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Supernumerary spacing of rainbows produced by an elliptical-cross-section cylinder. I. Theory

James A. Lock

A sequence of rainbows is produced in light scattering by a particle of high symmetry in the short-wavelength limit, and a supernumerary interference pattern occurs to one side of each rainbow. Using both a ray-tracing procedure and the Debye-series decomposition of first-order perturbation wave theory, I examine the spacing of the supernumerary maxima and minima as a function of the cylinder rotation angle when an elliptical-cross-section cylinder is normally illuminated by a plane wave. I find that the supernumerary spacing depends sensitively on the cylinder-cross-section shape, and the spacing varies sinusoidally as a function of the cylinder rotation angle for small cylinder ellipticity. I also find that relatively large uncertainties in the supernumerary spacing affect the rainbow angle only minimally.

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1. Introduction

When an electromagnetic plane wave is scattered by a high-symmetry object such as a sphere, spheroid, circular-cross-section cylinder, or an elliptical-cross-section cylinder whose size is much larger than the wavelength of light, a sequence of rainbows is observed in the scattering far zone. The various rainbows are labeled by index p , with the one-internal-reflection (i.e., primary) rainbow corresponding to $p = 2$, the two-internal-reflection (i.e., secondary) rainbow corresponding to $p = 3$, etc. The rainbow is an example of the fold caustic, the simplest of the structurally stable optical caustics.¹ A consequence of the rainbow's structural stability is that, if a spherical particle were to be slightly deformed into a prolate or oblate spheroid, or if a cylinder's circular cross section were to be slightly deformed into an ellipse, the resulting rainbow would distort in a number of ways. The rainbow scattering angle would shift, the spacing of the maxima and minima of the supernumerary interference pattern adjacent to the rainbow would change, and the relative intensity of the supernumerary maxima would change. But

structural stability requires that the basic rainbow morphology persists.

When a circular cylinder is deformed so that its cross section becomes slightly elliptical, one of the distortions of the $p = 2$ and $p = 3$ rainbows is described by the Möbius extension²⁻⁴ to ray theory. When the distorted cylinder is rotated about its axis so that the incident beam illuminates different portions of its elliptical surface, the rainbow scattering angle, to first order in the eccentricity of the ellipse, oscillates sinusoidally back and forth about the Descartes rainbow angle for a circular cylinder.

In this paper I examine another distortion of the $p = 2$ and $p = 3$ rainbows when the cross section of a cylinder is deformed from a circle into an ellipse. The angular spacing of the maxima and minima of the supernumerary interference pattern is standardly parameterized⁵ by the quantity h_p , which is defined in Eq. (2) below. One observes that as the cylinder is rotated about its axis and the rainbow angle oscillates back and forth, the supernumerary pattern adjacent to it also oscillates back and forth, but by an alternately slightly larger or slightly smaller amount. This corresponds to an expansion and contraction of the supernumeraries of the p rainbow, which is described mathematically by h_p being a function of the cylinder rotation angle ξ .

This paper proceeds as follows. In Section I briefly review the ray, Möbius, Airy, and complex angular momentum (CAM) theories of the rainbow and describe the appearance of rainbows in exact Rayleigh–Debye theory for scattering by a circular

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cylinder. In Sections 3 and 4 I calculate h_p using two different methods, one in the scattering near zone and the other in the far zone. In Section 3 I use a numerical ray-tracing procedure to determine the shape of the phase fronts exiting a circular- or elliptical-cross-section cylinder in the vicinity of the $p = 2$ and $p = 3$ rainbows. From the shape of the phase fronts, I obtain $h_2(\xi)$ and $h_3(\xi)$ and find that the coefficients of the Fourier-series decomposition of $h_2(\xi)$ and $h_3(\xi)$ scale as various powers of the eccentricity of the elliptical cross section. In Section 4 I determine $h_2(\xi)$ using the far-zone scattered intensity in the vicinity of the rainbow. To do this I first derive the wave theory scattering equations to first order in the perturbation of the shape of the cylinder's cross section from that of a circle. I then perform a Debye-series decomposition of the resulting partial-wave scattering amplitudes and find that the surface shape perturbation induces a coupling between partial waves at all the interactions of the partial waves with the cylinder surface. I then numerically compute $h_2(\xi)$ using first-order perturbation theory along with Airy or CAM modeling of the supernumerary intensity minima and compare the results with the ray theory results of Section 3.

Finally in Section 5 I summarize the results and comment on their significance.

2. Theories of the Rainbow

A. Rainbow in Ray Theory and Möbius Theory

I consider a family of parallel rays of wavelength λ and wave number $k = 2\pi/\lambda$ normally incident on a homogeneous circular cylinder of refractive index n and radius a . The scattering angle θ_p^D of the p rainbow is given in ray theory by⁶

$$\begin{aligned}\cos(\phi_i^D) &= [(n^2 - 1)/(p^2 - 1)]^{1/2}, \\ \sin(\phi_t^D) &= (1/n)\sin(\phi_i^D), \\ \theta_p^D &= (p - 1)\pi + 2\phi_i^D - 2p\phi_t^D.\end{aligned}\quad (1)$$

The same formulas are applicable to ray scattering by a sphere. The rainbow ray for a circular-cross-section cylinder or sphere is known as the Descartes ray and is denoted in Eqs. (1) by the superscript D . The angle that the incident and refracted Descartes ray makes with the normal to the cylinder surface as the ray enters and exits it is ϕ_i^D and ϕ_t^D , respectively. The shape of the phase fronts of the rays exiting a circular cylinder in the immediate vicinity of the p rainbow Descartes ray is⁵

$$Y = -h_p X^3/3a^2 + O(X^4/a^3), \quad (2)$$

where X is the distance measured along the exit plane of the cylinder, defined as being tangent to the cylinder and normal to the Descartes ray; Y is the distance measured parallel to the Descartes ray; and⁷

$$h_p = [(p^2 - 1)^2(p^2 - n^2)^{1/2}]/[p^2(n^2 - 1)^{3/2}]. \quad (3)$$

We next consider a cylinder with an elliptical cross section whose surface is given by

$$(x'^2/a^2) + (y'^2/b^2) = 1, \quad (4)$$

where x', y', z' is a coordinate system attached to the cylinder whose symmetry axis coincides with the z' axis. The cylinder's eccentricity is

$$\epsilon = (b/a) - 1. \quad (5)$$

Another set of coordinates x, y, z is fixed in the laboratory with $z = z'$. The cylinder is oriented so that the x' axis makes an angle ξ with the x axis (this is illustrated in Fig. 1 of Ref. 8). A family of incident rays propagates in the $-y$ direction before encountering the cylinder, and the scattering angle θ is measured clockwise from the $-y$ axis. The scattering angle of the $p = 2$ and $p = 3$ rainbows of the elliptical-cross-section cylinder is found in Möbius theory to be²⁻⁴

$$\begin{aligned}\theta_2^R(\xi) &= \theta_2^D - 8\epsilon \sin(\phi_t^D)\cos^3(\phi_t^D) \\ &\quad \times \cos(2\xi + \theta_2^D) + O(\epsilon^2), \\ \theta_3^R(\xi) &= \theta_3^D + 32\epsilon \sin(\phi_t^D)\cos^3(\phi_t^D) \\ &\quad \times \cos(2\phi_t^D)\cos(2\xi + \theta_3^D) + O(\epsilon^2).\end{aligned}\quad (6)$$

The details of the Möbius calculation are described more fully in Ref. 8.

B. Rainbow in Airy Theory

In the physical optics model of the rainbow, also known as Airy theory, the cubic phase front of the electric field exiting a circular cylinder in the vicinity of the rainbow is Fourier transformed to the scattering far zone.⁹ The square of the far-zone scattered electric field is

$$I(\theta) = (2\pi I_0 F/r)(ax^{1/3}/h_p^{2/3})\text{Ai}^2(-x^{2/3}\Delta/h_p^{1/3}), \quad (7)$$

where I_0 is the intensity of the plane wave, $x = ka$ is the cylinder size parameter, r is the distance from the cylinder axis to the far-zone position of the detector in the x, y plane, Ai is the Airy integral,¹⁰

$$\Delta = \theta - \theta_p^D, \quad (8)$$

and F is the appropriate combination of flat-surface transmission and reflection Fresnel coefficients evaluated at the incident and transmitted angles ϕ_i^D and ϕ_t^D of the Descartes ray. To derive Eq. (7) it is assumed that the incident electric field is polarized parallel to the cylinder axis (i.e., the TE polarization). If the incident electric field were instead polarized perpendicular to the cylinder axis (i.e., the TM polarization), a large, if not dominant, contribution to the scattered intensity is proportional to the square of the derivative of the Airy integral, $\text{Ai}'^2(-x^{2/3}\Delta/h_p^{1/3})$, because the internal reflections occur near the Brewster angle.¹¹

Fig. 1. Far-zone scattered intensity as a function of scattering angle in the vicinity of the $p = 2$ rainbow for a circular-cross-section cylinder with $n = 1.333$ and $x = 1000.0$ calculated by exact wave theory (solid curve) and the $p = 2$ Debye-series portion of exact wave theory (dashed curve).

C. Rainbow in Rayleigh–Debye Theory and Complex Angular Momentum Theory

The exact solution for scattering of a normally incident electromagnetic plane wave by an infinitely long homogeneous circular cylinder is written in terms of an infinite series of cylindrical multipole partial waves and is sometimes called Rayleigh theory¹² in analogy to the term Mie theory for scattering by a sphere. The decomposition of the partial-wave scattering amplitudes into the individual contributions of diffraction plus reflection ($p = 0$), transmission ($p = 1$), and transmission following $p - 1 \geq 1$ internal reflections is known as the Debye series.¹³ A number of rainbows are observed in the computed Rayleigh–Debye far-zone scattered intensity for a circular-cross-section cylinder^{14–16} with $x \gg 1$ (e.g., the $p = 2$ rainbow for $n = 1.333$ and $x = 1000.0$ is shown as the solid curve in Fig. 1). The scattered intensity in the vicinity of the rainbow, however, is modulated by scattered light because of other Debye-series contributions, such as reflection.¹⁷ Thus, for a careful examination of the p rainbow in isolation from these other contributions, the scattered intensity is computed with only the p term of the Debye series (e.g., the dashed curve in Fig. 1). The resulting intensity is closely approximated by the Airy theory intensity of Eq. (7), with the accuracy of the approximation improving as x increases.

CAM theory^{17–19} provides a more accurate modeling of the p rainbow of a circular-cross-section cylinder or a sphere than does Airy theory. In CAM theory the sum over partial waves is converted into an integral by use of the modified Watson transformation, and a uniform approximation of the saddle-point contributions to the integral is performed. The resulting scattered electric field in the vicinity of the rainbow is a linear combination of both the Airy integral and its derivative:

$$E(\theta) \propto \text{Ai}[-x^{2/3}\Delta/h_p^{1/3}u(\Delta)] - i v(\Delta)x^{-1/3}\text{Ai}'[-x^{2/3}\Delta/h_p^{1/3}u(\Delta)], \quad (9)$$

with

$$\begin{aligned} u(\Delta) &= 1 + u_1\Delta + u_2\Delta^2 + \dots, \\ v(\Delta) &= v_0 + v_1\Delta + v_2\Delta^2 + \dots \end{aligned} \quad (10)$$

The CAM coefficients u_j and v_j are functions of refractive index and p . The v_j coefficients also depend on x for $j \geq 1$. The function $v(\Delta)$ acts as a background for the rainbow so that the intensity at the supernumerary intensity minima for TE scattering is positive rather than zero. The function $u(\Delta)$ produces an increased stretching of the supernumerary pattern with respect to the predictions of Airy theory as Δ increases. Both of these behaviors are evident in Fig. 3 of Ref. 20 where the $p = 2$ rainbow in Mie theory is compared with Airy theory. The first few CAM coefficients are calculated in Ref. 17 for both the TE and the TM polarizations for electromagnetic scattering by a sphere with $n = 1.33$ and in Eq. (4.43) of Ref. 19 for scattering of scalar waves. CAM theory with $u(\Delta) = 1$ and $v(\Delta) = 0$ reduces to Airy theory. In Section 4 I use CAM theory, including only the coefficients u_1 and v_0 . I call this the lowest-order CAM correction to Airy theory.

3. Supernumerary Spacing Parameter h_p in Ray Theory

For an elliptical-cross-section cylinder, one would ideally like to obtain an analytical expression for $h_p(\xi)$ for the $p = 2$ and $p = 3$ rainbows to first order in the cylinder ellipticity ϵ using ray theory, analogous to the Möbius result of Eqs. (6) for the rainbow angle $\theta_p^R(\xi)$. I did not pursue this calculation, however, for the following reason. The Möbius theory derivation of the rainbow angle locates the position of the Descartes ray in the exit plane of the cylinder to first order in the eccentricity ϵ . Können⁴ has extended Möbius theory for the $p = 3$ rainbow to locate the position of the Descartes ray in the exit plane to order ϵ^2 . But the determination of the shape of the exiting cubic phase front requires that the position of the Descartes ray in the exit plane be known to at least order ϵ^4 .

Instead, I determined $h_2(\xi)$ and $h_3(\xi)$ using the following numerical ray-tracing procedure. For each cylinder rotation angle ξ , I considered a family of parallel rays normally incident on the cylinder at intervals of $\Delta\phi_i = 0.0002^\circ$. Each ray was propagated through the cylinder by use of the equations derived in Ref. 8, and the minimum deflection ray (i.e., the rainbow ray) was identified. Next, another family of seven rays centered on the rainbow ray at intervals of $\Delta\phi_i = 0.05^\circ$ was propagated through the cylinder and the length of the optical path L of each ray from the cylinder's entrance plane (defined as being normal to the incoming rainbow ray and tangent to the ellipse) to its exit plane (defined as being normal to the outgoing rainbow ray and tangent to the ellipse) was computed. The length of the optical

path of the four rays for which $\phi_i \geq \phi_i^R$ was then fitted to the form

$$L/a = L(R)/a + A_+(X/a)^2 - h_+(X/a)^3/3 + B_+(X/a)^4, \quad (11)$$

where X is the distance along the exit plane from the rainbow ray to the ray in question; $L(R)$ is the optical path length of the rainbow ray; and A_+ , h_+ , and B_+ are constants determined by the fitting procedure. Similarly, the length of the optical path of the four rays for which $\phi_i \leq \phi_i^R$ was fitted to the form

$$L/a = L(R)/a + A_-(X/a)^2 + h_-(X/a)^3/3 + B_-(X/a)^4. \quad (12)$$

Because the rainbow ray was located to within $\Delta\phi_i \leq 0.0002^\circ$, the values of A_+ and A_- were found to be exceedingly close to zero as required by Eq. (2). Because the $\Delta\phi_i$ interval for the set of seven rays near the rainbow ray was small, a correspondingly small portion of the exiting phase front was sampled. As a result, h_+ and h_- were found to be nearly identical, and B_+ and B_- differed only minimally. The super-numerary scaling parameter h_p was then taken to be

$$h_p(\xi) = (h_+ + h_-)/2. \quad (13)$$

After this procedure was carried out for $0^\circ \leq \xi \leq 360^\circ$ in intervals of $\Delta\xi = 1^\circ$ for both the $p = 2$ and $p = 3$ rainbows, the resulting functions $h_2(\xi)$ and $h_3(\xi)$ were decomposed into the Fourier series

$$h_2(\xi) = e_0 + \sum_{m=1}^{\infty} e_m \cos(m\xi) + \sum_{m=1}^{\infty} f_m \sin(m\xi), \quad (14)$$

$$h_3(\xi) = g_0 + \sum_{m=1}^{\infty} g_m \cos(m\xi) + \sum_{m=1}^{\infty} j_m \sin(m\xi). \quad (15)$$

An analogous Fourier-series decomposition of the rainbow angles $\theta_2^R(\xi)$ and $\theta_3^R(\xi)$ was performed in Ref. 8. The Fourier decomposition of $h_2(\xi)$ and $h_3(\xi)$ provides a useful diagnostic for our choice of $\Delta\phi_i$ for the family of incident rays used to locate the rainbow ray and for the family of seven rays used to determine the shape of the exiting phase front. If the cylinder has a circular cross section, all the Fourier coefficients for $m \neq 0$ should vanish. Similarly, if the cylinder has an elliptical cross section, all the odd- m Fourier coefficients should vanish because of the 180° rotational symmetry of the ellipse. I found that if $\Delta\phi_i$ were much larger than 0.0002° for the initial family of incident rays, the rainbow ray was not located with sufficient precision in the exit plane. As a result, the fitted values of A_+ and A_- were no longer near zero, and the odd- m Fourier coefficients exhibited a substantial amount of noise. Similarly, if $\Delta\phi_i$ was much larger than 0.05° for the family of seven rays in the vicinity of the rainbow ray, a larger than desired portion of the exiting wave front was sampled. The fitting of this larger portion to only order $(X/a)^4$ did not allow $h_2(\xi)$ and $h_3(\xi)$ to be determined with sufficient precision because higher powers of

X/a also contribute substantially to the shape of the phase front over the larger interval. As a consequence, the odd- m Fourier coefficients again exhibited a substantial amount of noise.

The Fourier coefficients of $h_2(\xi)$ and $h_3(\xi)$ exhibited the following scaling behavior in ϵ . For a circular-cross-section cylinder, e_0 and g_0 are equal to h_2 and h_3 , respectively. For an elliptical-cross-section cylinder, both e_0 and g_0 differ from the circular-cross-section value of h_2 and h_3 of Eq. (3) by a term approximately linear in ϵ , and the even- m coefficients e_m , f_m , g_m , and j_m for $m \geq 2$ are approximately proportional to $\epsilon^{m/2}$, with the constants of proportionality depending on refractive index. An identical scaling behavior in ϵ was previously found⁸ for the Fourier coefficients of the rainbow angle $\theta_2^R(\xi)$ and $\theta_3^R(\xi)$. As to refractive-index dependence, the $m \leq 2$ portion of Eq. (14) for the $p = 2$ rainbow was found to be well fit by the expression

$$h_2(\xi) \approx e_0 + 19\epsilon[\sin(\phi_i^D)]^{3/4}[\cos(\phi_i^D)]^{-10/3} \cos(2\xi + \Phi), \quad (16)$$

where the difference between e_0 and h_2 of Eq. (3) is approximately an order of magnitude smaller than the amplitude of the $\cos(2\xi + \Phi)$ term in approximation (16), and

$$\Phi \approx (250^\circ)n - 285^\circ. \quad (17)$$

For $1.25 \leq n \leq 1.7$, the amplitude of the $\cos(2\xi + \Phi)$ term agrees with the results of the numerical ray-tracing calculation to better than 2%, and Φ agrees to within approximately 5° . I attribute no fundamental theoretical basis to approximations (16) and (17). They are solely the result of our motivation to obtain an approximate equation for $h_2(\xi)$ to first order in ϵ that resembles the Möbius theory result of Eqs. (6) for $\theta_2^R(\xi)$.

In Fig. 2 I plot $h_2(\xi) - h_{\text{ave}}$ as a function of ξ for an elliptical-cross-section cylinder with $\epsilon = 0.0001$ and $n = 1.333$, where h_{ave} is the average value of $h_2(\xi)$ over the interval $0^\circ \leq \xi \leq 180^\circ$. The behavior of $h_2(\xi) - h_{\text{ave}}$ for $180^\circ \leq \xi \leq 360^\circ$ is identical to that for $0^\circ \leq \xi \leq 180^\circ$ because of the 180° rotational symmetry of the elliptical cross section. The small jaggedness of the solid curve is due to the approximate nature of our numerical ray-tracing procedure. The dashed curve is the $m = 2$ Fourier component of $h_2(\xi) - h_{\text{ave}}$ that, according to the scaling behavior in ϵ mentioned in the above paragraph, is the dominant contribution in the Möbius regime, $\epsilon \ll 1$. The results for $h_3(\xi)$ were similar to those of Fig. 2 and, together with Fig. 2, illustrate that to determine the shape of the exiting phase front, and from it $h_p(\xi)$, by numerical ray tracing requires the rainbow ray to be located accurately in the exit plane and the exiting phase front to be sampled over a sufficiently small interval about the rainbow ray.

Fig. 2. Deviation of the $p = 2$ supernumerary spacing parameter from its average value, $h_2(\xi) - h_{ave}$, as a function of the cylinder rotation angle ξ for an elliptical-cross-section cylinder with $n = 1.333$, $x = 1000.0$, and $\epsilon = 0.0001$. The solid curve is the prediction of ray theory, the dashed curve is the $m = 2$ Fourier component of the ray theory result, and the open and solid circles are the prediction of first-order perturbation theory along with the modeling of the supernumerary intensity minima with Airy theory and CAM theory, respectively.

4. Supernumerary Spacing Parameter h_p in First-Order Perturbation Rayleigh–Debye Theory

The computation of light scattering by a nonspherical particle or by a cylinder with a noncircular cross section is best handled for moderate size parameters and for moderate eccentricities by the so-called T -matrix method.^{21,22} This approach, however, is not appropriate for a detailed quantitative study of rainbows because the cylinder size parameter that is required, $x \sim 1000$ or greater, far exceeds the size parameter range for which T -matrix computations are numerically stable.²³ As an alternative, we employ the first-order perturbation theory extension to the exact Rayleigh–Debye partial-wave scattering equations. In Refs. 24–27, the equations for wave scattering by a nonspherical particle were derived to first order in the particle-shape perturbation, and in Ref. 28 the form of the equations was simplified substantially. Similarly, in Ref. 29 the equations for wave scattering by a cylinder with a noncircular cross section were derived to first order in the cylinder-shape perturbation. These first-order perturbation-theory equations can again be simplified as follows.

The surface of the cylinder is taken to be

$$r(\theta) = a[1 + \delta f(\theta)] \quad (18)$$

in polar coordinates where $a\delta f(\theta)$ is the surface-shape perturbation from a circle, normalized so that

$$f(\theta) \sim O(1). \quad (19)$$

First-order perturbation theory is an accurate approximation to exact wave scattering theory when³⁰

$$\delta \ll 1, \quad (20)$$

$$ka\delta \ll 2\pi. \quad (21)$$

Inequality (21) is usually the more restrictive of the two and limits the regime of applicability of first-order perturbation theory for large-radius cylinders to small perturbations.

The geometry of the plane wave used here is different from that for the ray theory in Section 3. The electric field strength of the plane wave is E_0 and it is normally incident on the cylinder with its propagation direction parallel to the x axis and its electric field polarized either parallel to the z axis (TE polarization) or parallel to the y axis (TM polarization). We define the polarization-dependent quantities α and β by

$$\alpha = \begin{cases} n & \text{for TE,} \\ 1 & \text{for TM,} \end{cases} \quad \beta = \begin{cases} 1 & \text{for TE} \\ n & \text{for TM} \end{cases} \quad (22)$$

and let

$$y = nka, \quad (23)$$

$$E_l = \alpha H_l^{(1)'}(x) J_l(y) - \beta H_l^{(1)}(x) J_l'(y), \quad (24)$$

where J_l are Bessel functions and $H_l^{(1)}$ and $H_l^{(2)}$ are Hankel functions of the first and second kind. The time dependence of the plane wave is $\exp(-i\omega t)$ so that $H_l^{(1)}$ and $H_l^{(2)}$ describe radially outgoing and incoming cylindrical multipole waves, respectively.³¹ By matching the tangential components of the incident, scattered, and interior electric and magnetic fields E_z , B_z , $E_\theta + \delta(df/d\theta)E_r$, and $B_\theta + \delta(df/d\theta)B_r$ at the surface of the cylinder to first order in δ , we obtain the far-zone scattered intensity:

$$I(\theta) = (2I_0/\pi kr) |F(\theta)|^2, \quad (25)$$

where the scattering amplitude $F(\theta)$ is

$$F(\theta) = \sum_{l=-\infty}^{\infty} b_l \exp(il\theta) \quad (26)$$

for a TE incident plane wave and

$$F(\theta) = \sum_{l=-\infty}^{\infty} a_l \exp(il\theta) \quad (27)$$

for a TM incident plane wave. The partial-wave scattering amplitudes a_l and b_l for a circular cylinder are given by

$$\left. \begin{matrix} a_l^{(0)} \\ b_l^{(0)} \end{matrix} \right\} = [\alpha J_l'(x) J_l(y) - \beta J_l(x) J_l'(y)] / E_l, \quad (28)$$

where the superscript (0) henceforth indicates a circular cross section. For completeness, the TE and TM partial-wave interior amplitudes are

$$\left. \begin{matrix} c_l^{(0)} \\ d_l^{(0)} \end{matrix} \right\} = 2i/(n^2 \pi x E_l). \quad (29)$$

When the surface of the cylinder is noncircular, the partial wave-scattering and interior amplitudes become, after a large amount of algebra,

$$\begin{aligned}
a_l &= a_l^{(0)} + [2i(n^2 - 1)\delta/\pi] \\
&\times \sum_{l'=-\infty}^{\infty} i^{l'-l} a_{l',l}^{(1)} + O(\delta^2), \\
b_l &= b_l^{(0)} + [2i(n^2 - 1)\delta/\pi] \\
&\times \sum_{l'=-\infty}^{\infty} i^{l'-l} b_{l',l}^{(1)} + O(\delta^2), \\
c_l &= c_l^{(0)} - [2i(n^2 - 1)\delta/(n^2\pi)] \\
&\times \sum_{l'=-\infty}^{\infty} i^{l'-l} c_{l',l}^{(1)} + O(\delta^2), \\
d_l &= d_l^{(0)} - [2i(n^2 - 1)\delta/(n^2\pi)] \\
&\times \sum_{l'=-\infty}^{\infty} i^{l'-l} d_{l',l}^{(1)} + O(\delta^2), \quad (30)
\end{aligned}$$

where the superscript (1) henceforth indicates a first-order correction in the perturbation strength δ . The first-order corrections are

$$\begin{aligned}
a_{l',l}^{(1)} &= \{[(l'^2/x^2)J_{l'}(y)J_l(y) \\
&+ J_{l'}'(y)J_l'(y)]I_{l',l} \\
&- (il'/x^2)J_{l'}(y)J_l(y)I_{l',l}'\}/(E_l E_l), \\
b_{l',l}^{(1)} &= [J_{l'}(y)J_l(y)I_{l',l}]/(E_l E_l), \\
c_{l',l}^{(1)} &= \{[(l'^2/xy)J_{l'}(y)H_l^{(1)}(x) \\
&+ J_{l'}'(y)H_l^{(1)'}(x)]I_{l',l} - (il'/xy)J_{l'}(y) \\
&\times H_l^{(1)}(x)I_{l',l}'\}/(E_l E_l), \\
d_{l',l}^{(1)} &= [J_{l'}(y)H_l^{(1)}(x)I_{l',l}]/(E_l E_l), \quad (31)
\end{aligned}$$

where

$$\begin{aligned}
I_{l',l} &= (1/2\pi) \int_0^{2\pi} d\theta f(\theta) \exp[i(l' - l)\theta], \\
I_{l',l}' &= (1/2\pi) \int_0^{2\pi} d\theta (df/d\theta) \exp[i(l' - l)\theta]. \quad (32)
\end{aligned}$$

The Fourier-series decomposition of the surface perturbation $f(\theta)$ is

$$\begin{aligned}
f(\theta) &= A_0 + \sum_{q=1}^{\infty} A_q \cos[q(\theta - \xi)] \\
&+ \sum_{q=1}^{\infty} B_q \sin[q(\theta - \xi)], \quad (33)
\end{aligned}$$

where again ξ is the rotation angle of the cylinder's x' axis with respect to the laboratory x axis. Inserting Eq. (33) into Eqs. (32), we obtain

$$\begin{aligned}
I_{l',l} &= A_0 \delta_{l',l} + (1/2) \sum_{q=1}^{\infty} (A_q - iB_q) \exp(-iq\xi) \delta_{l',l-q} \\
&+ (1/2) \sum_{q=1}^{\infty} (A_q + iB_q) \exp(iq\xi) \delta_{l',l+q}, \\
I_{l',l}' &= (1/2) \sum_{q=1}^{\infty} iq(A_q - iB_q) \exp(-iq\xi) \delta_{l',l-q} \\
&- (1/2) \sum_{q=1}^{\infty} iq(A_q + iB_q) \exp(iq\xi) \delta_{l',l+q}, \quad (34)
\end{aligned}$$

where $\delta_{l',l}$ is the Kronecker delta symbol. Equations (34) illustrate that two different incident partial waves, $l' = l + q$ and $l' = l - q$, are coupled to each scattered partial wave l by the Fourier component q of the surface perturbation. This is a simpler situation than for scattering by a perturbed sphere²⁸ where all the incident partial waves l' in the interval $l - q \leq l' \leq l + q$ are coupled with various strengths to each scattered partial wave l by the Fourier component q of the surface perturbation.

Before we can apply first-order perturbation theory to the p rainbow of an elliptical-cross-section cylinder illuminated by a TE-polarized plane wave, we must first perform a Debye-series decomposition of the partial-wave scattering amplitudes and retain only the p term in Eq. (26) so as to rid the scattered intensity of the influence of other scattering processes such as reflection. We denote the cylinder interior as region 1 and the exterior as region 2. We consider a radially incoming TE or TM cylindrical multipole wave in region 2 with the partial-wave number l' . When it encounters the cylinder surface, a portion $\mathcal{F}_{l',l}^{21}$ of the wave amplitude is transmitted into the cylinder as a different cylindrical multipole wave with the partial-wave number l . The remaining portion of the wave amplitude $\mathcal{R}_{l',l}^{22}$ is reflected from the surface as a radially outgoing cylindrical multipole wave with the partial-wave number l . Following the procedure in the appendix of Ref. 20 applied to the perturbed cylinder geometry, we obtain

$$\begin{aligned}
\mathcal{F}_{l',l}^{21} &= T_l^{21} \delta_{l',l} - [4i(n^2 - 1)\delta/\pi] l'^{-l} T_{l',l}^{21} + O(\delta^2), \\
\mathcal{R}_{l',l}^{22} &= R_l^{22} \delta_{l',l} - [4i(n^2 - 1)\delta/\pi] l'^{-l} R_{l',l}^{22} + O(\delta^2), \quad (35)
\end{aligned}$$

where

$$\begin{aligned}
T_l^{21} &= 4i/(n\pi x D_l), \\
R_l^{22} &= -[\alpha H_l^{(2)'}(x) H_l^{(2)}(y) - \beta H_l^{(2)}(x) H_l^{(2)'}(y)]/D_l, \quad (36)
\end{aligned}$$

with

$$D_l = \alpha H_l^{(1)'}(x) H_l^{(2)}(y) - \beta H_l^{(1)}(x) H_l^{(2)'}(y). \quad (37)$$

Equations (36) and (37) describe the conservation of partial-wave number for transmission and reflection

of a cylindrical multipole wave by a circular-cross-section cylinder and correspond to conservation of angular momentum in analogous expressions for quantum mechanical scattering. In addition, the quantities

$$\begin{aligned} T_{l',l}^{21} &= \{[(l'^2/xy)H_{l'}^{(2)}(y)H_l^{(1)}(x) \\ &\quad + H_{l'}^{(2)'}(y)H_l^{(1)'}(x)]I_{l',l} - (il'/xy) \\ &\quad \times H_{l'}^{(2)}(y)H_l^{(1)}(x)I_{l',l}'\}/(nD_l D_l), \\ R_{l',l}^{22} &= \{[(l'^2/x^2)H_{l'}^{(2)}(y)H_l^{(2)}(y) \\ &\quad + H_{l'}^{(2)'}(y)H_l^{(2)'}(y)]I_{l',l} - (il'/x^2)H_{l'}^{(2)}(y) \\ &\quad \times H_l^{(2)}(y)I_{l',l}'\}/(D_l D_l) \end{aligned} \quad (38)$$

for the TM polarization and

$$\begin{aligned} T_{l',l}^{21} &= [H_{l'}^{(2)}(y)H_l^{(1)}(x)I_{l',l}]/(nD_l D_l), \\ R_{l',l}^{22} &= [H_{l'}^{(2)}(y)H_l^{(2)}(y)I_{l',l}]/(D_l D_l) \end{aligned} \quad (39)$$

for the TE polarization describe the coupling between the partial waves l' and l of the incident and scattered light induced by the noncircular character of the surface shape. In quantum mechanical scattering, analogous expressions describe the target taking up some of the angular momentum of the projectile during the scattering process.

Similarly, when a TE or TM radially outgoing cylindrical multipole wave in region 1 with the partial-wave number l' encounters the cylinder surface, a portion $\mathcal{F}_{l',l}^{12}$ of the wave amplitude is transmitted to the exterior as a different cylindrical multipole wave with the partial-wave number l , and the remaining portion of the wave amplitude $\mathcal{R}_{l',l}^{11}$ is reflected back inside the cylinder as a different cylindrical multipole wave with the partial-wave number l . Again, following the procedure in the appendix of Ref. 20 applied to the perturbed cylinder geometry, we obtain

$$\begin{aligned} \mathcal{F}_{l',l}^{12} &= T_l^{12}\delta_{l',l} - [4i(n^2 - 1)\delta/\pi]i^{l'-l}T_{l',l}^{12} + O(\delta^2), \\ \mathcal{R}_{l',l}^{11} &= R_l^{11}\delta_{l',l} - [4i(n^2 - 1)\delta/\pi]i^{l'-l}R_{l',l}^{11} + O(\delta^2), \end{aligned} \quad (40)$$

where

$$\begin{aligned} T_l^{12} &= 4in/(\pi x D_l), \\ R_l^{11} &= -[\alpha H_l^{(1)'}(x)H_l^{(1)}(y) - \beta H_l^{(1)}(x)H_l^{(1)'}(y)]/D_l, \end{aligned} \quad (41)$$

$$\begin{aligned} T_{l',l}^{12} &= \{[(l'^2/xy)H_{l'}^{(1)}(x)H_l^{(2)}(y) \\ &\quad + H_{l'}^{(1)'}(x)H_l^{(2)'}(y)]I_{l',l} - (il'/xy)H_{l'}^{(1)}(x) \\ &\quad \times H_l^{(2)}(y)I_{l',l}'\}[n/(D_l D_l)], \\ R_{l',l}^{11} &= \{[(l'^2/y^2)H_{l'}^{(1)}(x)H_l^{(1)}(x) \\ &\quad + H_{l'}^{(1)'}(x)H_l^{(1)'}(x)]I_{l',l} - (il'/y^2)H_{l'}^{(1)}(x) \\ &\quad \times H_l^{(1)}(x)I_{l',l}'\}/(D_l D_l) \end{aligned} \quad (42)$$

for the TM polarization and

$$\begin{aligned} T_{l',l}^{12} &= [H_{l'}^{(1)}(x)H_l^{(2)}(y)I_{l',l}]/[n/(D_l D_l)], \\ R_{l',l}^{11} &= [H_{l'}^{(1)}(x)H_l^{(1)}(x)I_{l',l}]/(D_l D_l) \end{aligned} \quad (43)$$

for the TE polarization.

Now that we have obtained the partial-wave transmission and reflection Fresnel coefficients to first order in δ , again after a great amount of algebra, one finds that the partial-wave scattering amplitudes of Eqs. (28) and (31) can be written in terms of these Fresnel coefficients as

$$\begin{aligned} \left. \begin{matrix} a_{l',l}^{(0)} \\ b_{l',l}^{(0)} \end{matrix} \right\} &= (1/2)[1 - R_l^{22} - T_l^{21}(1 - R_l^{11})^{-1}T_l^{12}] \\ &= (1/2)[1 - R_l^{22} - \sum_{p=1}^{\infty} T_l^{21}(R_l^{11})^{p-1}T_l^{12}], \end{aligned} \quad (44)$$

$$\begin{aligned} \left. \begin{matrix} a_{l',l}^{(1)} \\ b_{l',l}^{(1)} \end{matrix} \right\} &= R_{l',l}^{22} + T_{l',l}^{21}(1 - R_l^{11})^{-1}T_l^{12} \\ &\quad + T_{l',l}^{21}(1 - R_{l',l}^{11})^{-1}R_{l',l}^{11}(1 - R_l^{11})^{-1}T_l^{12} \\ &\quad + T_{l',l}^{21}(1 - R_{l',l}^{11})^{-1}T_{l',l}^{12}, \\ &= R_{l',l}^{22} + \sum_{p=1}^{\infty} T_{l',l}^{21}(R_l^{11})^{p-1}T_l^{12} \\ &\quad + \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} T_{l',l}^{21}(R_{l',l}^{11})^s R_{l',l}^{11}(R_l^{11})^t T_l^{12} \\ &\quad + \sum_{p=1}^{\infty} T_{l',l}^{21}(R_{l',l}^{11})^{p-1}T_{l',l}^{12}, \end{aligned} \quad (45)$$

where the TE version of the coefficients is to be used in the decomposition of $b_{l',l}^{(0)}$ and $b_{l',l}^{(1)}$ and the TM version is to be used in the decomposition of $a_{l',l}^{(0)}$ and $a_{l',l}^{(1)}$. Equations (44) are the original series expansion of Debye,¹³ and Eqs. (45) are its extension to first order in δ for a cylinder with a noncircular cross section. The significance of Eqs. (45) is that it clearly illustrates the physical mechanism by which the noncircular component of the surface shape induces a coupling between the incoming and outgoing partial waves l' and l . The coupling is produced at any of the interactions of the cylindrical multipole waves with the surface, i.e., at the external reflection, at the initial transmission into the cylinder, at any of the internal reflections, and at the final transmission out of the cylinder. Because the partial-wave scattering amplitudes are calculated to only first order in δ , only one change of the partial-wave number is permitted in each term of Eqs. (45). The extension of perturbation theory to $O(\delta^n)$ would include up to n changes of the partial-wave number in each term.

For completeness, the Debye-series decomposition

of the partial-wave interior amplitudes of Eqs. (29) and (31) is found after much algebra to be

$$\left. \begin{aligned} c_l^{(0)} \\ d_l^{(0)} \end{aligned} \right\} &= (1/n) T_l^{21} (1 - R_l^{11})^{-1} \\ &= (1/n) \sum_{p=1}^{\infty} T_l^{21} (R_l^{11})^{p-1}, \quad (46)$$

$$\left. \begin{aligned} c_{l',l}^{(1)} \\ d_{l',l}^{(1)} \end{aligned} \right\} &= 2n T_{l',l}^{21} (1 - R_l^{11})^{-1} + 2n T_{l'}^{21} \\ &\quad \times (1 - R_{l'}^{11})^{-1} R_{l',l}^{11} (1 - R_l^{11})^{-1} \\ &= 2n \sum_{p=1}^{\infty} T_{l',l}^{21} (R_l^{11})^{p-1} \\ &\quad + 2n \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} T_{l'}^{21} (R_{l'}^{11})^s R_{l',l}^{11} (R_l^{11})^t, \quad (47)$$

where again the TE version of the Fresnel coefficients is to be used in the decomposition of $d_l^{(0)}$ and $d_{l',l}^{(1)}$, and the TM version is to be used in the decomposition of $c_l^{(0)}$ and $c_{l',l}^{(1)}$. The partial-wave coupling induced by the noncircular component of the surface shape is implicit in the T -matrix formalism^{32,33} and provides the physical explanation for the optimal positioning of a tightly focused laser beam on a microparticle when morphology-dependent resonances are excited.^{28,34,35}

Equations (45) are not especially convenient for numerical computations because they contain Bessel functions of negative order. If we assume that the cylinder surface shape is given in Fourier-series form and we substitute Eqs. (34) into Eqs. (31, 38), and (42), the scattering amplitude $F(\theta)$ for TE scattering can be simplified to

$$\begin{aligned} F(\theta) &= F^{(0)}(\theta) + [2i(n^2 - 1)\delta/\pi] A_0 Q_0(\theta) \\ &\quad + [2i(n^2 - 1)\delta/\pi] \sum_{q=1}^{\infty} i^q Q_1(q, \theta) \\ &\quad \times [A_q \cos(q\xi) - B_q \sin(q\xi)] \\ &\quad - [2i(n^2 - 1)\delta/\pi] \sum_{q=1}^{\infty} i^q Q_2(q, \theta) \\ &\quad \times [A_q \sin(q\xi) + B_q \cos(q\xi)]. \quad (48) \end{aligned}$$

The first term of Eq. (48),

$$F^{(0)}(\theta) = b_0^{(0)} + 2 \sum_{l=1}^{\infty} b_l^{(0)} \cos(l\theta), \quad (49)$$

is the scattering amplitude for a circular-cross-section cylinder in Rayleigh-Debye theory, and the expressions

$$\begin{aligned} Q_0(\theta) &= U_0 + 2 \sum_{l=1}^{\infty} U_l \cos(l\theta), \\ Q_1(q, \theta) &= \sum_{l=0}^{\infty} V_{l,q}^+ \{\cos(l\theta) + (-1)^q \cos[(l+q)\theta]\} \\ &\quad + \sum_{l=1}^{q-1} (-1)^l V_{l,q}^- \cos(l\theta), \end{aligned}$$

$$\begin{aligned} Q_2(q, \theta) &= \sum_{l=0}^{\infty} V_{l,q}^+ \{\sin(l\theta) - (-1)^q \sin[(l+q)\theta]\} \\ &\quad - \sum_{l=1}^{q-1} (-1)^l V_{l,q}^- \sin(l\theta) \end{aligned} \quad (50)$$

linear in δ in Eq. (48) with

$$\begin{aligned} U_l &= J_l^2(y)/E_l^2, \\ V_{l,q}^+ &= J_l(y) J_{q+l}(y)/(E_l E_{q+l}), \\ V_{l,q}^- &= J_l(y) J_{q-l}(y)/(E_l E_{q-l}) \end{aligned} \quad (51)$$

are the contribution to the scattering amplitude provided by the various Fourier components of the surface perturbation. The term proportional to A_0 corresponds to a perturbation in the form of a small increase of radius, and the terms proportional to A_q and B_q for $q \geq 1$ correspond to a shape perturbation. The one-internal-reflection portion of Eqs. (51) is

$$\begin{aligned} U_l &= T_{l,l}^{21} R_l^{11} T_l^{12} + T_l^{21} R_{l,l}^{11} T_l^{12} \\ &\quad + T_l^{21} R_l^{11} T_{l,l}^{12}, \\ V_{l,q}^+ &= T_{q+l,l}^{21} R_l^{11} T_l^{12} + T_{q+l}^{21} R_{q+l,l}^{11} T_l^{12} \\ &\quad + T_{q+l}^{21} R_{q+l}^{11} T_{q+l,l}^{12}, \\ V_{l,q}^- &= T_{q-l,l}^{21} R_l^{11} T_l^{12} + T_{q-l}^{21} R_{q-l,l}^{11} T_l^{12} \\ &\quad + T_{q-l}^{21} R_{q-l}^{11} T_{q-l,l}^{12}. \end{aligned} \quad (52)$$

The first, second, and third terms of $V_{l,q}^+$ and $V_{l,q}^-$ correspond to the change in the partial-wave number occurring at the initial refraction into the cylinder, at the internal reflection, and at the final refraction out of the cylinder, respectively. Because U_l corresponds only to an increase of radius, the partial-wave number is conserved at the initial refraction, at the internal reflection, and at the final refraction.

We now specify the cylinder-shape Fourier coefficients A_q and B_q . When the cylinder rotation angle is $\xi = 0^\circ$, the ellipse of Eq. (4) can be expressed in polar coordinates as

$$r(\theta) = a(1 + \epsilon)[1 + (2\epsilon + \epsilon^2)\cos^2(\theta)]^{-1/2}, \quad (53)$$

and the surface perturbation is

$$f(\theta) = r(\theta) - a. \quad (54)$$

The first few Fourier coefficients of Eq. (54) are

$$\begin{aligned} A_0 &= \epsilon/2 - 3\epsilon^2/16 + 3\epsilon^3/32 + \dots, \\ A_2 &= -\epsilon/2 + 5\epsilon^3/64 + \dots, \\ A_4 &= 3\epsilon^2/16 - 3\epsilon^3/32 + \dots, \\ A_6 &= -5\epsilon^3/64 + \dots, \\ B_2 &= B_4 = B_6 = 0. \end{aligned} \quad (55)$$

The leading term of the coefficients A_q for $q \geq 2$ is proportional to $\epsilon^{q/2}$. Because first-order perturbation theory is accurate to only order ϵ , the association

$$\delta = \epsilon/2, \quad A_0 = 1, \quad A_2 = -1 \quad (56)$$

describes the ellipse in first-order perturbation theory in a self-consistent way. At this level of approximation, an ellipse and a quadrupole deformation of a circle are equivalent. Similarly, a perturbation in the form of a small increase of radius is

$$\delta = \epsilon/2, \quad A_0 = 1. \quad (57)$$

In the numerical studies of these equations, I first performed the following test as a check of my analytical perturbation-theory formulas, the computer program that implements them, and to explore the limit of applicability of first-order perturbation theory. I computed the $p = 2$ portion of the scattered intensity for a circular-cross-section cylinder having $n = 1.333$ and $x = 1000.0$ using exact Rayleigh–Debye theory. The angle of the intensity minimum between the first and second supernumerary maxima $\theta_a^{\min}$ and the angle of the intensity minimum between the second and third supernumerary maxima $\theta_b^{\min}$, both measured in degrees, were determined to within $\pm 0.000\,005^\circ$. I then assumed that these angles correspond to the first and second zeros of the Airy integral,³⁶ $\text{Ai}(-2.338\,107)$ and $\text{Ai}(-4.087\,949)$, respectively. I determined the rainbow angle θ_2^R and the supernumerary spacing parameter h_2 from Eqs. (7) and (8) by

$$\begin{aligned} 2.338\,107 &= (\pi/180)x^{2/3}(\theta_a^{\min} - \theta_2^R)/h_2^{1/3}, \\ 4.087\,949 &= (\pi/180)x^{2/3}(\theta_b^{\min} - \theta_2^R)/h_2^{1/3}, \end{aligned} \quad (58)$$

or

$$\begin{aligned} \theta_2^R &= (2.336\,182)\theta_a^{\min} - (1.336\,182)\theta_b^{\min}, \\ h_2 &= (0.992\,283)(10^{-6})x^2(\theta_b^{\min} - \theta_a^{\min})^3. \end{aligned} \quad (59)$$

Using Eqs. (59), I found $\theta_2^R = 137.945\,68^\circ$ and $h_2 = 4.643\,221$, which differ only slightly from the ray theory values $\theta_2^R = 137.921\,89^\circ$ and $h_2 = 4.899\,194$ of Eqs. (1) and (3). The exact wave theory results differ slightly from the ray theory results because we are not in the $x \rightarrow \infty$ limit and because Rayleigh–Debye intensity is not perfectly fit by the square of an Airy integral. In particular, curved-surface Fresnel transmission and reflection coefficients rather than flat-surface Fresnel coefficients should be used in Eq. (7), the angle dependence of the Fresnel coefficients should be included, and terms of higher order than X^3/a^2 in the phase-front shape of Eq. (2) should be included.^{11,17}

These improvements are taken into account in CAM theory; when we model the intensity minima of the $p = 2$ rainbow using the lowest-order CAM correction to Airy theory, we obtain an improved agreement between first-order perturbation theory and ray theory. In Ref. 17, Khare and Nussenzveig obtained $u_1 = 0.202$ and $v_0 = 0.473$ for electromagnetic scattering of a TE-polarized plane wave by a sphere with $n = 1.33$. We use these values of u_1 and v_0 for scattering by both a circular cylinder, which should be appropriate, and by an elliptical-cross-section cylinder, which may not be entirely appropriate. The

presence of the Ai' term in relation (9) is of no consequence in the determination of $\theta_a^{\min}$ and $\theta_b^{\min}$ because the relative maxima and minima of Eq. (7) and the intensity corresponding to relation (9) are identical. But when the progressively increased stretching of the exact Rayleigh–Debye wave theory with respect to Airy theory is taken into account by replacing $x^{2/3}\Delta/h_p^{1/3}$ in Eqs. (58) with $(x^{2/3}\Delta/h_p^{1/3})(1 + 0.202\Delta)$, we obtain $\theta_2^R = 137.914\,91^\circ$ and $h_2 = 4.954\,462$. This makes up approximately 71% and 78%, respectively, of the difference between ray theory and first-order perturbation theory with Airy modeling of the supernumerary intensity minima.

I then determined the limit to the validity of first-order perturbation theory as follows. I repeated the procedure of Eqs. (59) using Airy theory to model the supernumerary intensity minima for a set of slightly larger circular cylinders using two different methods. First, I used exact Rayleigh–Debye theory for $n = 1.333$ and $x = (1000.0)(1 + \delta)$ with $\delta \ll 1$. Second, for the same set of circular cylinders I used first-order perturbation theory with $n = 1.333$, $x = 1000.0$, and with the perturbation being an increase of radius as in Eqs. (57). The values of $\theta_a^{\min}$ and $\theta_b^{\min}$ that I obtained using first-order perturbation theory and exact Rayleigh–Debye theory agreed to within $\pm 0.000\,02^\circ$ for $\delta \leq 0.0001$, or for $1000.0 \leq x \leq 1000.1$, which is in agreement with inequality (21). As long as $\delta \leq 0.0001$, both minimum intensity angles in perturbation theory increased by the same amount with respect to their Rayleigh–Debye values. So the resulting value of h_2 from Eqs. (59) remained identical to the Rayleigh–Debye result. But for much larger values of δ , the two minimum intensity angles in perturbation theory increased by differing amounts, leading to differences between the perturbation-theory and Rayleigh–Debye values of h_2 . Thus I consider $\delta = 0.0001$ for $x = 1000.0$ to be a safe upper limit for the accuracy of first-order perturbation theory to determine h_2 for a circular-cross-section cylinder.

I also tried to determine h_2 from the first two supernumerary maxima of the perturbation-theory intensity. Again I associated these maxima with the first maximum and the first minimum of the Airy integral,³⁶ $\text{Ai}(-1.018\,793)$ and $\text{Ai}(-3.248\,198)$, respectively. Using this method, I found that the safe upper limit for first-order perturbation theory was now an order of magnitude smaller than it was when the intensity minima were used. This is due to the fact that first-order perturbation theory introduced a slowly varying background intensity superposed on the rainbow. This shifted the angles of the broad intensity maxima of Fig. 1 by larger differing amounts than it did for the sharp intensity minima. Thus I obtained the best performance from first-order perturbation theory when I analyzed the first two intensity minima of the supernumerary interference pattern.

I then used first-order perturbation theory with Eqs. (56) and (59) and with Airy modeling of the

Fig. 3. Deviation of the $p = 2$ rainbow angle from its average value, $\theta_2^R(\xi) - \theta_{ave}^R$, as a function of the cylinder rotation angle ξ for an elliptical-cross-section cylinder with $n = 1.333$, $x = 1000.0$, and $\epsilon = 0.0001$. The solid curve is the prediction of ray theory, and the open circles are the prediction of first-order perturbation theory along with the modeling of the supernumerary intensity minima with Airy theory.

intensity minima to obtain $\theta_2^R(\xi)$ and $h_2(\xi)$ for an elliptical-cross-section cylinder having $n = 1.333$, $x = 1000.0$, $\epsilon = 0.0001$, and $\delta = 0.000\ 05$ as a function of the cylinder rotation angle ξ . For the Airy theory modeling, we assumed that the scattered intensity in the vicinity of the rainbow is given by

$$I(\theta, \xi) \propto \text{Ai}^2\{-x_{ave}^{2/3}[\theta - \theta_p^R(\xi)]/h_p(\xi)^{1/3}\}, \quad (60)$$

where x_{ave} is the average value of the size parameter of the elliptical cross section. The resulting behavior of $h_2(\xi) - h_{ave}$ and $\theta_2^R(\xi) - \theta_{ave}^R$ is shown in Figs. 2 and 3, respectively. In obtaining these graphs I took into account the difference in the geometry of the plane wave in ray theory and Rayleigh–Debye theory. In ray theory the plane wave propagates in the $-y$ direction, it is incident on a cylinder with eccentricity ϵ , and the scattering angle θ is measured clockwise from the $-y$ axis; however, in first-order perturbation Rayleigh–Debye theory the incident plane wave propagates in the x direction so that it sees a cylinder of eccentricity $-\epsilon$, and the scattering angle is measured counterclockwise from the x axis so that the observer sees the angle $-\theta$. In Fig. 3, $\theta_2^R(\xi) - \theta_{ave}^R$ is compared with that obtained with the ray-tracing procedure of Section 2. The results are virtually identical and agree exactly with the Möbius theory prediction of Eqs. (6). The value of θ_{ave}^R for the elliptical-cross-section data of Fig. 3 is $137.945\ 39^\circ$ which compares favorably with $\theta_2^R = 137.945\ 68^\circ$ obtained from exact Rayleigh–Debye theory for a circular-cross-section cylinder with $x_{ave} = 1000.05$.

In Fig. 2 for $h_2(\xi) - h_{ave}$, the prediction of ray theory is the solid curve, the $m = 2$ Fourier component of the ray theory result is the dashed curve, and the results of first-order perturbation theory with Airy theory modeling of the rainbow intensity minima are the open circles. The comparison between the first-order perturbation theory and ray theory

results is not as close as it was for the rainbow angle. The amplitude of the oscillation of h_2 in perturbation theory is approximately 17% higher than that for ray theory, the oscillation is approximately 20° out of phase with ray theory, and the oscillation is not quite sinusoidal. But in another sense the results are encouraging because we are comparing $h_2(\xi)$ obtained by two entirely different methods. The ray theory results were obtained from the phase-front curvature in the near zone whereas I obtained the wave theory results from analyzing the far-zone rainbow. I obtained $h_{ave} = 4.643\ 972$ for the elliptical-cross-section perturbation-theory data of Fig. 2, in comparison with $h_2 = 4.643\ 188$ obtained using exact Rayleigh–Debye theory for a circular-cross-section cylinder having $x_{ave} = 1000.05$.

I verified that the difference between ray theory and first-order perturbation theory for $h_2(\xi) - h_{ave}$ in Fig. 2 was not due to the perturbation theory calculation being beyond its upper limit for accuracy for an elliptical surface shape by decreasing δ by an order of magnitude, repeating the calculation, and obtaining the same results. Similarly, I verified that the differences in Fig. 2 were not due to numerical inaccuracies in determining the intensity minima $\theta_a^{\min}$ and $\theta_b^{\min}$. An uncertainty $\delta\theta^{\min}$ in degrees in either $\theta_a^{\min}$ or $\theta_b^{\min}$ produces an uncertainty

$$\delta h_2 \sim (0.029\ 923)\delta\theta^{\min}(h_2 x_{ave})^{2/3} \quad (61)$$

in h_2 and an uncertainty

$$\delta\theta_2^R \sim \delta\theta^{\min} \quad (62)$$

in θ_2^R . When $\delta\theta^{\min} \approx 0.000\ 01^\circ$ and $x_{ave} = 1000.05$, we find that $\delta h_2 \sim 0.000\ 086$, which is approximately 2.2% of the amplitude of oscillation of $h_2(\xi)$ in Fig. 2, and $\delta\theta_2^R \sim 0.000\ 01^\circ$, which is approximately 0.076% of the amplitude of oscillation of $\theta_2^R(\xi)$ in Fig. 3. These uncertainties are negligible. Thus I conclude that the difference between the first-order perturbation theory with Airy theory modeling of the supernumerary intensity minima and the ray theory results for $h_2(\xi) - h_{ave}$ are due to the fact that the wave theory intensity is not perfectly fit by the square of an Airy integral and, in particular, that the intensity minima in wave theory do not correspond exactly to the zeros of the Airy integral of Eq. (7).

The sensitivity analysis described in the preceding paragraph also indicates that h_2 is approximately 29 times more sensitive (i.e., $2.2\%/0.076\% \approx 29$) to any inaccuracy in the determination $\theta_a^{\min}$ and $\theta_b^{\min}$ than is the position of the $p = 2$ rainbow. A consequence of this is that a sizable error in either the calculation or the measurement of h_p can occur without greatly affecting the calculated or measured value of the rainbow angle. Said a different way, with Eqs. (58) written as

$$\theta_2^R(\xi) = \theta_a^{\min}(\xi) - 2.338\ 107(180/\pi)h_2(\xi)^{1/3}/x_{ave}^{2/3}, \quad (63)$$

approximately 97% of the ξ dependence of θ_2^R is produced by the ξ dependence of $\theta_a^{\min}$ when $x_{\text{ave}} \approx 1000.0$ whereas only approximately 3% is produced by the ξ dependence of h_2 . This is why the comparison between first-order perturbation theory and ray theory for the rainbow angle was so good in Fig. 3. These differing sensitivities also have great significance for my experimental measurement of $h_2(\xi)$ and $h_3(\xi)$ to be reported separately.

We have already seen that use of the lowest-order CAM correction to Airy theory to model the intensity minima of the $p = 2$ rainbow of a circular-cross-section cylinder eliminated most of the difference in θ_2^R and h_2 between first-order perturbation theory and ray theory. We do not obtain the same degree of success, however, for an elliptical-cross-section cylinder. Use of CAM modeling of the rainbow intensity minima in Eqs. (58), the results for $\theta_2^R(\xi) - \theta_{\text{ave}}^R$ are identical to those of Airy rainbow modeling. Also, we obtain $\theta_{\text{ave}}^R = 137.914\ 62^\circ$ and $h_{\text{ave}} = 4.954\ 793$ for the elliptical-cross-section cylinder which are extremely close to the CAM theory results $\theta_2^R = 137.914\ 91^\circ$ and $h_2 = 4.954\ 462$ for the circular-cross-section cylinder discussed above. But the amplitude of oscillation of $h_2(\xi) - h_{\text{ave}}$ by use of CAM modeling of the rainbow minima (with the coefficients u_1 and v_0 for a sphere) grows to 28% larger than that for ray theory. The CAM results are shown as the solid circles in Fig. 2. These results illustrate that, although CAM modeling of the rainbow minima improves the absolute value of h_2 for an elliptical-cross-section cylinder, it does not improve the ξ dependence of h_2 . If, however, the CAM formalism of Refs. 17–19 were to be extended to elliptical-cross-section cylinders so that the ξ dependence of u_1 is included, the CAM predictions would again presumably be closer to ray theory than the predictions of first-order perturbation theory with Airy modeling of the rainbow minima.

5. Conclusion

The feature of the rainbow that has historically received the most attention by theorists is the far-zone rainbow scattering angle θ_p^R . But both the ray theory and the first-order perturbation wave theory calculations described here indicate that the supernumerary spacing parameter h_p is a much more sensitive and delicate feature of the rainbow than is θ_p^R . Using ray theory I found that $h_p(\xi)$ oscillates sinusoidally as a function of ξ for a small cylinder ellipticity ϵ , just as the rainbow angle did in Möbius theory. Although I was not able to obtain an analytic expression for $h_p(\xi)$ for $\epsilon \ll 1$, a good approximation in this regime is given by approximations (16) and (17) for $p = 2$ and $1.25 \leq n \leq 1.7$. As ϵ increases, higher Fourier components become important, and the oscillation is no longer exactly sinusoidal.

In the first-order perturbation version of wave theory, we found in Eqs. (34) that each Fourier component of the cylinder's surface shape for $q \geq 1$ couples two different incident partial waves to a single interior or scattered partial wave. The Debye-series de-

composition of the perturbation-theory partial-wave scattering amplitudes of Eqs. (45) and (47) illustrates the physical mechanism of the partial-wave coupling in a clear way. The coupling occurs at any of the interactions of the multipole waves with the cylinder surface. The partial-wave-changing Fresnel coefficients for reflection and transmission are given in Eqs. (35) (38–40) (42), and (43). To determine $h_p(\xi)$ using first-order perturbation theory, we also had to model the minima of the supernumerary interference pattern using either Airy theory or the lowest-order CAM correction to Airy theory. We found that although CAM modeling for spheres produced a better agreement with ray theory for h_2 for a circular-cross-section cylinder and h_{ave} for an elliptical-cross-section cylinder, the Airy modeling produced a better agreement for the oscillation of $h_2(\xi)$. We also found that reasonably large uncertainties can occur in $h_p(\xi)$ without seriously affecting the calculated value of the rainbow angle $\theta_p^R(\xi)$. Conversely, the angles of the supernumerary maxima or minima need to be measured with extreme accuracy to determine $h_p(\xi)$, which strongly affects experiments that attempt to measure $h_p(\xi)$.

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