Analytical Solutions for Vertical Flow in Unsaturated, Rooted Soils with Variable Surface Fluxes

Fasong Yuan
*Cleveland State University*, f.yuan06@csuohio.edu

Zhiming Lu
*Los Alamos National Laboratory*, zhiming@lanl.gov

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ABSTRACT

Analytical solutions to Richards’ equation have been derived to describe the distribution of pressure head, water content, and fluid flow for rooted, homogeneous soils with varying surface fluxes. The solutions assume that (i) the constitutive relations for the hydraulic conductivity and water content as function of the pressure head are exponential, (ii) the initial water content distribution is a steady-state distribution, and (iii) the root water uptake is a function of depth. Three simple forms of root water uptake are considered, that is, uniform, stepwise, and exponential functional forms. The lower boundary of the rooted soil profile studied is a water table, while at the upper boundary time-dependent surface fluxes are specified, either infiltration or evaporation. Application of the Kirchhoff transformation allows us to linearize Richards’ equation and derive exact solutions. The steady-state solution is given in a closed form and the transient solution has the form of an infinite series. The solutions are used to simulate the hydraulic behavior of the rooted soils under different conditions of root uptake and surface flux. The restricted assumptions for the solutions may limit the applicability, but the solutions are relatively flexible and easy to implement compared to other analytical and numerical schemes. The analytical solutions provide a reliable and convenient means for evaluating the accuracy of various numerical schemes, which usually require sophisticated algorithms to overcome convergence and mass balance problems.


MATHEMATICAL FORMULATION

For one-dimensional flow in unsaturated soils with root water uptake, the flow equation can be written as

\[ \frac{\partial}{\partial z} \left( K(\psi) \frac{\partial \psi}{\partial z} + 1 \right) - S(z) = C(\psi) \frac{\partial \psi}{\partial t}, \quad 0 \leq z \leq L \]  

subject to an initial condition
Richards' equation can be linearized as

\[ \frac{d^2 \phi}{dz^2} + \frac{\alpha}{D} \frac{d \phi}{dt} + S(z) = 0 \]

with initial and boundary conditions

\[ \phi(z,0) = \phi_s(z), \]

and boundary conditions

\[ \phi(0,t) = \phi_1 \]

\[ \left[ K(\phi) \frac{d \phi}{dz} + 1 \right]_{z=L} = -q_1(t) \]

where \( \phi \) is the pressure head (L), \( K(\phi) \) is the hydraulic conductivity (LT\(^{-1}\)), \( C(\phi) = \frac{\partial \phi}{\partial \phi} \) is the differential water content, \( \psi \) is the volumetric water content, \( S \) represents the root water uptake (T\(^{-1}\)), \( z \) is the vertical coordinate pointing upward (L) (see Fig. 1), \( \psi_0 \) is the initial pressure head specified in the domain (L), \( \psi_1 \) is the prescribed pressure head at the lower boundary (L), \( q_1(t) \) is the time-dependent flux at the upper boundary (negative flux for infiltration, LT\(^{-1}\)), and \( t \) is the time (T).  

For mathematical convenience, we choose exponential models to describe the dependence of the hydraulic conductivity on the pressure head, that is, \( K(\phi) = K e^{\alpha \phi} \) and \( \theta = \theta_1 + (\theta_2 - \theta_1)e^{\alpha \phi} \). The latter leads to \( C(\phi) = \frac{\partial \psi}{\partial \phi} = \alpha (\theta_2 - \theta_1)e^{\alpha \phi} \). Here \( K_1 \) is the hydraulic conductivity (LT\(^{-1}\)) at saturation, \( \theta_1 \) is the water content at saturation (L\(^3\) L\(^{-3}\)), and \( \theta_2 \) is the residual water content (L\(^3\) L\(^{-3}\)). And \( \alpha \) is the soil pore-size distribution parameter (L\(^{-1}\)), which represents the reduction rate of the hydraulic conductivity and water content as \( \psi \) is usually negative in unsaturated soils. Using the Kirchhoff transformation (Gardner, 1958; Lu and Zhang, 2004),

\[ \Phi(z,t) = \int_0^z K(\psi) d\phi = \frac{K(\psi)}{\alpha} \]

Richards' equation can be linearized as

\[ \frac{d^2 \Phi}{dz^2} + \frac{\alpha}{D} \frac{d \Phi}{dt} + S(z) = 0 \]

where \( \Phi \) is called the matrix flux potential (L\(^2\) T\(^{-1}\)), \( D = K_1/\alpha(\theta_2 - \theta_1) \) is the soil moisture diffusivity (L\(^2\) T\(^{-1}\)).

In this study, we assume that the initial soil water distribution is a steady state rather than a uniform profile. In the following sections, we will derive the steady-state solution and then use it as an initial condition for a transient solution.

### Steady-State Solutions

The steady-state matrix flux potential \( \phi_s \) satisfies the ordinary differential equation:

\[ \frac{d^2 \phi}{dz^2} + \frac{\alpha}{D} \frac{d \phi}{dt} = S(z) = 0 \]

and the boundary conditions

\[ \phi_s(0) = \frac{K_s}{\alpha} \exp(\alpha \psi_1) \]

\[ \left[ \frac{d \phi_s}{dz} + \alpha \phi_s \right]_{z=L} = -q_0 \]

where \( q_0 \) is the surface flux at the time \( t = 0 \). Let

\[ \phi_s = \phi - q_0 - \frac{K_s}{\alpha(1 + \alpha L)} z - \frac{K_s}{\alpha} \exp(\alpha \psi_1) \]

then the steady-state equation and its boundary conditions become

\[ \frac{d^2 \phi}{dz^2} + \frac{\alpha}{D} \frac{d \phi}{dt} - \left[ S(z) + \alpha \right] = 0 \]

\[ \phi(0) = 0 \]

\[ \left[ \frac{d \phi}{dz} + \alpha \phi \right]_{z=L} = 0 \]

where \( A = \alpha(q_0 + K_s e^{\alpha \psi_1})/(1 + \alpha L) \). The solution to Eq. [14] to [16] can be expressed formally as

\[ \phi(z) = \int_0^L G(z, x) [S(x) + A] dx \]

where the Green function \( G(z, x) \) for this case is defined as

\[ G(z, s) = \left\{ \begin{array}{ll} \frac{\exp(-\alpha z)}{\alpha} [1 - \exp(\alpha s)] & 0 \leq s \leq z \leq L \\ \frac{1}{\alpha} [\exp(-\alpha z) - 1] & 0 \leq z \leq s \leq L \end{array} \right. \]

Combining Eq. [13], [17], and [18], one has

\[ \Phi(z) = \frac{K_s}{\alpha} \exp[\alpha(\psi_1 - z)] + \frac{q_0}{\alpha} \exp(-\alpha z) - 1 
+ \int_0^L G(z, x) S(x) dx \]

Equation [19] gives a general solution to steady verti-
cal flow problems. For any given uptake term \( S \) as a function of \( z \), the corresponding steady-state solution for the matrix flux potential \( \Phi \) can be derived by carrying out the integral in Eq. [19]. For complicated functional forms of the uptake term \( S \), the integral in Eq. [19] may need to be evaluated numerically. However, for some particular uptake functions, the steady-state solution can be derived analytically through Eq. [19] as follows.

### Uniform Root Uptake

In the simplest case, the root uptake term is a constant \( S(z) = S_0 > 0 \) for all \( 0 \leq z \leq L \). Integrating Eq. [19] yields

\[
\Phi_s(z) = \frac{K_0 \exp{[\alpha(\psi_0 - z)]}}{\alpha} + \frac{q_0}{\alpha} \exp(-\alpha z) - 1 + \frac{S_0}{\alpha^2} \left[(zL - 1) \exp(-\alpha z) - \alpha(L - z) - 1\right] \tag{20}
\]

For \( z = 0 \), the matrix flux potential \( \Phi_s(0) = K_0 \exp(\alpha \psi_0) / \alpha \), which is independent of the root uptake \( S \) and of course consistent with Eq. [8] and [11].

### Step Functions

In general, the depth of the root zone is less than that of the vadose zone; that is, the root uptake takes place only in the upper portion of the vadose zone. In this case, the root uptake may be approximated by \( S(z) = S_0 H(z - L_1) \), where \( H(z - L_1) \) is the Heaviside function defined as \( H(z - L_1) = 0 \) for \( 0 \leq z \leq L_1 \) and \( H(z - L_1) = 1 \) for \( L_1 \leq z \leq L \). Integrating Eq. [19] yields the steady-state solution

\[
\Phi_s(z) = \begin{cases} \frac{K_0 \exp{[\alpha(\psi_0 - z)]}}{\alpha} + \frac{q_0}{\alpha} \exp(-\alpha z) - 1 + \frac{S_0(L - L_1)}{\alpha^2} \left[(zL - 1) \exp(-\alpha z) - \alpha(L - z) - 1\right] & \text{for } 0 \leq z \leq L_1 \\ \frac{K_0 \exp{[\alpha(\psi_0 - z)]}}{\alpha} + \frac{q_0}{\alpha} \exp(-\alpha z) - 1 + \frac{S_0}{\alpha^2} \left[(zL - 1) \exp(-\alpha z) - \alpha(L - z) - 1\right] & \text{for } L_1 \leq z \leq L \end{cases} \tag{21}
\]

It is easy to check that both the steady-state solution \( \Phi_s \) and its first-order derivative are continuous at \( z = L_1 \). In the case that \( L_1 = 0 \), that is, uniform root uptake in the entire domain, from the second part of the solution we can verify that the above solution reduces to Eq. [20]. On the other hand, if \( L_1 = L \), that is, no uptake at all, from the first part of this solution, we can easily see that the term with uptake disappears.

More generally, if the uptake function \( S(z) \) is defined as a piecewise step function on \( 0 = z_0 \leq z_1 \leq \cdots \leq z_n = L \) as \( S(z) = \sum_{j=1}^{n} S_j H(z - z_{j-1}) H(0, z - z_{j-1}) \), the steady-state solution can be written as, for any \( z_{k-1} \leq z \leq z_k \),

\[
\Phi_s(z) = \frac{K_0 \exp{[\alpha(\psi_0 - z)]}}{\alpha} + \frac{q_0}{\alpha} \exp(-\alpha z) - 1 + \exp(-\alpha z) \sum_{j=1}^{k-1} S(z_j - z_{j-1}) - \frac{1}{\alpha} \sum_{j=k}^{n} S(z_j - z_{j-1}) - \frac{S_0}{\alpha^2} \left[(zL - 1) \exp(-\alpha z) - \alpha(L - z) - 1\right] \tag{22}
\]

This particular case is of interest in connection with observed root length or root mass in individual layers.

### Exponential Uptake

The distribution function of root uptake may be expressed in an exponential form (Raats, 1974; Rubin and Or, 1993; Schoups and Hopmans, 2002). \( S(z) = S_0 \exp[\beta(z - L_1)] \) where \( S_0 \) is the maximum uptake at the land surface \( (T^{-1}) \) and \( \beta \) is a constant \( (L^{-1}) \) representing the rate of reduction in root uptake. Carrying out the integral in Eq. [19] yields

\[
\Phi_s(z) = \frac{K_0 \exp{[\alpha(\psi_0 - z)]}}{\alpha} + \frac{q_0}{\alpha} \exp(-\alpha z) - 1 + \frac{S_0[\exp[\beta(z - L)] - \exp(-\alpha z - \beta L)] - \exp(-\alpha z - 1)}{\alpha \beta} \tag{23}
\]

The steady-state pressure head and water content can be computed from \( \psi = (1/\alpha) \ln(\alpha \Phi_s/K_0) \) and \( \theta = \psi + (\theta_0 - \theta_s) \Phi_s/K_0 \).

In case with \( L \) approaching infinity, Eq. [23] becomes

\[
\Phi_s(Z) = -\frac{q_0 - S_0[\alpha + \beta - \alpha \exp(-BZ)]}{\alpha \beta (\alpha + \beta)} \tag{24}
\]

where \( Z = L - z \) is the depth below the land surface. Given the assumption that hydraulic conductivity is a linear function of water content \( \theta \) or matrix flux potential \( \Phi_s \), Eq. [24] has the similar form of the solution given by Raats (1976, Eq. [16]).

### Transient Solutions

The steady-state solution \( \Phi_s \) is now taken as the initial condition \( \Phi_i \) for the transient problem Eq. [6] through [9]. Taking the Laplace transformation, we have the ordinary differential equation

\[
\frac{d^2 \Phi}{dz^2} + \frac{d \Phi}{dz} - \frac{s}{D} \Phi + \Phi_i - \frac{S}{s} = 0 \tag{25}
\]

with boundary conditions

\[
\Phi(0) = \frac{\Phi_i(z)}{s} \tag{26}
\]

\[
\left[ \frac{d \Phi}{dz} + \alpha \Phi \right]_{z=L} = -\tilde{q}(s) \tag{27}
\]

where \( s \) is the Laplace-transform complex variable, \( \Phi = \)
and is given by soils are assumed to be 0.45 and 0.20 cm$^3$ cm$^{-1}$ at any time can be derived from to the assumption that the uptake term $q$ does not depend on the root uptake, which is due to distributions of the pressure head, water content, and 

\begin{equation}
L(z,t) = \Phi(z) + 8D\exp\left[\frac{-\alpha(L - z)}{2}\right]
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} \left(\frac{\lambda_n^2 + \frac{\alpha^2}{4}}{2\alpha + \alpha^2 + 4L\lambda_n^2}\right)G(t)
\end{equation}

\begin{equation}
G(t) = \left[q_0(t) - q_0(\tau)\right]\exp\left[-D\left(\lambda_n^2 + \frac{\alpha^2}{4}\right)(t - \tau)\right]d\tau
\end{equation}

where $\lambda_n$ is the $nth$ positive root of equation $\sin(\lambda L) + (2\lambda/\alpha)\cos(\lambda L) = 0$. Note that the transient part in Eq. [28] does not depend on the root uptake, which is due to the assumption that the uptake term $S(z)$ is time-independent. The flux water flow below the land surface at any time can be derived from $q(z,t) = d\Phi/dz + \alpha\Phi$ and is given by

\begin{equation}
q(z,t) = q_0 + \int_0^t S(x)dx - 8D\exp\left[\frac{\alpha(L - z)}{2}\right]
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} \frac{\lambda_n^2 + \frac{\alpha^2}{4}}{2\alpha + \alpha^2 + 4L\lambda_n^2}G(t)\sin(\lambda_n L) - D\left(\lambda_n^2 + \frac{\alpha^2}{4}\right)\int_0^t S(x)dx
\end{equation}

In the case that $q_1$ is a constant, Eq. [28] can be simplified to

\begin{equation}
\Phi(z,t) = \Phi_{s0}(z) - 8(q_0 - q_1)\exp\left[\frac{\alpha(L - z)}{2}\right]
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} \frac{\sin(\lambda_n L)\sin(\lambda_n z)}{2\alpha + \alpha^2 + 4L\lambda_n^2}\exp\left[-D\left(\lambda_n^2 + \frac{\alpha^2}{4}\right)t\right]
\end{equation}

where $\Phi_{s0}$ is the final steady-state solution of the transient problem with surface flux $q_1$ and can be obtained on replacing $q_0$ in $\Phi$ by $q_1$. Correspondingly, Eq. [30] can be simplified to

\begin{equation}
q(z,t) = q_1 + \int_0^t S(x)dx + 4\alpha(q_0 - q_1)\exp\left[\frac{\alpha(L - z)}{2}\right]
\end{equation}

\begin{equation}
\sum_{n=1}^{\infty} \frac{\sin(\lambda_n L)\sin(\lambda_n z) + 2\alpha\cos(\lambda_n z)}{2\alpha + \alpha^2 + 4L\lambda_n^2}\exp\left[-D\left(\lambda_n^2 + \frac{\alpha^2}{4}\right)t\right]
\end{equation}

**ILLUSTRATIVE EXAMPLES AND DISCUSSION**

In this section, we will discuss the analytical solutions through numerical examples, in which we compute the distributions of the pressure head, water content, and water flux across a 100-cm soil profile with the lower boundary confined by the water table (Fig. 1). The water content at saturation and residual water content of the soils are assumed to be 0.45 and 0.20 cm$^3$ cm$^{-1}$ (Srivastava and Yeh, 1991). The hydraulic conductivity at saturation is taken as 1.0 cm h$^{-1}$. The initial water content profile is assumed to be a steady-state profile with a surface influx of 0.1 cm h$^{-1}$; that is, $q_0 = -0.1$. Both constant and varying surface fluxes are considered for the upper boundary conditions.

**Constant Surface Flux**

In this case we assume that a constant infiltration of 0.9 cm h$^{-1}$ (i.e., $q_1 = -0.9$) occurs and lasts for at least a few days. The transient distribution of the pressure head and the water content can be computed based on the solution (31) and the exponential hydraulic parameter models. Figures 2 and 3 show the computed distri-
Fig. 3. Wetting profiles of (a) pressure head and (b) water content for soils without root water uptake ($\alpha = 0.1$ cm$^{-1}$).

Fig. 4. Comparison of wetting profiles in rooted soils under constant surface flux ($q_l = -0.9$ cm h$^{-1}$). (a) $\alpha = 0.01$ cm$^{-1}$, $S_o = 0.02$ h$^{-1}$, (b) $\alpha = 0.1$ cm$^{-1}$ and $S_o = 0.0025$ h$^{-1}$.

Distributions of the pressure head and the water content for homogeneous soils with $\alpha = 0.01$ cm$^{-1}$ and $\alpha = 0.1$ cm$^{-1}$, respectively, for a period of 50 h. Note that the root uptake is ignored in the two examples. The calculated results are exactly the same as those of Srivastava and Yeh (1991). Both the pressure head and water content profiles are similar in shape because of the similar form of the exponential hydraulic parameter model used. The soil water moves faster in the soils with $\alpha = 0.01$ cm$^{-1}$, but the time needed to reach the steady state is nearly the same (about 50 h) due to the same surface boundary condition considered. This is especially the case near the soil surface where the soil water content approaches the steady state faster than further down in the soil.

In the presence of root water uptake, we consider a rooted soil profile with a maximum root depth of 40 cm (i.e., $L_r = 60$ cm in Fig. 1) and assume that the distribution of root water uptake can be described by the Heaviside function. The maximum water uptake at the land surface ($S_o$) is taken as 0.02 h$^{-1}$ for $\alpha = 0.01$ cm$^{-1}$, and 0.0025 h$^{-1}$ for $\alpha = 0.1$ cm$^{-1}$. Figures 4a and 4b show changes in the water content distribution for such rooted
soils during a period over 30 to 50 h. The initial water content profile for the rooted soils with $\alpha = 0.01 \text{ cm}^{-1}$ is much drier than that without root uptake (Fig. 2b). The initial moisture profile approaches a new steady state approximately 30 h after the beginning of the increase in infiltration rate. On the other hand, the water content profile of the rooted soils with $\alpha = 0.1 \text{ cm}^{-1}$ is similar to that without root uptake (Fig. 3b), as the root uptake component is relatively small, which accounts for $\sim 11\%$ of the infiltration.

**Time-Dependent Surface Flux**

In reality, the upper boundary conditions always vary with time as a result of agricultural practices and weather forcing, for example, irrigation, rainfall and evaporation, etc. Here we consider that the surface flux is an exponentially decaying function of time, namely $q_1(t) = q_0 + \delta \exp(kt)$ where $\delta = -0.8 \text{ cm h}^{-1}$ and $k = -0.1 \text{ h}^{-1}$. This simple surface flux model allows $q_1$ to approach $q_0$ when $t$ becomes sufficiently large (Fig. 5c). The moisture contents at any time and depth are computed through Eq. [28] using the exponential surface flux model and the results are presented in Fig. 5a and 5b for the rooted soils. Both of the rooted soils receive the same amount of water from the upper boundary, but exhibit rather different patterns of the water content distributions. The soil profile with $\alpha = 0.01 \text{ cm}^{-1}$ is on average wetter than the soil with $\alpha = 0.1 \text{ cm}^{-1}$ even though the amount of

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**Fig. 5.** Comparison of soil water distribution in rooted soils under varying surface fluxes $q_1(t)$. (a) $\alpha = 0.01 \text{ cm}^{-1}$ and $S_0 = 0.02 \text{ h}^{-1}$. (b) $\alpha = 0.1 \text{ cm}^{-1}$ and $S_0 = 0.0025 \text{ h}^{-1}$. (c) Exponential surface input function.
Fig. 6. Temporal development of water flows. (a) $\alpha = 0.01$ cm$^{-1}$, $S_0 = 0.02$ h$^{-1}$, and $k = 0$. (b) $\alpha = 0.01$ cm$^{-1}$, $S_0 = 0.02$ h$^{-1}$, and $k = -0.1$. (c) $\alpha = 0.1$ cm$^{-1}$, $S_0 = 0.0025$ h$^{-1}$, and $k = 0$. (d) $\alpha = 0.1$ cm$^{-1}$, $S_0 = 0.0025$ h$^{-1}$, and $k = -0.1$. Infiltration ($q_1$) is in thick solid curves, flow at interface between root zone and subsoil ($q_2$) in thin solid curves, and flow near the water table ($q_3$) in dashed curves. Note that $k$ is a constant in $q_1(t) = q_0 + 5\exp(kt)$, $\delta = -0.8$ cm h$^{-1}$.

water loss through root uptake is larger than that received from infiltration. This is because the larger root water uptake in the soil with $\alpha = 0.01$ cm$^{-1}$ favors the capillary rise that brings water from the water table into root zones. On the other hand, the impact from the rapid change in $q_1$ on the soil moisture content is much deeper in the soil with $\alpha = 0.1$ cm$^{-1}$ and the response time increases with the increasing depth.

To evaluate the transient water flow in response to changes in the surface flux, we use Eq. [30] to compute the flow ($q_2$) at the interface between the root zone and subsoil and the flow at the water table ($q_3$) (Fig. 6). In the rooted soil with $\alpha = 0.01$ cm$^{-1}$ (Fig. 6a and 6b) the difference between $q_2$ and $q_3$ is relatively small. Both $q_2$ and $q_3$ approach $-0.1$ cm h$^{-1}$ for the constant surface flux and 0.7 cm h$^{-1}$ for the exponentially decaying surface flux when $t > 50$ h. Note that the positive values of $q_2$ and $q_3$ suggest that water moves upward, that is, capillary rise. In the cases of constant surface flux (Fig. 6a and 6c) the absolute value of $q_2$ is always not less than that of $q_3$, while in the cases of varying surface flux (Fig. 6b and 6d) the absolute value of $q_2$ is not always greater than that of $q_3$. It is easy to check the mass is conservative in all the cases. Additionally, the response time of $q_2$ is usually shorter than that of $q_3$. The time lag is likely associated with the hydraulic conductivity.

**SUMMARY AND CONCLUSIONS**

We solved Richards’ equation for water flow in unsaturated, rooted soils under time-dependent varying upper boundary conditions. The analytical solutions are based on assumptions that (i) the hydraulic conductivity and water content are exponential functions of the pressure head, (ii) the initial water contents are in steady state, and (iii) the distribution of root water uptake is a function of depth. Both steady state and transient solutions are given and discussed through illustrative examples. Equation [28] gives an alternative single form of the one-dimensional solutions of Basha (2000, their Eq. [24], [26], [38], and [51]). The analytical solutions are validated by comparing the computed pressure head and water content using other analytical solutions (Srivastava and Jim Yeh, 1991). The analytical solutions are useful to predict the vertical distribution of the water content and the water flux. This analytical solution is not applicable in cases where the exponential hydraulic parameter model is not appropriate. An implicit assumption of a shallow water table with a fixed depth is needed for the solutions, that is, water table does not rise with infiltration or fall with root water uptake. Another limitation of the analytical solutions is imposed by the assumption related to the sink term of root water uptake. In reality, the distribution of root uptake is not only a function of depth but also related to other factors, for example, water content, salinity, and even plant physiological parameters. Nevertheless, the analytical solutions provide an additional tool for validating and/or checking the accuracy of numerical schemes.

**APPENDIX A**

In equations [25]–[27], let $\Phi = \phi + \Phi_r/s$, we obtain equations for the new variable $\phi$...
The characteristic equation for Eq. [A1] is

\[ \lambda^2 + \alpha \lambda - \frac{s}{D} = 0 \]  

and its two solutions are

\[ \lambda_{1,2} = \frac{\alpha}{2} + \sqrt{\frac{s}{D} + \frac{\alpha^2}{4}} = \frac{\alpha}{2} \pm \Delta \]

The general solution of \( \phi \) can be written as

\[ \phi(z) = C_1 \exp(\lambda_1 z) + C_2 \exp(\lambda_2 z) \]

where \( C_1 \) and \( C_2 \) are constants to be determined. Using boundary conditions [A2] and [A3], one can solve these two constants

\[ C_1 = -C_2 = \frac{\exp(\alpha L/2)}{2} \frac{q_0/s - q_1(s)}{\sqrt{s/D} + \alpha \sinh(\Delta L)} + \alpha \cosh(\Delta L) \]

The Laplace transformed variable \( \Phi \) can be written as

\[ \Phi(z) = \frac{\Phi(z)}{s} + \exp\left(\frac{\alpha(L - z)}{2}\right) \frac{[q_0/s - q_1(s)]\sinh(\Delta z)}{\sqrt{s/D} + \alpha \sinh(\Delta L)} + \alpha \cosh(\Delta L) \]

The solution of Eq. [6] to [9] is given as

\[ \Phi(z, t) = \Phi(z) + 8D \exp\left[\frac{[\alpha(L - z)]}{2}\right] \sum_{n=1}^{\infty} \left[ q_0 - q_1(t) \right] * L^{-}[F(s)] \]

where the symbol \( * \) represents convolution, and

\[ F(s) = \frac{\sinh(\Delta z)}{\sqrt{s/D} + \alpha \sinh(\Delta L)} + \alpha \cosh(\Delta L) \]

Finding the inverse of Laplace transformation of function \( F(s) \) is equivalent to finding the sum of all residues of \( e^{\eta} F(s) \) at poles of \( F(s) \), at which the denominator of \( F(s) \) is zero; that is,

\[ \frac{\alpha}{2} \sqrt{s/D} + \alpha \sinh(\Delta L) + \alpha \cosh(\Delta L) = 0 \]

We get all poles of \( F(s) \) by setting \( \Delta = i\lambda_n \), or \( s = -D(\lambda_n^2 + \frac{\alpha^2}{4}) \), where \( \lambda_n \) satisfies

\[ \sin(\lambda_n^2 L) + \frac{2\alpha}{\alpha} \cos(\lambda_n^2 L) = 0 \]

Since the residue of \( e^{\eta} F(s) \) at \( s = -D(\lambda_n^2 + \frac{\alpha^2}{4}) \) is

\[ \text{Res}\left\{\exp(st)F(s), -D\left(\lambda_n^2 + \frac{\alpha^2}{4}\right)\right\} = \frac{\left(\lambda_n^2 + \frac{\alpha^2}{4}\right) \sin(\lambda_n^2 L) \sin(\lambda_n^2 z)}{2\alpha + \alpha^2 L + 4L\lambda_n^2} \exp\left[-D\left(\lambda_n^2 + \frac{\alpha^2}{4}\right)t\right] \]

the inverse transformation of \( F(s) \) can be derived as

\[ L^{-1}[F(s)] = \frac{\left(\lambda_n^2 + \frac{\alpha^2}{4}\right) \sin(\lambda_n^2 L) \sin(\lambda_n^2 z)}{2\alpha + \alpha^2 L + 4L\lambda_n^2} \exp\left[-D\left(\lambda_n^2 + \frac{\alpha^2}{4}\right)t\right] \]

Finally, we solve the Kirchhoff transformed variable \( \Phi \)

\[ \Phi(z, t) = \Phi(z) + 8D \exp\left[\frac{[\alpha(L - z)]}{2}\right] \sum_{n=1}^{\infty} \left[ q_0 - q_1(t) \right] * L^{-1}[F(s)] \]

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