\bar{G}}(\boldsymbol{r}_{S_1}|\boldsymbol{r}) + [\mathbf{n}_{12}\times\mathbf{E}^{(2)}_{+}(\boldsymbol{r}_{S_1})]\cdot[\nabla\times\bar{\bar{G}}(\boldsymbol{r}_{S_1}|\boldsymbol{r})]\}\,dS_1 \quad \text{for } \boldsymbol{r}\in\begin{cases}\text{region 2,}\\ \text{regions 1 and 3.}\end{cases} \tag{27}$$

The continuity of the tangential components of the fields $(\mathbf{E}_{+},\mathbf{H}_{+})$ at the core-coating interface S_1 is ensured by the boundary conditions

$$\mathbf{n}_{12}\times\mathbf{E}^{(2)}_{+}(\boldsymbol{r}_{S_1}) = \mathbf{n}_{12}\times\mathbf{E}^{(1)}_{-}(\boldsymbol{r}_{S_1}), \tag{28}$$

$$\mathbf{n}_{12}\times\mathbf{H}^{(2)}_{+}(\boldsymbol{r}_{S_1}) = \mathbf{n}_{12}\times\mathbf{H}^{(1)}_{-}(\boldsymbol{r}_{S_1}). \tag{29}$$

The quantities $(\mathbf{n}_{12}\times\mathbf{E}^{(1)}_{-})$ and $(\mathbf{n}_{12}\times\mathbf{H}^{(1)}_{-})$ are obtained from the fields, which are assumed to be expanded in the following form throughout region 1:

$$\mathbf{E}^{(1)}(\boldsymbol{r}) = \sum_{n'=1}^{\infty}\sum_{m'=-n'}^{+n'}\left[D_{m'n'}\mathbf{M}^{(1)}_{m'n'}(k_1\boldsymbol{r}) + C_{m'n'}\mathbf{N}^{(1)}_{m'n'}(k_1\boldsymbol{r})\right], \tag{30}$$

$$\mathbf{H}^{(1)}(\boldsymbol{r}) = -i\frac{k_1}{\omega\mu_0}\sum_{n'=1}^{\infty}\sum_{m'=-n'}^{+n'}\left[D_{m'n'}\mathbf{N}^{(1)}_{m'n'}(k_1\boldsymbol{r}) + C_{m'n'}\mathbf{M}^{(1)}_{m'n'}(k_1\boldsymbol{r})\right]. \tag{31}$$

By restricting r to lie outside S_2, substituting Eqs. (28) and (29) into Eq. (27) and utilizing Eqs. (9), (10), (30), and (31), and invoking the following orthogonalities of the VSWFs defined in Ref. [21],

$$\int_{S_2}\left\{\mathbf{n}_{32}\times\mathbf{M}^{(3)}_{mn}(k_2\boldsymbol{r}_{S_2})\cdot\begin{bmatrix}\mathbf{M}^{(3)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\\ \mathbf{N}^{(3)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\end{bmatrix} + \mathbf{n}_{32}\times\mathbf{N}^{(3)}_{mn}(k_2\boldsymbol{r}_{S_2})\cdot\begin{bmatrix}\mathbf{M}^{(3)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\\ \mathbf{N}^{(3)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\end{bmatrix}\right\}dS_2 = 0, \tag{32}$$

$$\int_{S_2}\left\{\mathbf{n}_{32}\times\mathbf{M}^{(1)}_{mn}(k_2\boldsymbol{r}_{S_2})\cdot\begin{bmatrix}\mathbf{M}^{(1)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\\ \mathbf{N}^{(1)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\end{bmatrix} + \mathbf{n}_{32}\times\mathbf{N}^{(1)}_{mn}(k_2\boldsymbol{r}_{S_2})\cdot\begin{bmatrix}\mathbf{M}^{(1)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\\ \mathbf{N}^{(1)}_{-m'n'}(k_2\boldsymbol{r}_{S_2})\end{bmatrix}\right\}dS_2 = 0, \tag{33}$$

$$(-1)^m ik_2^2 \int_{S_2} \left\{ \mathbf{n}_{32} \times \mathbf{M}_{mn}^{(3)}(k_2\boldsymbol{r}_{S_2}) \cdot \begin{bmatrix} \mathbf{M}_{-m'n'}^{(1)}(k_2\boldsymbol{r}_{S_2}) \\ \mathbf{N}_{-m'n'}^{(1)}(k_2\boldsymbol{r}_{S_2}) \end{bmatrix} + \mathbf{n}_{32} \times \mathbf{N}_{mn}^{(3)}(k_2\boldsymbol{r}_{S_2}) \cdot \begin{bmatrix} \mathbf{M}_{-m'n'}^{(1)}(k_2\boldsymbol{r}_{S_2}) \\ \mathbf{N}_{-m'n'}^{(1)}(k_2\boldsymbol{r}_{S_2}) \end{bmatrix} \right\} dS_2 = \begin{pmatrix} 0 \\ \delta_{mm'}\delta_{nn'} \end{pmatrix}, \tag{34}$$

the coefficients (α_{mn},β_{mn}) for the fields in region 2 are related to the coefficients (C_{mn},D_{mn}) for the fields in region 1 by

$$\begin{bmatrix} \beta_{mn} \\ \cdots \\ \alpha_{mn} \end{bmatrix} = -\left[Q_{mn,m'n'}^{(1,1),S_1}(k_1,k_2)\right] \begin{bmatrix} D_{m'n'} \\ \cdots\cdot \\ C_{m'n'} \end{bmatrix}. \tag{35}$$

By restricting r to lie inside S_1, the coefficients $(\gamma_{mn},\delta_{mn})$ are related to (C_{mn},D_{mn}) by

$$\begin{bmatrix} \delta_{mn} \\ \cdots \\ \gamma_{mn} \end{bmatrix} = \left[Q_{mn,m'n'}^{(1,3),S_1}(k_1,k_2)\right] \begin{bmatrix} D_{m'n'} \\ \cdots\cdot \\ C_{m'n'} \end{bmatrix}. \tag{36}$$

In Eqs. (35) and (36), $Q^{(1,1),S_1}(k_1,k_2)$ and $Q^{(1,3),S_1}(k_1,k_2)$ are similar in form to $Q^{(1,1),S_2}(k_2,k_3)$ and $Q^{(1,3),S_2}(k_2,k_3)$, respectively, except that the matrix elements contain an integration over S_1 instead of over S_2, the surface normal $\mathbf{n}_{23}$ is replaced by $\mathbf{n}_{12}$, and the wave numbers k_2 and k_3 in the calculation of $Q^{(1,1),S_2}(k_2,k_3)$ and $Q^{(1,3),S_2}(k_2,k_3)$ are replaced by k_1 and k_2, respectively. For the remainder of this paper, column vectors or matrices for which partial-wave subscripts are suppressed are written in boldface.

Substituting Eqs. (35) and (36) into Eqs. (13) and (24) and eliminating the column vector containing $\boldsymbol{C}$ and $\boldsymbol{D}$ give a direct relation between the incident and scattered fields,

$$\begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{A} \end{bmatrix} = -[\boldsymbol{T}] \begin{bmatrix} \boldsymbol{G}^{\text{TE}} \\ \boldsymbol{G}^{\text{TM}} \end{bmatrix}, \tag{37}$$

where

$$\boldsymbol{T} = \{[\boldsymbol{Q}^{(1,1),S_2}(k_2,k_3)] - [\boldsymbol{Q}^{(3,1),S_2}(k_2,k_3)][\boldsymbol{T}^{S_1}]\} \times \{[\boldsymbol{Q}^{(1,3),S_2}(k_2,k_3)] - [\boldsymbol{Q}^{(3,3),S_2}(k_2,k_3)][\boldsymbol{T}^{S_1}]\}^{-1} \tag{38}$$

and

$$[\boldsymbol{T}^{S_1}] = [\boldsymbol{Q}^{(1,1),S_1}(k_1,k_2)][\boldsymbol{Q}^{(1,3),S_1}(k_1,k_2)]^{-1}. \tag{39}$$

For a homogeneous particle, $[\boldsymbol{T}^{S_1}] = 0$ and Eq. (38) reduces to the homogenous particle T matrix [22], namely, $[\boldsymbol{T}] = [\boldsymbol{Q}^{(1,1),S_2}(k_2,k_3)][\boldsymbol{Q}^{(1,3),S_2}(k_2,k_3)]^{-1}$. Substitution of Eqs. (35) and (36) into Eq. (13) gives the relation between the interior coefficients and the incident amplitudes,

$$\begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{C} \end{bmatrix} = [\boldsymbol{U}]^{-1} \begin{bmatrix} \boldsymbol{G}^{\text{TE}} \\ \boldsymbol{G}^{\text{TM}} \end{bmatrix}, \tag{40}$$

where

$$[\boldsymbol{U}] = \{[\boldsymbol{Q}^{(1,3),S_2}(k_2,k_3)][\boldsymbol{Q}^{(1,3),S_1}(k_1,k_2)] - [\boldsymbol{Q}^{(3,3),S_2}(k_2,k_3)][\boldsymbol{Q}^{(1,1),S_1}(k_1,k_2)]\}^{-1}. \tag{41}$$

Finally, substitution of Eq. (40) into Eqs. (35) and (36) gives the relation between the coating field coefficients and the incident amplitudes,

$$\begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{\alpha} \end{bmatrix} = -[\boldsymbol{Q}^{(1,1),S_1}(k_1,k_2)][\boldsymbol{U}]^{-1} \begin{bmatrix} \boldsymbol{G}^{\text{TE}} \\ \boldsymbol{G}^{\text{TM}} \end{bmatrix}, \tag{42}$$

$$\begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\gamma} \end{bmatrix} = [\boldsymbol{Q}^{(1,3),S_1}(k_1,k_2)][\boldsymbol{U}]^{-1} \begin{bmatrix} \boldsymbol{G}^{\text{TE}} \\ \boldsymbol{G}^{\text{TM}} \end{bmatrix}. \tag{43}$$

III. DEBYE SERIES FOR A COATED PARTICLE

A. Homogeneous particle

Since both the notation and the procedure for developing the Debye series for a coated particle depend crucially on that used for a homogeneous particle [12], we first briefly review the homogeneous particle case. In the Debye-series formulation, the spherical Hankel functions of the first kind, $h_n^{(1)}$, and the second kind, $h_n^{(2)}$, are used to denote the outgoing and the incoming traveling waves, respectively. As the incident wave interacts with a homogeneous particle for the first time (with the scattering order $p = 0$), the incident (i) and transmitted (t) waves are incoming, and the VSWFs of the fourth type $(\mathbf{M}_{mn}^{(4)},\mathbf{N}_{mn}^{(4)})$ are used for their description. The externally reflected (r) wave is outgoing, and, hence, the VSWFs of the third type $(\mathbf{M}_{mn}^{(3)},\mathbf{N}_{mn}^{(3)})$ are used for its description. However, when the incident wave interacts with the particle for the $(p+1)^{\text{th}}$ time (with the scattering order $p \geqslant 1$), the incident and transmitted waves are outgoing while the internally reflected wave is incoming. The types of VSWFs used in this case must change accordingly to match the character of the radially propagating waves.

The incident, externally reflected, and transmitted fields at the order $p = 0$ are described by

$$\mathbf{E}_{p=0}^{(\text{i})} = \sum_{n_0=1}^{\infty} \sum_{m_0=-n_0}^{+n_0} \left[G_{m_0n_0}^{\text{TE}} \mathbf{M}_{m_0n_0}^{(4)}(k_3\boldsymbol{r}) + G_{m_0n_0}^{\text{TM}} \mathbf{N}_{m_0n_0}^{(4)}(k_3\boldsymbol{r})\right], \tag{44}$$

$$\mathbf{E}_{p=0}^{(\text{r})} = \sum_{n_1=1}^{\infty} \sum_{m_1=-n_1}^{+n_1} \left[W_{m_1n_1}^{323,\text{TE}}(p=0)\mathbf{M}_{m_1n_1}^{(3)}(k_3\boldsymbol{r}) + W_{m_1n_1}^{323,\text{TM}}(p=0)\mathbf{N}_{m_1n_1}^{(3)}(k_3\boldsymbol{r})\right], \tag{45}$$

$$\mathbf{E}_{p=0}^{(\text{t})} = \sum_{n_1=1}^{\infty} \sum_{m_1=-n_1}^{+n_1} \left[W_{m_1n_1}^{32,\text{TE}}(p=0)\mathbf{M}_{m_1n_1}^{(4)}(k_2\boldsymbol{r}) + W_{m_1n_1}^{32,\text{TM}}(p=0)\mathbf{N}_{m_1n_1}^{(4)}(k_2\boldsymbol{r})\right], \tag{46}$$

respectively. By using the Green's dyadic expansion in terms of the VSWFs of the fourth and third kinds, the reflected and transmitted fields are related to the incident field by [12]

$$\begin{bmatrix} \boldsymbol{W}^{323,\text{TE}}(p=0) \\ \boldsymbol{W}^{323,\text{TM}}(p=0) \end{bmatrix} = -[\boldsymbol{T}^{S_2,D_0}(k_2,k_3)] \begin{bmatrix} \boldsymbol{G}^{\text{TE}} \\ \boldsymbol{G}^{\text{TM}} \end{bmatrix}, \tag{47}$$

and

$$\begin{bmatrix} \boldsymbol{W}^{32,\text{TE}}(p=0) \\ \boldsymbol{W}^{32,\text{TM}}(p=0) \end{bmatrix} = [\boldsymbol{U}^{S_2,D_0}(k_2,k_3)]^{-1} \begin{bmatrix} \boldsymbol{G}^{\text{TE}} \\ \boldsymbol{G}^{\text{TM}} \end{bmatrix}, \tag{48}$$

respectively, where the elements of the transition matrices $\boldsymbol{T}^{S_2,D_0}$ and $\boldsymbol{U}^{S_2,D_0}$ contain surface integrals over S_2. Their form was given previously [12]. Equations (47) and (48) describe the total externally reflected and transmitted electric waves of order $p=0$.

It is also useful to follow the propagation and the evolution of a single incoming partial wave as it interacts repeatedly with the coated particle's surface. For a single incident TE partial wave (m_0,n_0), the transmission amplitudes $T^{32,\mathrm{TE/TE}}_{m_1n_1,m_0n_0}$ and $T^{32,\mathrm{TM/TE}}_{m_1n_1,m_0n_0}$ can be determined by assuming the incident partial wave has unit amplitude (namely, $G^{\mathrm{TM}}_{mn}=0$ and $G^{\mathrm{TE}}_{mn}=0$ except $G^{\mathrm{TE}}_{mn}=1$ for $m=m_0$ and $n=n_0$). Then, from Eqs. (47) and (48), the transmitted field coefficients for the single TE incident partial wave are

$$\begin{bmatrix}\boldsymbol{T}^{32,\mathrm{TE/TE}}\\ \boldsymbol{T}^{32,\mathrm{TM/TE}}\end{bmatrix}=[\boldsymbol{U}^{S_2,D_0}]^{-1}\begin{bmatrix}\boldsymbol{G}^{\mathrm{TE}}\\ \mathbf{0}\end{bmatrix},\tag{49}$$

and the externally reflected field coefficients are

$$\begin{bmatrix}\boldsymbol{R}^{323,\mathrm{TE/TE}}\\ \boldsymbol{R}^{323,\mathrm{TM/TE}}\end{bmatrix}=-[\boldsymbol{T}^{S_2,D_0}]\begin{bmatrix}\boldsymbol{G}^{\mathrm{TE}}\\ \mathbf{0}\end{bmatrix}.\tag{50}$$

For a single incident TM partial wave (m_0,n_0), the coefficients $(R^{323,\mathrm{TE/TM}}_{m_1n_1,m_0n_0},R^{323,\mathrm{TM/TM}}_{m_1n_1,m_0n_0})$ and $(T^{32,\mathrm{TE/TM}}_{m_1n_1,m_0n_0},T^{32,\mathrm{TM/TM}}_{m_1n_1,m_0n_0})$ are obtained in a similar way, starting with $G^{\mathrm{TM}}_{m_0n_0}=1$ and neglecting all other partial waves. By summing over the contributions from all the incident partial waves, the transmitted and externally reflected field coefficients for the complete incident beam, analogous to Eqs. (47) and (48), are expressed by

$$\begin{bmatrix}W^{323,\mathrm{TE}}_{m_1n_1}(p=0)\\ W^{323,\mathrm{TM}}_{m_1n_1}(p=0)\end{bmatrix}=\sum_{n_0=1}^{\infty}\sum_{m_0=-n_0}^{+n_0}\begin{bmatrix}R^{323,\mathrm{TE/TE}}_{m_1n_1,m_0n_0} & R^{323,\mathrm{TE/TM}}_{m_1n_1,m_0n_0}\\ R^{323,\mathrm{TM/TE}}_{m_1n_1,m_0n_0} & R^{323,\mathrm{TM/TM}}_{m_1n_1,m_0n_0}\end{bmatrix}\begin{bmatrix}G^{\mathrm{TE}}_{m_0n_0}\\ G^{\mathrm{TM}}_{m_0n_0}\end{bmatrix},\tag{51}$$

$$\begin{bmatrix}W^{32,\mathrm{TE}}_{m_1n_1}(p=0)\\ W^{32,\mathrm{TM}}_{m_1n_1}(p=0)\end{bmatrix}=\sum_{n_0=1}^{\infty}\sum_{m_0=-n_0}^{+n_0}\begin{bmatrix}T^{32,\mathrm{TE/TE}}_{m_1n_1,m_0n_0} & T^{32,\mathrm{TE/TM}}_{m_1n_1,m_0n_0}\\ T^{32,\mathrm{TM/TE}}_{m_1n_1,m_0n_0} & T^{32,\mathrm{TM/TM}}_{m_1n_1,m_0n_0}\end{bmatrix}\begin{bmatrix}G^{\mathrm{TE}}_{m_0n_0}\\ G^{\mathrm{TM}}_{m_0n_0}\end{bmatrix}.\tag{52}$$

As the wave further propagates inside the particle toward the external medium ($p\geqslant 1$), the incident wave amplitudes are $I^{\mathrm{TE}}_{m_pn_p}(p)$ and $I^{\mathrm{TM}}_{m_pn_p}(p)$, and the transmitted wave and internally reflected wave are described by

$$\mathbf{E}^{(\mathrm{i})}_{p\geqslant 1}=\sum_{n_p=1}^{\infty}\sum_{m_p=-n_p}^{+n_p}\left[I^{\mathrm{TE}}_{m_pn_p}(p)\mathbf{M}^{(3)}_{m_pn_p}(k_2\boldsymbol{r})+I^{\mathrm{TM}}_{m_pn_p}(p)\mathbf{N}^{(3)}_{m_pn_p}(k_2\boldsymbol{r})\right],\tag{53}$$

$$\mathbf{E}^{(\mathrm{t})}_{p\geqslant 1}=\sum_{n_{p+1}=1}^{\infty}\sum_{m_{p+1}=-n_{p+1}}^{+n_{p+1}}\left[W^{23,\mathrm{TE}}_{m_{p+1}n_{p+1}}(p)\mathbf{M}^{(3)}_{m_{p+1}n_{p+1}}(k_3\boldsymbol{r})+W^{23,\mathrm{TM}}_{m_{p+1}n_{p+1}}(p)\mathbf{N}^{(3)}_{m_{p+1}n_{p+1}}(k_3\boldsymbol{r})\right],\tag{54}$$

and

$$\mathbf{E}^{(\mathrm{r})}_{p\geqslant 1}=\sum_{n_{p+1}=1}^{\infty}\sum_{m_{p+1}=-n_{p+1}}^{+n_{p+1}}\left[W^{232,\mathrm{TE}}_{m_{p+1}n_{p+1}}(p)\mathbf{M}^{(4)}_{m_{p+1}n_{p+1}}(k_2\boldsymbol{r})+W^{232,\mathrm{TM}}_{m_{p+1}n_{p+1}}(p)\mathbf{N}^{(4)}_{m_{p+1}n_{p+1}}(k_2\boldsymbol{r})\right],\tag{55}$$

respectively. Note that for the order $p=1$, the transmitted field of order $p=0$ is used as the incident field; namely, we have $I^{\mathrm{TE}}_{m_pn_p}=T^{32,\mathrm{TE}}_{m_{p-1}n_{p-1}}$ and $I^{\mathrm{TM}}_{m_pn_p}=T^{32,\mathrm{TM}}_{m_{p-1}n_{p-1}}$ in Eq. (53). For $p\geqslant 2$, the reflected field of order $(p-1)$ is used as the incident field; namely, we have $I^{\mathrm{TE}}_{m_pn_p}=R^{232,\mathrm{TE}}_{m_{p-1}n_{p-1}}$ and $I^{\mathrm{TM}}_{m_pn_p}=R^{232,\mathrm{TM}}_{m_{p-1}n_{p-1}}$. Again, the transmitted and internally reflected waves are related to the incident wave of Eq. (53) by a matrix [12],

$$\begin{bmatrix}\boldsymbol{W}^{23,\mathrm{TE}}(p)\\ \boldsymbol{W}^{23,\mathrm{TM}}(p)\end{bmatrix}=[\boldsymbol{U}^{S_2,D_p}(k_2,k_3)]^{-1}\times\begin{cases}\begin{bmatrix}\boldsymbol{W}^{32,\mathrm{TE}}(p-1)\\ \boldsymbol{W}^{32,\mathrm{TM}}(p-1)\end{bmatrix}, & p=1,\\[2ex] \begin{bmatrix}\boldsymbol{W}^{232,\mathrm{TE}}(p-1)\\ \boldsymbol{W}^{232,\mathrm{TM}}(p-1)\end{bmatrix}, & p\geqslant 2,\end{cases}\tag{56}$$

$$\begin{bmatrix}\boldsymbol{W}^{232,\mathrm{TE}}(p)\\ \boldsymbol{W}^{232,\mathrm{TM}}(p)\end{bmatrix}=-[\boldsymbol{T}^{S_2,D_p}(k_2,k_3)]\times\begin{cases}\begin{bmatrix}\boldsymbol{W}^{32,\mathrm{TE}}(p-1)\\ \boldsymbol{W}^{32,\mathrm{TM}}(p-1)\end{bmatrix}, & p=1,\\[2ex] \begin{bmatrix}\boldsymbol{W}^{232,\mathrm{TE}}(p-1)\\ \boldsymbol{W}^{232,\mathrm{TM}}(p-1)\end{bmatrix}, & p\geqslant 2.\end{cases}\tag{57}$$

Since the matrices $\boldsymbol{U}^{S_2,D_p}$ and $\boldsymbol{T}^{S_2,D_p}$ are independent of p for $p\geqslant 1$, they only need to be evaluated once.

Alternatively, the transmitted and internally reflected total fields of Eqs. (56) and (57) at any order can be obtained by summing over the contributions from all the individual incident partial waves, namely,

$$\begin{bmatrix}W^{23,\mathrm{TE}}_{m_{p+1}n_{p+1}}(p)\\ W^{23,\mathrm{TM}}_{m_{p+1}n_{p+1}}(p)\end{bmatrix}=\sum_{n_p=1}^{\infty}\sum_{m_p=-n_p}^{+n_p}\begin{bmatrix}T^{23,\mathrm{TE/TE}}_{m_{p+1}n_{p+1},m_pn_p} & T^{23,\mathrm{TE/TM}}_{m_{p+1}n_{p+1},m_pn_p}\\ T^{23,\mathrm{TM/TE}}_{m_{p+1}n_{p+1},m_pn_p} & T^{23,\mathrm{TM/TM}}_{m_{p+1}n_{p+1},m_pn_p}\end{bmatrix}\times\begin{cases}\begin{bmatrix}W^{32,\mathrm{TE}}_{m_pn_p}(p-1)\\ W^{32,\mathrm{TM}}_{m_pn_p}(p-1)\end{bmatrix}, & p=1,\\[2ex] \begin{bmatrix}W^{232,\mathrm{TE}}_{m_pn_p}(p-1)\\ W^{232,\mathrm{TM}}_{m_pn_p}(p-1)\end{bmatrix}, & p\geqslant 2,\end{cases}\tag{58}$$

$$\begin{bmatrix}W^{232,\mathrm{TE}}_{m_{p+1}n_{p+1}}(p)\\ W^{232,\mathrm{TM}}_{m_{p+1}n_{p+1}}(p)\end{bmatrix}=\sum_{n_p=1}^{\infty}\sum_{m_p=-n_p}^{+n_p}\begin{bmatrix}R^{232,\mathrm{TE/TE}}_{m_{p+1}n_{p+1},m_pn_p} & R^{232,\mathrm{TE/TM}}_{m_{p+1}n_{p+1},m_pn_p}\\ R^{232,\mathrm{TM/TE}}_{m_{p+1}n_{p+1},m_pn_p} & R^{232,\mathrm{TM/TM}}_{m_{p+1}n_{p+1},m_pn_p}\end{bmatrix}$$

$$\times \begin{cases} \begin{bmatrix} W^{32,\mathrm{TE}}_{m_p n_p}(p-1) \\ W^{32,\mathrm{TM}}_{m_p n_p}(p-1) \end{bmatrix}, & p=1, \\ \begin{bmatrix} W^{232,\mathrm{TE}}_{m_p n_p}(p-1) \\ W^{232,\mathrm{TM}}_{m_p n_p}(p-1) \end{bmatrix}, & p \geqslant 2. \end{cases} \tag{59}$$

For a single incident TE partial wave (m_1,n_1) at the order $p \geqslant 1$, the transmitted and reflected field coefficients $(T^{23,\mathrm{TE/TE}}_{m_2n_2,m_1n_1}, T^{23,\mathrm{TM/TE}}_{m_2n_2,m_1n_1})$ and $(R^{232,\mathrm{TE/TE}}_{m_2n_2,m_1n_1}, R^{232,\mathrm{TM/TE}}_{m_2n_2,m_1n_1})$ are calculated by setting the incident amplitude to unity $(I^{\mathrm{TE}}_{m_1n_1}=1)$, neglecting other incident partial waves, and using Eqs. (56) and (57), respectively. Similarly, for a single TM partial wave (m_1,n_1), the amplitudes $(R^{232,\mathrm{TE/TM}}_{m_2n_2,m_1n_1}, R^{232,\mathrm{TM/TM}}_{m_2n_2,m_1n_1})$ and $(T^{23,\mathrm{TE/TM}}_{m_2n_2,m_1n_1}, T^{23,\mathrm{TM/TM}}_{m_2n_2,m_1n_1})$ can be obtained by setting $I^{\mathrm{TM}}_{m_1n_1}=1$ and neglecting all other incident partial waves.

B. Influence of the core

The influence of the core (c) on composite particle scattering can be described by introducing a multiple-scattering operator $\Im_c$ that acts on an incoming wave in the coating with incident amplitudes $W^{\mathrm{TE}}_{m_pn_p}(p)$ and $W^{\mathrm{TM}}_{m_pn_p}(p)$ to produce its complete interaction with the core. Knowing $W^{\mathrm{TE}}_{m_pn_p}(p)$ and $W^{\mathrm{TM}}_{m_pn_p}(p)$, the action of $\Im_c$ produces both outgoing waves externally reflected from the core and the outgoing waves that have been transmitted out of the core, namely,

$$\Im_c\left(\begin{bmatrix} \boldsymbol{W}^{\mathrm{TE}}(p) \\ \boldsymbol{W}^{\mathrm{TM}}(p) \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{W}^{212,\mathrm{TE}}(p'=0,p) \\ \boldsymbol{W}^{212,\mathrm{TM}}(p'=0,p) \end{bmatrix} + \sum_{p'=1}^{\infty} \begin{bmatrix} \boldsymbol{W}^{12,\mathrm{TE}}(p',p) \\ \boldsymbol{W}^{12,\mathrm{TM}}(p',p) \end{bmatrix}, \tag{60}$$

where $(p'+1)$ counts the number of successive interactions of partial waves at the core-coating interface after $(p+1)$ prior interactions at the coating-medium interface. The amplitudes $\boldsymbol{W}^{212,\mathrm{TE}}(0,p)$ and $\boldsymbol{W}^{212,\mathrm{TM}}(0,p)$ stand for the portion of the incident incoming wave externally reflected from the core-coating interface back into the coating (region 2). They contain the contribution from all incoming partial waves in the coating having the order p and incident partial-wave amplitudes $\boldsymbol{W}^{21,\mathrm{TE}}(p)$ and $\boldsymbol{W}^{21,\mathrm{TM}}(p)$. The amplitudes $\boldsymbol{W}^{12,\mathrm{TE}}(p',p)$ and $\boldsymbol{W}^{12,\mathrm{TM}}(p',p)$ stand for the portion of the partial waves transmitted from the core (region 1) to the coating (region 2) after $(p'+1)$ interactions at the core-coating interface [i.e., transmission into the coating after $(p'-1)$ internal reflections in the core]. When all the partial waves are added together at order p', these portions are evaluated exactly the same way they were for a particle (the core) in an external medium (the coating), namely,

$$\begin{bmatrix} \boldsymbol{W}^{212,\mathrm{TE}}(p',p) \\ \boldsymbol{W}^{212,\mathrm{TM}}(p',p) \end{bmatrix} = -[\boldsymbol{T}^{S_1,D_0}(k_1,k_2)] \begin{bmatrix} \boldsymbol{W}^{\mathrm{TE}}(p) \\ \boldsymbol{W}^{\mathrm{TM}}(p) \end{bmatrix}, \quad p'=0, \tag{61}$$

$$\begin{bmatrix} \boldsymbol{W}^{12,\mathrm{TE}}(p',p) \\ \boldsymbol{W}^{12,\mathrm{TM}}(p',p) \end{bmatrix} = [\boldsymbol{U}^{S_1,D_{p'}}(k_1,k_2)]^{-1} \times \begin{cases} \begin{bmatrix} \boldsymbol{W}^{21,\mathrm{TE}}(p'-1,p) \\ \boldsymbol{W}^{21,\mathrm{TM}}(p'-1,p) \end{bmatrix}, & p'=1, \\ \begin{bmatrix} \boldsymbol{W}^{121,\mathrm{TE}}(p'-1,p) \\ \boldsymbol{W}^{121,\mathrm{TM}}(p'-1,p) \end{bmatrix}, & p' \geqslant 2. \end{cases} \tag{62}$$

After $(p'+1)$ interactions at the core-coating interface, the transmitted field coefficients $[\boldsymbol{W}^{21,\mathrm{TE}}(p',p), \boldsymbol{W}^{21,\mathrm{TM}}(p',p)]$ and the internally reflected field coefficients $[\boldsymbol{W}^{121,\mathrm{TE}}(p',p), \boldsymbol{W}^{121,\mathrm{TM}}(p',p)]$ are related to the incident amplitudes $[\boldsymbol{W}^{\mathrm{TE}}(p), \boldsymbol{W}^{\mathrm{TM}}(p)]$ by

$$\begin{bmatrix} \boldsymbol{W}^{21,\mathrm{TE}}(p',p) \\ \boldsymbol{W}^{21,\mathrm{TM}}(p',p) \end{bmatrix} = [\boldsymbol{U}^{S_1,D_0}(k_1,k_2)]^{-1} \begin{bmatrix} \boldsymbol{W}^{\mathrm{TE}}(p) \\ \boldsymbol{W}^{\mathrm{TM}}(p) \end{bmatrix}, \quad p'=0, \tag{63}$$

$$\begin{bmatrix} \boldsymbol{W}^{121,\mathrm{TE}}(p',p) \\ \boldsymbol{W}^{121,\mathrm{TM}}(p',p) \end{bmatrix} = -[\boldsymbol{T}^{S_1,D_{p'}}(k_1,k_2)] \times \begin{cases} \begin{bmatrix} \boldsymbol{W}^{21,\mathrm{TE}}(p'-1,p) \\ \boldsymbol{W}^{21,\mathrm{TM}}(p'-1,p) \end{bmatrix}, & p'=1, \\ \begin{bmatrix} \boldsymbol{W}^{121,\mathrm{TE}}(p'-1,p) \\ \boldsymbol{W}^{121,\mathrm{TM}}(p'-1,p) \end{bmatrix}, & p' \geqslant 2. \end{cases} \tag{64}$$

In Eqs. (61)–(64), the elements of the matrices $\boldsymbol{T}^{S_1,D_0}, \boldsymbol{U}^{S_1,D_0}, \boldsymbol{T}^{S_1,D_{p'}}$, and $\boldsymbol{U}^{S_1,D_{p'}}$ contain surface integrals over S_1. They are evaluated in the same way as that for a homogeneous nonspherical particle [12].

For a single incident partial wave, the action of the $\Im_c$ operator is expressed by

$$\Im_c\left(\begin{bmatrix} W^{\mathrm{TE}}_{m_pn_p}(p) \\ W^{\mathrm{TM}}_{m_pn_p}(p) \end{bmatrix}\right) = \begin{Bmatrix} \left[\sum_{n'_1=1}^{\infty}\sum_{m'_1=-n'_1}^{+n'_1} W^{212,\mathrm{TE}}_{m'_1n'_1}(p'=0,p) + \sum_{p'=1}^{\infty}\sum_{n'_{p'+1}=1}^{\infty}\sum_{m'_{p'+1}=-n'_{p'+1}}^{+n'_{p'+1}} W^{12,\mathrm{TE}}_{m'_{p'+1}n'_{p'+1}}(p',p)\right] \\ \left[\sum_{n'_1=1}^{\infty}\sum_{m'_1=-n'_1}^{+n'_1} W^{212,\mathrm{TM}}_{m'_1n'_1}(p'=0,p) + \sum_{p'=1}^{\infty}\sum_{n'_{p'+1}=1}^{\infty}\sum_{m'_{p'+1}=-n'_{p'+1}}^{+n'_{p'+1}} W^{12,\mathrm{TM}}_{m'_{p'+1}n'_{p'+1}}(p',p)\right] \end{Bmatrix}, \tag{65}$$

$$\begin{bmatrix} W^{212,\mathrm{TE}}_{m'_1 n'_1}(p',p) \\ W^{212,\mathrm{TM}}_{m'_1 n'_1}(p',p) \end{bmatrix} = \sum_{m'_0=1}^{\infty} \sum_{m'_0=-n'_0}^{+n'_0} \begin{bmatrix} R^{212,\mathrm{TE/TE}}_{m'_1 n'_1, m'_0 n'_0} & R^{212,\mathrm{TE/TM}}_{m'_1 n'_1, m'_0 n'_0} \\ R^{212,\mathrm{TM/TE}}_{m'_1 n'_1, m'_0 n'_0} & R^{212,TM/TM}_{m'_1 n'_1, m'_0 n'_0} \end{bmatrix} \begin{bmatrix} W^{\mathrm{TE}}_{m_p n_p}(p) \\ W^{\mathrm{TM}}_{m_p n_p}(p) \end{bmatrix}, \quad p' = 0, \tag{66}$$

$$\begin{bmatrix} W^{12,\mathrm{TE}}_{m'_{p'+1} n'_{p'+1}}(p',p) \\ W^{12,\mathrm{TM}}_{m'_{p'+1} n'_{p'+1}}(p',p) \end{bmatrix} = \sum_{n'_{p'}=1}^{\infty} \sum_{m'_{p'}=-n'_{p'}}^{+n'_{p'}} \begin{bmatrix} T^{12,\mathrm{TE/TE}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} & T^{12,\mathrm{TE/TM}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} \\ T^{12,\mathrm{TM/TE}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} & T^{12,\mathrm{TM/TM}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} \end{bmatrix} \times \begin{cases} \begin{bmatrix} W^{21,\mathrm{TE}}_{m'_{p'} n'_{p'}}(p'-1,p) \\ W^{21,\mathrm{TM}}_{m'_{p'} n'_{p'}}(p'-1,p) \end{bmatrix}, & p' = 1, \\ \begin{bmatrix} W^{121,\mathrm{TE}}_{m'_{p'} n'_{p'}}(p'-1,p) \\ W^{121,\mathrm{TM}}_{m'_{p'} n'_{p'}}(p'-1,p) \end{bmatrix}, & p' \geqslant 2, \end{cases} \tag{67}$$

where

$$\begin{bmatrix} W^{21,\mathrm{TE}}_{m'_1 n'_1}(p',p) \\ W^{21,\mathrm{TM}}_{m'_1 n'_1}(p',p) \end{bmatrix} = \sum_{n'_0=1}^{\infty} \sum_{m'_0=-n'_0}^{+n'_0} \begin{bmatrix} T^{21,\mathrm{TE/TE}}_{m'_1 n'_1, m'_0 n'_0} & T^{21,\mathrm{TE/TM}}_{m'_1 n'_1, m'_0 n'_0} \\ T^{21,\mathrm{TM/TE}}_{m'_1 n'_1, m'_0 n'_0} & T^{21,\mathrm{TM/TM}}_{m'_1 n'_1, m'_0 n'_0} \end{bmatrix} \begin{bmatrix} W^{\mathrm{TE}}_{m_p n_p}(p) \\ W^{\mathrm{TM}}_{m_p n_p}(p) \end{bmatrix}, \quad p' = 0, \tag{68}$$

$$\begin{bmatrix} W^{121,\mathrm{TE}}_{m'_{p'+1} n'_{p'+1}}(p',p) \\ W^{121,\mathrm{TM}}_{m'_{p'+1} n'_{p'+1}}(p',p) \end{bmatrix} = \sum_{n'_p=1}^{\infty} \sum_{m'_p=-n'_p}^{+n'_p} \begin{bmatrix} R^{121,\mathrm{TE/TE}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} & R^{121,\mathrm{TE/TM}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} \\ R^{121,\mathrm{TM/TE}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} & R^{121,\mathrm{TM/TM}}_{m'_{p'+1} n'_{p'+1}, m'_{p'} n'_{p'}} \end{bmatrix} \times \begin{cases} \begin{bmatrix} W^{21,\mathrm{TE}}_{m'_{p'} n'_{p'}}(p'-1,p) \\ W^{21,\mathrm{TM}}_{m'_{p'} n'_{p'}}(p'-1,p) \end{bmatrix}, & p' = 1, \\ \begin{bmatrix} W^{121,\mathrm{TE}}_{m'_{p'} n'_{p'}}(p'-1,p) \\ W^{121,\mathrm{TM}}_{m'_{p'} n'_{p'}}(p'-1,p) \end{bmatrix}, & p' \geqslant 2. \end{cases} \tag{69}$$

C. Composite scattering

1. Scattered field

By referring to Eqs. (47), (48), (56), and (57) for a homogeneous particle and by considering the influence of the core, the Debye series for the scattered field of a coated particle when all partial waves contribute at order p is expressed as

$$\begin{bmatrix} \boldsymbol{W}^{323,\mathrm{TE}}(p) \\ \boldsymbol{W}^{323,\mathrm{TM}}(p) \end{bmatrix} = -[\boldsymbol{T}^{D_0,S_2}(k_2,k_3)] \begin{bmatrix} \boldsymbol{G}^{\mathrm{TE}} \\ \boldsymbol{G}^{\mathrm{TM}} \end{bmatrix}, \quad p = 0, \tag{70}$$

$$\begin{bmatrix} \boldsymbol{W}^{32,\mathrm{TE}}(p) \\ \boldsymbol{W}^{32,\mathrm{TM}}(p) \end{bmatrix} = [\boldsymbol{U}^{D_0,S_2}(k_2,k_3)]^{-1} \begin{bmatrix} \boldsymbol{G}^{\mathrm{TE}} \\ \boldsymbol{G}^{\mathrm{TM}} \end{bmatrix}, \quad p = 0, \tag{71}$$

$$\begin{bmatrix} \boldsymbol{W}^{23}_{\mathrm{TE}}(p) \\ \boldsymbol{W}^{23}_{\mathrm{TM}}(p) \end{bmatrix} = [\boldsymbol{U}^{S_2,D_p}(k_2,k_3)]^{-1} \times \begin{cases} \Im_c\left(\begin{bmatrix} \boldsymbol{W}^{32,\mathrm{TE}}(p-1) \\ \boldsymbol{W}^{32,\mathrm{TM}}(p-1) \end{bmatrix}\right), & p = 1, \\ \Im_c\left(\begin{bmatrix} \boldsymbol{W}^{232,\mathrm{TE}}(p-1) \\ \boldsymbol{W}^{232,\mathrm{TM}}(p-1) \end{bmatrix}\right), & p \geqslant 2, \end{cases} \tag{72}$$

where

$$\begin{bmatrix} \boldsymbol{W}^{232,\mathrm{TE}}(p) \\ \boldsymbol{W}^{232,\mathrm{TM}}(p) \end{bmatrix} = -[\boldsymbol{T}^{S_2,D_p}(k_2,k_3)] \times \begin{cases} \Im_c\left(\begin{bmatrix} \boldsymbol{W}^{32,\mathrm{TE}}(p-1) \\ \boldsymbol{W}^{32,\mathrm{TM}}(p-1) \end{bmatrix}\right), & p = 1, \\ \Im_c\left(\begin{bmatrix} \boldsymbol{W}^{232,\mathrm{TE}}(p-1) \\ \boldsymbol{W}^{232,\mathrm{TM}}(p-1) \end{bmatrix}\right), & p \geqslant 2. \end{cases} \tag{73}$$

Alternatively, by referring to Eqs. (51), (52), (58), and (59) for a homogeneous particle and considering the influence of the core, the Debye series for the scattered field of a coated particle for a single incident partial wave is

$$\begin{bmatrix} W^{323,\mathrm{TE}}_{m_1 n_1}(p) \\ W^{323,\mathrm{TM}}_{m_1 n_1}(p) \end{bmatrix} = \sum_{n_0=1}^{\infty} \sum_{m_0=-n_0}^{+n_0} \begin{bmatrix} R^{323,\mathrm{TE/TE}}_{m_1 n_1, m_0 n_0} & R^{323,\mathrm{TE/TM}}_{m_1 n_1, m_0 n_0} \\ R^{323,\mathrm{TM/TE}}_{m_1 n_1, m_0 n_0} & R^{323,\mathrm{TM/TM}}_{m_1 n_1, m_0 n_0} \end{bmatrix} \times \begin{bmatrix} G^{\mathrm{TE}}_{m_0 n_0} \\ G^{\mathrm{TM}}_{m_0 n_0} \end{bmatrix}, \quad p = 0, \tag{74}$$

$$\begin{bmatrix} W^{32,\mathrm{TE}}_{m_1n_1}(p) \\ W^{32,\mathrm{TM}}_{m_1n_1}(p) \end{bmatrix} = \sum_{n_0=1}^{\infty}\sum_{m_0=-n_0}^{+n_0} \begin{bmatrix} T^{32,\mathrm{TE/TE}}_{m_1n_1,m_0n_0} & T^{32,\mathrm{TE/TM}}_{m_1n_1,m_0n_0} \\ T^{32,\mathrm{TM/TE}}_{m_1n_1,m_0n_0} & T^{32,\mathrm{TM/TM}}_{m_1n_1,m_0n_0} \end{bmatrix} \times \begin{bmatrix} G^{\mathrm{TE}}_{m_0n_0} \\ G^{\mathrm{TM}}_{m_0n_0} \end{bmatrix}, \quad p = 0, \tag{75}$$

$$\begin{bmatrix} W^{232,\mathrm{TE}}_{m_{p+1}n_{p+1}}(p) \\ W^{232,\mathrm{TM}}_{m_{p+1}n_{p+1}}(p) \end{bmatrix} = \sum_{n_p=1}^{\infty}\sum_{m_p=-n_p}^{+n_p} \begin{bmatrix} R^{232,\mathrm{TE/TE}}_{m_{p+1}n_{p+1},m_pn_p} & R^{232,\mathrm{TE/TM}}_{m_{p+1}n_{p+1},m_pn_p} \\ R^{232,\mathrm{TM/TE}}_{m_{p+1}n_{p+1},m_pn_p} & R^{232,\mathrm{TM/TM}}_{m_{p+1}n_{p+1},m_pn_p} \end{bmatrix} \times \begin{cases} \Im_c\left(\begin{bmatrix} W^{32,\mathrm{TE}}_{m_pn_p}(p-1) \\ W^{32,\mathrm{TM}}_{m_pn_p}(p-1) \end{bmatrix}\right), & p = 1, \\ \Im_c\left(\begin{bmatrix} W^{232,\mathrm{TE}}_{m_pn_p}(p-1) \\ W^{232,\mathrm{TM}}_{m_pn_p}(p-1) \end{bmatrix}\right), & p \geqslant 2, \end{cases} \tag{76}$$

$$\begin{bmatrix} W^{23,\mathrm{TE}}_{m_{p+1}n_{p+1}}(p) \\ W^{23,\mathrm{TM}}_{m_{p+1}n_{p+1}}(p) \end{bmatrix} = \sum_{n_p=1}^{\infty}\sum_{m_p=-n_p}^{+n_p} \begin{bmatrix} T^{23,\mathrm{TE/TE}}_{m_{p+1}n_{p+1},m_pn_p} & T^{23,\mathrm{TE/TM}}_{m_{p+1}n_{p+1},m_pn_p} \\ T^{23,\mathrm{TM/TE}}_{m_{p+1}n_{p+1},m_pn_p} & T^{23,\mathrm{TM/TM}}_{m_{p+1}n_{p+1},m_pn_p} \end{bmatrix} \times \begin{cases} \Im_c\left(\begin{bmatrix} W^{32,\mathrm{TE}}_{m_pn_p}(p-1) \\ W^{32,\mathrm{TM}}_{m_pn_p}(p-1) \end{bmatrix}\right), & p = 1, \\ \Im_c\left(\begin{bmatrix} W^{232,\mathrm{TE}}_{m_pn_p}(p-1) \\ W^{232,\mathrm{TM}}_{m_pn_p}(p-1) \end{bmatrix}\right), & p \geqslant 2. \end{cases} \tag{77}$$

Knowing the influence of the core on the wave in the coating before it is transmitted back into the medium, the Debye series for the Mie coefficients in Eq. (3) is then

$$A_{mn} = -\frac{1}{2}\left[G^{\mathrm{TM}}_{mn} - W^{323,\mathrm{TM}}_{mn}(0) - \sum_{p=1}^{\infty} W^{23,\mathrm{TM}}_{mn}(p)\right], \tag{78}$$

$$B_{mn} = -\frac{1}{2}\left[G^{\mathrm{TE}}_{mn} - W^{323,\mathrm{TE}}_{mn}(0) - \sum_{p=1}^{\infty} W^{23,\mathrm{TE}}_{mn}(p)\right]. \tag{79}$$

2. *Field in the coating*

Due to scattering of the radially incoming wave in the coating by the core, there are both incoming and outgoing waves in region 2 so that, for the order $p = 0$,

$$\mathbf{E}^{(2)}(\boldsymbol{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\left\{W^{32,\mathrm{TE}}_{mn}(0)\mathbf{M}^{(4)}_{mn}(k_2\boldsymbol{r}) + W^{32,\mathrm{TM}}_{mn}(0)\mathbf{N}^{(4)}_{mn}(k_2\boldsymbol{r}) + \Im_c\left(\begin{bmatrix} W^{32,\mathrm{TE}}_{mn}(0) \\ W^{32,\mathrm{TM}}_{mn}(0) \end{bmatrix}\right)^{\mathrm{TE}}\mathbf{M}^{(3)}_{mn}(k_2\boldsymbol{r}) + \Im_c\left(\begin{bmatrix} W^{32,\mathrm{TE}}_{mn}(0) \\ W^{32,\mathrm{TM}}_{mn}(0) \end{bmatrix}\right)^{\mathrm{TM}}\mathbf{N}^{(3)}_{mn}(k_2\boldsymbol{r})\right\}, \tag{80}$$

and, for $p \geqslant 1$,

$$\mathbf{E}^{(2)}(\boldsymbol{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{+n}\left\{W^{232,\mathrm{TE}}_{mn}(p)\mathbf{M}^{(4)}_{mn}(k_2\boldsymbol{r}) + W^{232,\mathrm{TM}}_{mn}(p)\mathbf{N}^{(4)}_{mn}(k_2\boldsymbol{r}) + \Im_c\left(\begin{bmatrix} W^{232,\mathrm{TE}}_{mn}(p) \\ W^{232,\mathrm{TM}}_{mn}(p) \end{bmatrix}\right)^{\mathrm{TE}}\mathbf{M}^{(3)}_{mn}(k_2\boldsymbol{r}) + \Im_c\left(\begin{bmatrix} W^{232,\mathrm{TE}}_{mn}(p) \\ W^{232,\mathrm{TM}}_{mn}(p) \end{bmatrix}\right)^{\mathrm{TM}}\mathbf{N}^{(3)}_{mn}(k_2\boldsymbol{r})\right\}, \tag{81}$$

where the superscript TE (TM) of the total argument of $\Im_c$ means the TE (TM) part of the result. By collecting the contributions from all individual partial waves, the Mie amplitudes $(\alpha_{mn}, \beta_{mn}, \gamma_{mn}, \delta_{mn})$ in Eqs. (9) and (10) for the wave in the coating have the following Debye-series form:

$$\begin{bmatrix} \gamma_{mn} \\ \delta_{mn} \end{bmatrix} = \begin{bmatrix} W^{32,\mathrm{TM}}_{mn}(p=0) \\ W^{32,\mathrm{TE}}_{mn}(p=0) \end{bmatrix} + \sum_{p=1}^{\infty}\begin{bmatrix} W^{232,\mathrm{TM}}_{mn}(p) \\ W^{232,\mathrm{TE}}_{mn}(p) \end{bmatrix}, \tag{82}$$

$$\begin{bmatrix} \alpha_{mn} \\ \beta_{mn} \end{bmatrix} = -\frac{1}{2}\left\{\begin{bmatrix} \gamma_{mn} \\ \delta_{mn} \end{bmatrix} - \Im_c\left(\begin{bmatrix} W^{32,\mathrm{TE}}_{mn}(p=0) \\ W^{32,\mathrm{TM}}_{mn}(p=0) \end{bmatrix}\right) - \sum_{p=1}^{\infty}\Im_c\left(\begin{bmatrix} W^{232,\mathrm{TE}}_{mn}(p) \\ W^{232,\mathrm{TM}}_{mn}(p) \end{bmatrix}\right)\right\}. \tag{83}$$

3. *Field in the core*

At order p, the contribution to the interior field in the core by the incoming wave with amplitudes $\boldsymbol{W}^{21,\mathrm{TE}}(p)$ and $\boldsymbol{W}^{21,\mathrm{TM}}(p)$ that is transmitted from the coating into the core can be determined by introducing another operator $\Im_t$,

$$\Im_t\left(\begin{bmatrix} \boldsymbol{W}^{\mathrm{TE}}(p) \\ \boldsymbol{W}^{\mathrm{TM}}(p) \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{W}^{21,\mathrm{TE}}(p'=0,p) \\ \boldsymbol{W}^{21,\mathrm{TM}}(p'=0,p) \end{bmatrix} + \sum_{p'=1}^{\infty}\begin{bmatrix} \boldsymbol{W}^{121,\mathrm{TE}}(p',p) \\ \boldsymbol{W}^{121,\mathrm{TM}}(p',p) \end{bmatrix}, \tag{84}$$

where $\boldsymbol{W}^{21,\mathrm{TE}}(p',p)$ and $\boldsymbol{W}^{21,\mathrm{TM}}(p',p)$, along with $\boldsymbol{W}^{121,\mathrm{TE}}(p',p)$ and $\boldsymbol{W}^{121,\mathrm{TM}}(p',p)$, are evaluated by Eqs. (63) and (64), respectively, for a full-scattering order, and by Eqs. (68) and (69), respectively, for a single partial wave. The operator $\Im_t$ describes the total field in the core as the sum of the field transmitted into it from the coating plus the field that has internally reflected p' times within the core after being transmitted in. By use of $\Im_t$, the Debye series for the interior coefficients C_{mn} and D_{mn} in Eqs. (30) and (31) is

$$\begin{bmatrix} D_{mn} \\ C_{mn} \end{bmatrix} = \Im_t \left(\begin{bmatrix} W_{mn}^{32,\mathrm{TE}}(p=0) \\ W_{mn}^{32,\mathrm{TM}}(p=0) \end{bmatrix} \right) + \sum_{p=1}^{\infty} \Im_t \left(\begin{bmatrix} W_{mn}^{232,\mathrm{TE}}(p) \\ W_{mn}^{232,\mathrm{TM}}(p) \end{bmatrix} \right). \quad (85)$$

IV. VERIFICATION OF DEBYE SERIES FOR A COATED PARTICLE

In Refs. [12] and [17], it was pointed out that that in order to verify the Debye series for scattering of a shaped beam by a particle, it is not enough to compute the numerical value of all the partial-wave transmission and reflection amplitudes. One must also show that when these reflection and transmission amplitudes are added together, they produce the exact partial-wave scattering and interior amplitudes of the Mie theory. The procedure used to accomplish this verification for a homogeneous particle (region 1) in an exterior medium (region 2) can be summarized as follows. In analogy to the propagation and the evolution of a single partial wave inside the particle described in Sec. III, consider a single incoming partial wave (m,n) in medium 2 with polarization $i = \mathrm{TE,TM}$ and amplitude $G^i/2$ that is incident on the 12 interface. The portion $\boldsymbol{R}^{212,j/i}$ is reflected back into region 2 as an outgoing wave with partial-wave number (m',n') and polarization $j = \mathrm{TE,TM}$, and the portion $\boldsymbol{T}^{21,j/i}$ is transmitted into region 1 as an incoming wave with partial-wave number (m',n') and polarization j. For the remainder of this section, the initial and final partial-wave numbers (m,n) and (m',n') of the reflection and transmission amplitude matrices are suppressed. Consider also an outgoing partial wave in medium 1 that is incident on the 12 interface with amplitude $J^{i'}/2$ with $i' = \mathrm{TE,TM}$. The portion $\boldsymbol{R}^{121,j'/i'}$ is reflected back into region 1 in any partial-wave channel as an incoming wave with the polarization j', and the portion $\boldsymbol{T}^{12,j'/i'}$ is transmitted into region 2 in any partial-wave channel as an outgoing wave. The wave amplitudes for these two situations are then added together in region 1 and region 2. The constraint that the total wave in region 1 must be a standing wave of the first type, which is finite at the origin, determines $J^{i'}$ in terms of G^i. When the total wave in region 2 is rewritten as a standing wave of the first type plus an outgoing wave, the coefficient of the outgoing wave is the Debye-series expansion of the partial-wave scattering amplitude matrices $\boldsymbol{B}^{\mathrm{TE}/i}$ and $\boldsymbol{A}^{\mathrm{TM}/i}$ of Eq. (3). The matrices $\boldsymbol{B}$ and $\boldsymbol{A}$ couple the final partial wave (m',n') to the initial partial wave (m,n) and the final polarization state TE or TM to the initial polarization state i when the amplitude of the incident standing wave of the first type is G^i as in Eq. (1). In addition, when the total wave in region 2 is written as an outgoing wave plus an incoming wave, the amplitude of the outgoing wave in region 2 and the amplitude of the standing wave in region 1 are simply related to the amplitude of the incoming wave in region 2. These two conditions are of great significance for the coated particle geometry. The details of the homogeneous particle derivation with all the partial-wave subscripts present are given in Refs. [12] and [17].

A similar procedure is now outlined for deriving the Debye series for the coated sphere geometry. However, new ingredients are needed beyond those used for the homogeneous particle case. Since the details of the calculation are rather involved, the procedure is only outlined here, and then the appropriate results are stated and are physically interpreted. The single outgoing partial wave in region 1 (the core) that is incident on the 12 interface is again described by the incident amplitude $J^{i'}/2$, and the single incoming partial wave in region 2 (now the coating) incident on the 12 interface is given the amplitude $K^{i''}/2$. Consider also a single outgoing partial wave in region 2 that is incident on the 23 interface with polarization i''' and amplitude $L^{i'''}/2$. The portion $\boldsymbol{R}^{232,j'''/i'''}$ is reflected back into region 2 with any partial-wave number as an incoming wave with the polarization j''', and the portion $\boldsymbol{T}^{23,j'''/i'''}$ is transmitted into region 3 (the external medium) in any partial-wave channel as an outgoing wave. Lastly, consider the single incoming partial wave in region 3 with polarization i and amplitude $G^i/2$ incident on the 23 interface. The portion $\boldsymbol{R}^{323,j/i}$ is reflected back into region 3 as an outgoing wave with any polarization and partial-wave number, and the portion $\boldsymbol{T}^{32,j/i}$ is transmitted into region 2 as an incoming wave.

We now add the amplitudes for these four situations in regions 1, 2, and 3. Since the wave in region 1 must again be a standing wave of the first type, the amplitudes $J^{i'}$ may be written in terms of the $K^{i''}$ in exactly the same way they were for the homogeneous particle problem, namely,

$$J^i = \sum_{i''} \boldsymbol{S}^{i/i''} K^{i''}, \quad (86)$$

where $i'' = \mathrm{TE,TM}$,

$$\begin{aligned} \boldsymbol{S}^{\mathrm{TE}/j} = {} & [\boldsymbol{I} - \boldsymbol{P}^{121,\mathrm{TE/TE}} \boldsymbol{R}^{121,\mathrm{TE/TM}} \boldsymbol{P}^{121,\mathrm{TM/TM}} \boldsymbol{R}^{121,\mathrm{TM/TE}}]^{-1} \\ & \times [\boldsymbol{P}^{121,\mathrm{TE/TE}} \boldsymbol{R}^{121,\mathrm{TE/TM}} \boldsymbol{P}^{121,\mathrm{TM/TM}} \boldsymbol{T}^{21,\mathrm{TM}/j} \\ & + \boldsymbol{P}^{121,\mathrm{TE/TE}} \boldsymbol{T}^{21,\mathrm{TE}/j}], \end{aligned} \quad (87)$$

$$\begin{aligned} \boldsymbol{S}^{\mathrm{TM}/j} = {} & [\boldsymbol{I} - \boldsymbol{P}^{121,\mathrm{TM/TM}} \boldsymbol{R}^{121,\mathrm{TM/TE}} \boldsymbol{P}^{121,\mathrm{TE/TE}} \boldsymbol{R}^{121,\mathrm{TE/TM}}]^{-1} \\ & \times [\boldsymbol{P}^{121,\mathrm{TM/TM}} \boldsymbol{R}^{121,\mathrm{TM/TE}} \boldsymbol{P}^{121,\mathrm{TE/TE}} \boldsymbol{T}^{21,\mathrm{TE}/j} \\ & + \boldsymbol{P}^{121,\mathrm{TM/TM}} \boldsymbol{T}^{21,\mathrm{TM}/j}], \end{aligned} \quad (88)$$

$$\boldsymbol{P}^{121,\mathrm{TE/TE}} = \left(\boldsymbol{I} - \boldsymbol{R}^{121,\mathrm{TE/TE}}\right)^{-1}, \quad (89)$$

$$\boldsymbol{P}^{121,\mathrm{TM/TM}} = \left(\boldsymbol{I} - \boldsymbol{R}^{121,\mathrm{TM/TM}}\right)^{-1}, \quad (90)$$

and $\boldsymbol{I}$ is the identity matrix. The matrices $\boldsymbol{P}$ of Eqs. (89) and (90), when expanded as a series in powers of $\boldsymbol{R}$, describe an infinite number of successive polarization-preserving internal reflections of the wave in the core but with the partial-wave number changing at each internal reflection [17]. The matrix $\boldsymbol{S}$ describes an incoming wave in the coating being transmitted into the core, internally reflecting there any number of times, and remaining in the core at the end. It is analogous to the operator $\Im_t$ of Eq. (84) except that all the scattering orders p' have been explicitly summed by the matrix inverses in Eqs. (87)–(90). The superscript j in Eqs. (87) and (88) describes the polarization state of the partial wave at the start of the process, and the superscript TE or TM describes the polarization state at the end of the process. If the polarization state after transmission into the core is TE and the final polarization in the core is also TE, the partial wave makes an even number of polarization changes upon internal reflection, $\boldsymbol{R}^{\mathrm{TE/TM}}$ or $\boldsymbol{R}^{\mathrm{TM/TE}}$, along the way. If, on the other hand, the polarization after transmission is TM and the final polarization is TE, the wave makes an odd number of polarization changes. At each successive internal reflection, a contribution to all final partial waves is produced from the initial partial wave due to partial-wave coupling.

The total wave in the coating consists of a radially outgoing portion plus a radially incoming portion. The coefficient matrix of each portion $(\boldsymbol{F}^{\text{out}})^{i/j}$ and $(\boldsymbol{F}^{\text{in}})^{i/j}$ has components with both values of the initial and final polarizations and all partial-wave numbers. This form for the wave in the coating differs slightly from Eqs. (9) and (10) where the wave is expressed as the sum of a standing wave plus an outgoing wave. The form of Eqs. (9) and (10) provides a somewhat simpler derivation of the partial-wave-scattering amplitudes in the context of the EBCM formalism, whereas the form of an incoming plus outgoing wave used in this section provides a somewhat simpler derivation of the Debye series. The two sets of results for the fields in the coating, Eqs. (42) and (43), and Eqs. (105) and (106), are straightforward linear combinations of each other.

Irrespective of the boundary conditions at the 23 interface, the boundary conditions at the 12 interface for the coated particle geometry do not change from what they were for the homogeneous particle geometry. This means that although the numerical value of the amplitude of the incoming wave in the coating $(\boldsymbol{F}^{\text{in}})^{i/j}$ is strongly dependent on the details of the 23 interface, the amplitude of the outgoing wave in the coating $(\boldsymbol{F}^{\text{out}})^{i/j}$ is expressed in terms of $(\boldsymbol{F}^{\text{in}})^{i/j}$ in exactly the same way it was for the homogeneous particle problem. This is analogous to the action of the operator $\Im_c$ of Eq. (60). This constraint allows us to determine the amplitudes $L^{i'''}$ in terms of the amplitudes G^i giving

$$L^{i'''} = \sum_i \boldsymbol{Z}^{i'''/i} G^i, \tag{91}$$

where

$$\boldsymbol{Z}^{\text{TE}/j} = [\boldsymbol{I} - \boldsymbol{Y}^{\text{TE/TE}}\boldsymbol{X}^{\text{TE/TM}}\boldsymbol{Y}^{\text{TM/TM}}\boldsymbol{X}^{\text{TM/TE}}]^{-1} \times [\boldsymbol{Y}^{\text{TE/TE}}\boldsymbol{X}^{\text{TE/TM}}\boldsymbol{Y}^{\text{TM/TM}}\boldsymbol{\Gamma}^{\text{TM}/j} + \boldsymbol{Y}^{\text{TE/TE}}\boldsymbol{\Gamma}^{\text{TE}/j}], \tag{92}$$

$$\boldsymbol{Z}^{\text{TM}/j} = [\boldsymbol{I} - \boldsymbol{Y}^{\text{TM/TM}}\boldsymbol{X}^{\text{TM/TE}}\boldsymbol{Y}^{\text{TE/TE}}\boldsymbol{X}^{\text{TE/TM}}]^{-1} \times [\boldsymbol{Y}^{\text{TM/TM}}\boldsymbol{X}^{\text{TM/TE}}\boldsymbol{Y}^{\text{TE/TE}}\boldsymbol{\Gamma}^{\text{TE/j}} + \boldsymbol{Y}^{\text{TM/TM}}\boldsymbol{\Gamma}^{\text{TM}/j}], \tag{93}$$

and

$$\boldsymbol{\Omega}^{i/j} = \boldsymbol{R}^{212,i/k} + \sum_k \boldsymbol{T}^{12,i/k}\boldsymbol{S}^{k/j}, \tag{94}$$

$$\boldsymbol{X}^{i/j} = \sum_k \boldsymbol{\Omega}^{i/k}\boldsymbol{R}^{232,k/j}, \tag{95}$$

$$\boldsymbol{\Gamma}^{i/j} = \sum_k \boldsymbol{\Omega}^{i/k}\boldsymbol{T}^{32,k/j}, \tag{96}$$

$$\boldsymbol{Y}^{\text{TE/TE}} = (\boldsymbol{I} - \boldsymbol{X}^{\text{TE/TE}})^{-1}, \tag{97}$$

$$\boldsymbol{Y}^{\text{TM/TM}} = (\boldsymbol{I} - \boldsymbol{X}^{\text{TM/TM}})^{-1}. \tag{98}$$

The amplitude matrix $\boldsymbol{\Omega}$ of Eq. (94) describes two trajectories of an initially incoming partial wave in the coating that ends up as an outgoing wave in the coating after interacting with the core. As in Eq. (60), it can reflect off the core, or it can be transmitted from the coating into the core, then successively internally reflect in the core any number of times, and finally be transmitted back into the coating. The amplitude matrix $\boldsymbol{X}$ of Eq. (95) also describes two trajectories of an initially outgoing partial wave in the coating that ends up as an outgoing wave in the coating after it interacts with both the 23 interface and the core. It can reflect off the 23 interface, then reflect off the core, ending up back in the coating. It can also reflect off the 23 interface, then be transmitted into the core, successively internally reflect in the core any number of times, and finally be transmitted back into the coating. The amplitude matrix $\boldsymbol{\Gamma}$ of Eq. (96) describes two trajectories of an initially incoming partial wave in the exterior medium that ends up as an outgoing wave in the coating after interacting with the core. It can be transmitted into the coating, then reflect off the core, ending up back in the coating. It can also be transmitted from the external medium into the coating, then be transmitted from the coating into the core, internally reflect any number of times in the core, and finally be transmitted back into the coating. These matrices differ from the $\boldsymbol{T}$ and $\boldsymbol{U}$ matrices of the EBCM formulation of Sec. III since they described the effect of a single interaction with an interface, whereas Eqs. (95) and (96) describe a single interaction with the 23 interface along with a multiple-scattering interaction with the core. The amplitude matrix $\boldsymbol{Z}$ describes an initially incoming partial wave in the exterior medium that is transmitted into the coating, then multiple scatters from the core, and eventually ends up as an outgoing wave in the coating. Equations (92) and (93) have exactly the same polarization-changing multiple-scattering structure as do Eqs. (87) and (88) with $\boldsymbol{Z}$ replacing $\boldsymbol{S}$, $\boldsymbol{Y}$ replacing $\boldsymbol{P}$, $\boldsymbol{X}$ replacing $\boldsymbol{R}$, and $\boldsymbol{\Gamma}$ replacing $\boldsymbol{T}$, with the amplitude matrices $\boldsymbol{Y}$ describing all numbers of polarization-preserving multiple-scattering interactions $\boldsymbol{X}$.

At this point, if the wave in region 3 is rewritten as a standing wave of the first type plus an outgoing wave, the coefficient of the outgoing wave is the complete Mie partial-wave-scattering amplitude matrix, which now has the Debye-series expansion

$$\boldsymbol{B}^{\text{TE}/j} = -\frac{1}{2}\left[\boldsymbol{I}\delta_{\text{TE}/j} - \boldsymbol{R}^{323,\text{TE}/j} - \sum_k \boldsymbol{T}^{23,\text{TE}/k}\boldsymbol{Z}^{k/j}\right], \tag{99}$$

$$\boldsymbol{A}^{\text{TM}/j} = -\frac{1}{2}\left[\boldsymbol{I}\delta_{\text{TM}/j} - \boldsymbol{R}^{323,\text{TM}/j} - \sum_k \boldsymbol{T}^{23,\text{TM}/k}\boldsymbol{Z}^{k/j}\right], \tag{100}$$

when the amplitude of the incident wave is G^i and where δ is the Kronecker delta symbol. In Eqs. (99) and (100), diffraction occurs only in the polarization-preserving channels TE/TE and TM/TM, exterior reflection occurs in all polarization channels, and the last term describes all possible wave trajectories of a partial wave that is transmitted from the exterior medium into the coated particle and is eventually transmitted back out after multiple scattering with both the 23 interface and the core. The difference between Eqs. (99) and (100) and Eqs. (78) and (79) is that while sums over the scattering orders p and p' still need to be carried out in the latter equations, in the former equations, those sums have already been evaluated exactly in terms of the matrix inverses contained in the $\boldsymbol{Z}$ matrix of Eqs. (92) and (93).

The amplitudes $K^{i''}$ must still be determined in order to obtain the full partial-wave amplitudes in the core and the coating. The final constraint we employ to determine these amplitudes is that the boundary conditions at the 23 interface do not affect the relation between the coefficient of the incoming wave in the coating and the coefficient of the standing wave in the core. This relation is produced only by the boundary conditions at the 12 interface and is the same as it was for a homogeneous particle. This gives

$$K^{i''} = \sum_{i} \boldsymbol{V}^{i''/i} G^{i}, \tag{101}$$

where

$$\boldsymbol{V}^{i/j} = \boldsymbol{T}^{32,i/j} + \sum_{k} \boldsymbol{R}^{232,i/k} \boldsymbol{Z}^{k/j}. \tag{102}$$

The amplitude matrix $\boldsymbol{V}$ describes two trajectories of an initially incoming wave in the exterior medium that ends up as an incoming wave in the coating. The wave can be transmitted from the exterior medium to the coating, or it can be transmitted into the coating, multiply scatter from the core into the coating, and finally reflect from the 23 interface back into the coating. The amplitude matrices $\boldsymbol{X}$, $\boldsymbol{\Gamma}$, and $\boldsymbol{V}$ form a complete description of the interaction of a partial wave at the 23 interface. The matrix $\boldsymbol{X}$ describes an initially outgoing wave in region 2 that ends up as an outgoing wave in region 2 while the simpler matrix $\boldsymbol{R}^{232}$ describes a partial wave that is initially an outgoing wave and finally an incoming wave in region 2. The matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{V}$ describe an initially incoming wave in region 3 that ends up either as an outgoing or as an incoming wave in region 2 after interacting with the core.

By substituting Eq. (99), the partial-wave core amplitude matrices are then

$$\boldsymbol{D}^{\mathrm{TE}/j} = \sum_{k} \boldsymbol{S}^{\mathrm{TE}/k} \boldsymbol{V}^{k/j}, \tag{103}$$

$$\boldsymbol{C}^{\mathrm{TM}/j} = \sum_{k} \boldsymbol{S}^{\mathrm{TM}/k} \boldsymbol{V}^{k/j}. \tag{104}$$

Lastly, the coefficient matrix of the total incoming wave in the coating is

$$(\boldsymbol{F}^{\mathrm{in}})^{i/j} = \boldsymbol{V}^{i/j}, \tag{105}$$

and the coefficient matrix of the total outgoing wave in the coating is

$$(\boldsymbol{F}^{\mathrm{out}})^{i/j} = \boldsymbol{Z}^{i/j} + \sum_{k} \boldsymbol{\Omega}^{i/k} \boldsymbol{V}^{k/j}. \tag{106}$$

Again, the difference between Eqs. (103)–(106) and Eqs. (82), (83), and (85) is that sums over the scattering orders p and p' still need to be carried out in the latter equations, whereas in the former equations, those sums have already been explicitly evaluated in terms of the matrix inverses contained in the $\boldsymbol{Z}$ and $\boldsymbol{S}$ matrices of Eqs. (87), (88), (92), and (93). As a check of these equations, in the coated sphere limit where there is no partial wave or polarization coupling, equation pairs (87) and (88), (94) and (95), (99) and (100), (103) and (104), and (105) and (106) reduce to Eqs. (43), (38), (37), (42), and (44) plus (45) of Ref. [13], respectively. Similarly, when partial-wave coupling is permitted but polarization coupling is not, the results reduce to the analogous expressions for scalar-wave scattering that we derived but have not given here. When each matrix inverse is expanded as an infinite series and is inserted in Eqs. (99) and (100) for the partial-wave-scattering amplitudes, and when the resulting terms are collected together into the number of interactions $p + 1$ at the 23 interface and the number of interactions $p' + 1$ at the 12 interface, the results of this section are identical order by order to the results of Secs. III B and III C, which is known as the order of scattering formalism. The results of this section complete the verification of the Debye-series decomposition by exactly summing all the scattering orders.

Lastly, we conjecture that the approach used in this section can be generalized to increasing numbers of layers in the following way. For each new layer, one begins the calculation with two additional situations beyond those considered with one fewer layer, namely, a single incoming partial wave in the exterior region incident on the outermost interface along with its transmitted and reflected portions, and an outgoing wave in the outermost coating incident on the outermost interface along with its transmitted and reflected portions. The waves of all the enumerated situations are then added together in each region. The additional constraints used to determine the two new wave amplitudes are that (i) the relation between the incoming and outgoing waves in the outermost coating and (ii) the relation between the incoming wave in the outermost coating and the wave amplitude inside the remaining portion of the composite particle are exactly the same as they were when the total particle had one fewer layer. This construction is an elaboration of the progressive iteration scheme [23,24] for determining the scattering amplitudes of a multilayer sphere.

V. ONE-INTERNAL-REFLECTION SCATTERING BY A COATED SPHEROID

In order to test the formalism developed in Sec. III, we applied it to scattering with one internal reflection of a linearly polarized incident plane wave with $\lambda = 0.5145$ μm by a coated particle having a spheroidal core and whose overall shape was also spheroidal. The refractive index of the core particle was taken to be $n_1 = 1.334 + 1.2 \times 10^{-9}i$, and for reasons to be explained later, the refractive index of the coating was taken to be $n_2 = 1.20$. The scattering geometry is illustrated in Fig. 2. However, before examining the scattering results for a coated spheroid, several simpler cases were considered first in order to test our computer programs based on the formalism of Sec. III.

We first set the spheroid major and minor axes equal to each other so as to have a coated sphere. The core radius was $a_{12} = 10$ μm, and the coating thicknesses considered were $\delta = 0.0$, 0.5, 1.0, and 1.5 μm. For this geometry, there are two first-order rainbows [13]. The dimmer α rainbow has its internal reflection at the 12 interface, and the brighter β rainbow has its internal reflection at the 23 interface. For TE-polarized incident light, the one-internal-reflection portion of the Debye series was computed, and the location of the first-order α and β rainbows was determined using the so-called 0.4393 intensity fitting procedure of Ref. [17]. This

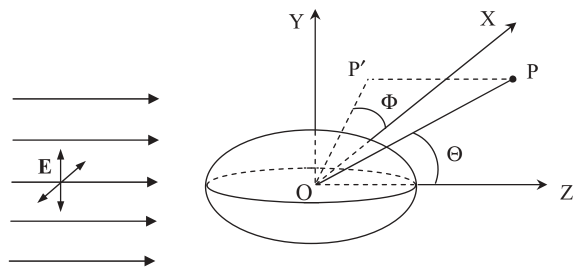


FIG. 2. Geometry of a plane wave with its propagation direction in the horizontal plane incident on a spheroid whose symmetry axis is in the vertical direction. The scattering angle Θ is measured with respect to the Z direction, and Φ is the azimuthal scattering angle. The plane wave can be either vertically or horizontally polarized.

procedure assumes the scattered intensity in the vicinity of the rainbow conform to the square of an Airy function and, thus, directly locates the angle where the argument of the Airy function is zero (i.e., the rainbow angle Θ_G). This differs from the more commonly used procedure of locating the principal maximum of the rainbow and then applying the Airy correction to extrapolate back to the location of the zero argument [25]. The angular shift of the α and β rainbows with respect to the Descartes rainbow angle Θ_D as a function of coating thickness is plotted in Fig. 3. The results, thus, contain a systematic -0.23° shift due to the small-particle-size influence to the rainbow angle described in Fig. 3(a) of Ref. [17]. The angular shift of the α rainbow due to the coating is slightly more negative, and the angular shift of the β rainbow is slightly more positive than the first-order geometric optics prediction of Ref. [13]. The exact form of the second-order geometric optics coating shift is not known. However, its size should be on the order of the first-order coating shift multiplied by δ/a_{12}.

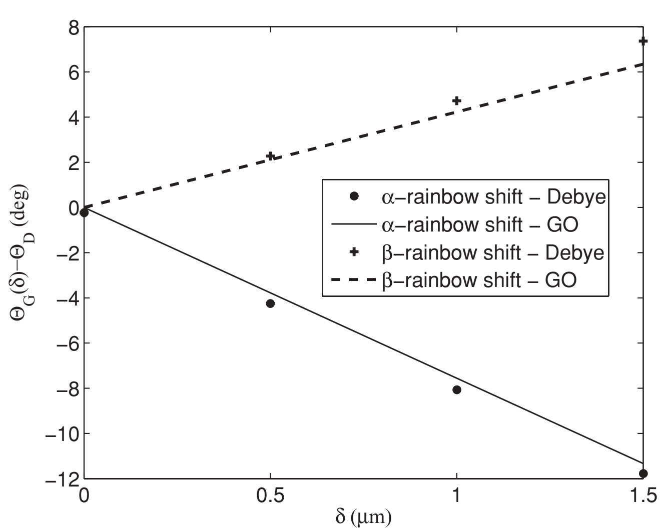
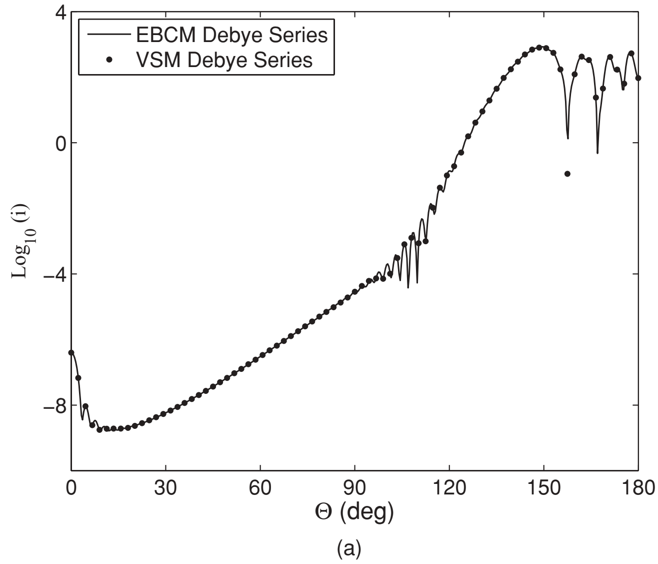


FIG. 3. Shift in the scattering angle of the first-order α and β rainbows of a coated sphere as a function of the coating thickness δ obtained from the appropriate one-internal-reflection term of the Debye series. The refractive index of the core and the coating are $n_1 = 1.334 + 1.2 \times 10^{-9}i$ and $n_2 = 1.20$. respectively. The incident light has wavelength $\lambda = 0.5145$ μm, and the core has radius $a_{12} = 10.0$ μm. The curves labeled GO are the first-order geometric optics coating shift of Ref. [13].

When $\delta = 1.5$ μm, the second-order geometric optics shift of the α and β rainbows should then be on the order of $\pm 1.7^\circ$ and $\pm 1.0^\circ$. Our results obtained from the one-internal-reflection portion of the Debye series fall within these limits. For comparison, in Fig. 3(a) of Ref. [13], the maximum value of δ/a_{12} considered was 0.035, which is about one-quarter of the δ/a_{12} interval considered here. Thus, the deviation of the computed results from the first-order geometrical optics shift was not evident in those data. The refractive index of the coating was taken here to be less than that of the core because if the coating's refractive index had instead been greater, the coating shift of the α and β rainbows would both have been negative, resulting in a smaller angular distance between them and, thus, greater interference between them.

We then set the coating refractive index equal to that of the core ($n_2 = n_1 = 1.334 + 1.2 \times 10^{-9}i$) and computed the one-internal-reflection portion of the scattered intensity for the resulting homogeneous spheroid. We considered scattering of a polarized plane wave by a prolate spheroid with $a_{12} = 4.0$ μm, $a_{23} = 5.0$ μm, $a_{12}/b_{12} = a_{23}/b_{23} = 1.05$, and by an oblate spheroid with $b_{12} = 4.0$ μm, $b_{23} = 5.0$ μm, $b_{12}/a_{12} = b_{23}/a_{23} = 1.05$. The symmetric axis of each spheroid is oriented along the beam propagation direction. The intensity is graphed in the scattering plane normal to the polarization direction of the incident wave. As indicated in Fig. 4, the resulting intensity as a function of scattering angle using the formalism of Sec. III exactly matches the results obtained using the VSM-based Debye series for a homogeneous spheroid [17].

We then examined one-internal-reflection scattering by a coated spheroid. We considered a horizontally polarized plane wave whose propagation direction is again horizontal. It is scattered by an oblate spheroid whose symmetry axis is vertical. We computed the scattered intensity in the vertical plane since we wanted to study the combined effects of the coating shift and the Möbius shift on the angular position of the first-order α and β rainbows. We chose $b_{12} = 10.0$ μm for the equatorial radius of the core and the four coating thicknesses parameterized by (i) $b_{23} = 10.0$ μm, $b_{12}/a_{12} = b_{23}/a_{23} = 1.00$, (ii) $b_{23} = 10.5$ μm, $b_{12}/a_{12} = b_{23}/a_{23} = 1.03$, (iii) $b_{23} = 11.0$ μm, $b_{12}/a_{12} = b_{23}/a_{23} = 1.06$, and (iv) $b_{23} = 11.5$ μm, $b_{12}/a_{12} = b_{23}/a_{23} = 1.09$. Both the coating thickness and the ellipticity of the particle's perimeter increase from case to case. The resulting shift of the α and β rainbows with respect to the Descartes rainbow angle as a function of coating thickness is given in Fig. 5 and again contains the systematic -0.23° small-particle shift of Ref. [17]. The Möbius shift of the rainbow angle for a coated spheroid is not known. However, since the coating thicknesses considered here are less than 15% of the core's equatorial radius, we felt that the effect of the coating would be small and, thus, compared our results to the first-order Möbius correction for a homogeneous spheroid [26] with refractive index $n = 1.334$. Similarly, the first-order geometric optics coating shift is known only for a coated sphere, and not for a coated spheroid. Nonetheless, since the spheroid ellipticities considered here are relatively small, we used the coated sphere correction. When these shifts are taken into account, the observed α rainbow shift falls somewhat farther below, and the β rainbow falls somewhat farther above the combined shift curves than they did in



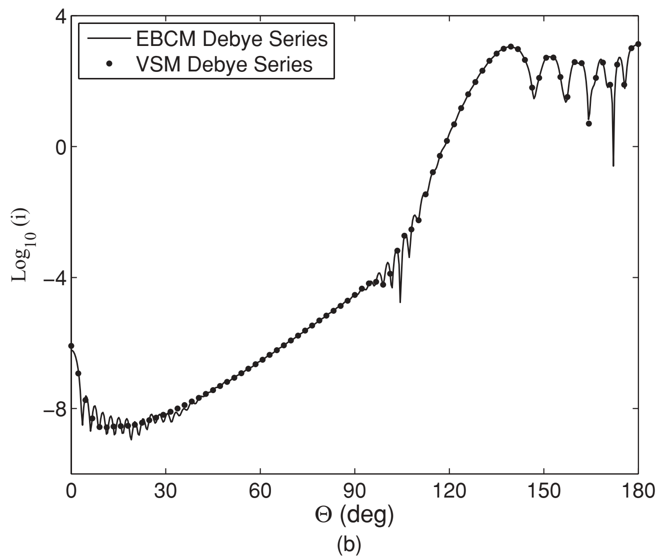


FIG. 4. Agreement of the VSM-based and EBCM-based Debye series in calculating the scattering of a polarized plane wave (a) by a prolate spheroid with $a_{12} = 4.0\ \mu$m, $a_{23} = 5.0\ \mu$m, $a_{12}/b_{12} = a_{23}/b_{23} = 1.05$, and $n_2 = n_1 = 1.334 + 1.2 \times 10^{-9}i$ and (b) by an oblate spheroid with $b_{12} = 4.0\ \mu$m, $b_{23} = 5.0\ \mu$m, $b_{12}/a_{12} = b_{23}/a_{23} = 1.05$, and $n_2 = n_1 = 1.334 + 1.2 \times 10^{-9}i$. The symmetry axis of each spheroid is oriented along the beam propagation direction. The intensity is plotted for the scattering plane $\Phi = 90^\circ$, which is normal to the polarization direction of the incident wave.

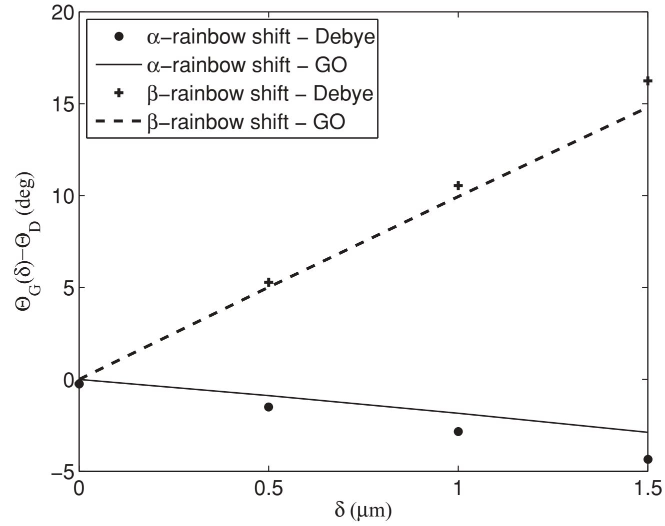


FIG. 5. Shift in the scattering angle of the first-order α and β rainbows of a coated spheroid measured in the vertical plane as a function of the coating thickness δ obtained from the appropriate one-internal-reflection term of the Debye series. The refractive index of the core and the coating are $n_1 = 1.334 + 1.2 \times 10^{-9}i$ and $n_2 = 1.20$, respectively. The incident light has wavelength $\lambda = 0.5145\ \mu$m and is horizontally polarized, and the core has radius $b_{12} = 10.0\ \mu$m and ellipticity ranging from $b_{12}/a_{12} = 1.0$ to 1.09. The curves labeled GO are the first-order geometric optics coating shift for a coated sphere of Ref. [13] plus the first-order Möbius correction for a homogeneous spheroid of Ref. [26].

Fig. 3 where the coating shift alone was considered. Like the coating shift, the Möbius shift of the rainbow angle is a geometric optics correction and is known only to first order in the spheroid ellipticity. It was found in Figs. 8 and 9 of Ref. [17] that for $b/a = 1.09$, the total wave optics Möbius shift for a homogeneous oblate spheroid was about 89% of the first-order geometrical optics Möbius shift due to the increasing importance of terms containing higher powers of the spheroid ellipticity. This additional correction would produce a slightly better fit in Fig. 5 for the α rainbow and a slightly worse fit for the β rainbow. As was the case in Fig. 3, the rainbow shifts computed using the one-internal-reflection portion of the Debye series nominally agree with the combined shift curve when an order of magnitude estimate of the second-order geometric optics coating correction is included. Thus, the homogeneous spheroid Möbius shift and the coating shift for a coated sphere appear to be reasonable approximations for the coated spheroid geometry, and the shifts are approximately additive for $b/a \leqslant 1.09$.

Lastly, the scattered intensity was computed when a vertically polarized plane wave whose propagation direction is in the horizontal plane is incident on a coated oblate spheroid whose symmetry axis is again vertical. The coated spheroid has $b_{12} = 10.0\ \mu$m and $b_{12}/a_{12} = 1.30$ for the core and $b_{23} = 11.0\ \mu$m and $b_{23}/a_{23} = 1.30$ for the outside surface. Scattering by a homogeneous spheroid having this axis ratio corresponds to the focal section [27,28] of the hyperbolic umbilic diffraction caustic (HUFS), which was studied in Ref. [29] for $b_{12} = 6.0\ \mu$m using the Debye-series decomposition of Asano's extension [30,31] of LMT with spheroidal wave functions, and in Ref. [12] for $b_{12} = 10.91\ \mu$m using the Debye-series decomposition of the EBCM formulation. At the HUFS condition, the transverse cusp diffraction caustic that first appeared for lower values of b/a merges into the first-order rainbow to produce a caustic with a higher degree of focusing than either the transverse cusp or the rainbow had individually. We examine the homogeneous spheroid HUFS geometry here in order to discover the effect that the twinning of the first-order rainbow into α and β components has on this caustic. A plot of the one-internal-reflection intensity corresponding to the α rainbow near the horizontal plane

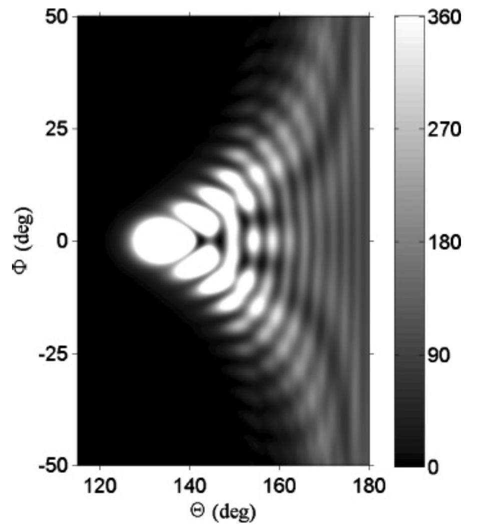


FIG. 6. Scattered intensity of the one-internal-reflection portion of the Debye series corresponding to the α rainbow as a function of the azimuthal angle Φ and the scattering angle Θ. The incident light has wavelength $\lambda = 0.5145$ μm and is vertically polarized, and the core has $b_{12} = 10.0$ μm, $b_{23} = 11.0$ μm, and $b_{12}/a_{12} = b_{23}/a_{23} = 1.30$, corresponding to the hyperbolic umbilic focal section of a homogeneous spheroid. The refractive index of the core and coating are $n_1 = 1.334 + 1.2 \times 10^{-9}i$ and $n_2 = 1.20$, respectively.

as a function of the azimuthal angle Φ and scattering angle Θ is shown in Fig. 6, the one-internal-reflection intensity corresponding to the β rainbow is shown in Fig. 7, and the intensity corresponding to the magnitude squared of the α amplitude plus the β amplitude is shown in Fig. 8. First, we examine the observed scattering angle of the principal peak of the caustic on the $\Phi = 0°$ horizontal plane where the cross section of the coated spheroid is circular and is, thus, free of the Möbius shift. The principal peak of the α and β rainbows was found in Figs. 6 and 7 to occur at $\Theta^{\alpha} = 136.02° \pm 0.06°$ and $\Theta^{\beta} = 147.06° \pm 0.06°$. In comparison, the Descartes rainbow angle for $n = 1.334$ is $\Theta_D = 138.07°$, the systematic

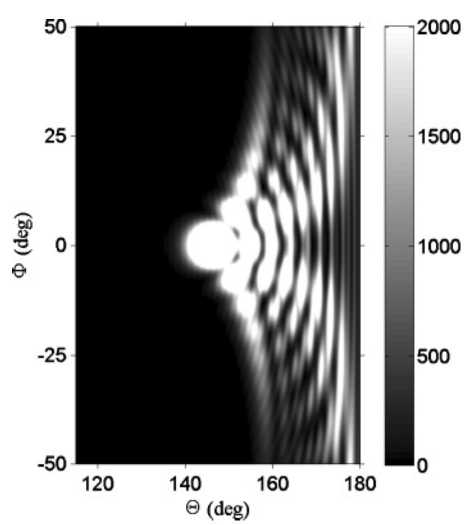


FIG. 7. Same as Fig. 6 except the intensity of the one-internal-reflection portion of the Debye series corresponding to the β rainbow is plotted.

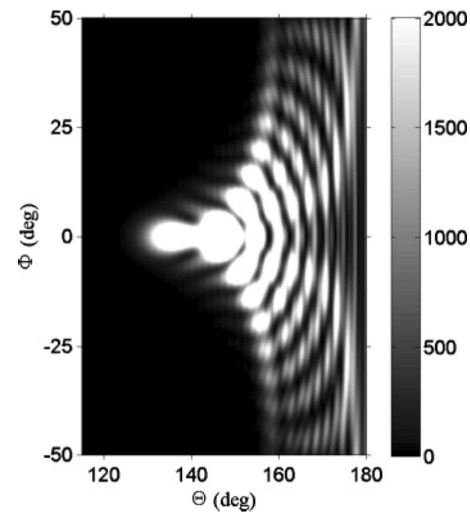


FIG. 8. Same as Fig. 6 except the intensity of the one-internal-reflection portion of the Debye series corresponding to the α rainbow plus that of the β rainbow is plotted.

small-particle shift of Ref. [17] is $-0.23°$, the coating shift is $-7.56°$ and $4.23°$ for the α and β rainbows, and the Airy shift from the zero argument position of an Airy function to the position of the principal maximum for the HUFS is $+6.38°$ and $+5.98°$. The predicted scattering angle of the principal rainbow maximum on the horizontal axis is then $\Theta^{\alpha} = 136.66°$ and $\Theta^{\beta} = 148.05°$. These predictions are $0.64°$ and $0.99°$ larger than the angles observed in Figs. 6 and 7, respectively. The difference is relatively small.

The plot of the shape of the β rainbow in Fig. 7 strongly resembles Fig. 7 of Ref. [29] and Fig. 6 of Ref. [12] if it was replotted with Φ as a function of Θ in rectangular coordinates. In particular, in Fig. 7, the angle between the $\Phi = 0°$ axis and the line joining the principal interference maximum on the axis with the first, second, and third maxima in the first line of off-axis interference maxima parallel to the caustic curve are $\xi = 62°$, $64°$, and $67°$, whereas in Fig. 7 of Ref. [29] they are $\xi = 53°$, $65°$, and $70°$. These sets of angles are reasonably close to each other, indicating that we are seeing the same caustic structure for the coated and uncoated spheroids. It should be noted that the size parameter of the spheroid in Fig. 7 is noticeably larger than that of Ref. [29], so the interference maxima have a smaller angular extent than in Ref. [29]. The conclusion of Ref. [29] was that the size parameter examined there is too small to resolve the diffraction structure of the HUFS, but it does exhibit some of the diffraction structure of the transverse cusp that is merging into the HUFS. This conclusion pertains as well to the β rainbow of the coated spheroid of Fig. 7.

However, the plot of the α rainbow in Fig. 6 has an entirely different appearance. Near the principal maximum, the arms of the interference pattern appear to be V shaped rather than curved, which is indicative of the HUFS in Fig. 4(d) of Ref. [27] rather than the transverse cusp in Fig. 4(c) of Ref. [27]. For a homogeneous spheroid of refractive index 1.334, the predicted angle [28] between the $\Phi = 0°$ axis and the line joining the principal on-axis interference maximum with the first line of the off-axis interference maxima parallel to the caustic curve is $\xi = 21.1^\circ$, whereas in Fig. 6, the corresponding angle is $\xi = 38^\circ$. It appears that the coating causes the α rainbow of the coated spheroid to qualitatively resemble the HUFS and also substantially increases its opening angle. Lastly, Fig. 8 shows the interference between the brighter β rainbow and the dimmer α rainbow for the coated spheroid at the ellipticity of the HUFS of a homogenous spheroid. The focal point of the combined caustic is now keyhole shaped, and a stronger interference structure is present due to the α rainbow than was the case in Fig. 7 for the β rainbow alone.

VI. CONCLUSION

When light is scattered by a particle composed of a number of distinct parts, it is of interest to understand the role each part has in producing the total scattered intensity, whether the parts are a number of inclusions or seeds distributed throughout the composite particle, or whether they are a number of concentric or nearly concentric layers covering a core particle. Over the years, much effort has been invested in understanding light scattering by composite objects. Some of this effort has been directed toward (i) the exact solution of the appropriate electromagnetic boundary value problems, and some has been directed toward (ii) the construction of a number of approximate models that can handle complicated geometries lying beyond the range of situations amenable to exact calculations. The method for determining the various Debye-series terms of light scattering by a coated nonspherical particle described here is of the first type. The results presented in the numerical verification of the method in Sec. V, along with the realization of how large parameter space is for this geometry, provide a strong hint at the wide variety of phenomena that can be both described and understood using the EBCM-based Debye series.

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