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James A. Lock

Cleveland State University, j.lock@csuohio.edu

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Negative acceleration components for a relativistic particle

J. A. Lock

In relativistic mechanics it is of great importance to distinguish between those properties of a physical system that are frame independent or intrinsic to it and those properties that are contrived or that depend for their occurrence on observing the system from a particular reference frame. In recent years there has been renewed interest in the three-vector Newtonian acceleration

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} \quad (1)$$

of a particle acted upon by an external force, and under what circumstances various components of $\mathbf{a}$ are opposite in direction to the applied force. Recent examinations of the Newtonian acceleration under an external force have been limited to very special cases such as for a two-dimensional¹ or three-dimensional² motion at the instant for which $v_x(t) = v_y(t)$ or $v_x(t) = v_y(t) = v_z(t)$, or for the case when the force is time independent.³ It is our purpose: (i) to demonstrate that a number of the conclusions arrived at numerically for the above special cases are more generally true for an arbitrary time-dependent applied force $\mathbf{F}(t)$ and (ii) to point out which of these properties of the Newtonian acceleration are rotationally invariant and which properties depend for their manifestation on being in a particular reference frame. Specifically we find that the component of $\mathbf{a}(t)$ in the direction of $\mathbf{F}(t)$ is always positive, implying that for a given coordinate system, all three components of $\mathbf{a}(t)$ can never be opposite in sign to the respective components of $\mathbf{F}(t)$. However, whether zero, one, or two of the acceleration components are directed opposite to the force components depends on which reference frame one is observing the particle from.

We consider a relativistic particle of rest mass m_0 with initial three-momentum $\mathbf{p}_0$ acted upon by the arbitrary time-dependent external force $\mathbf{F}(t)$. Integrating Newton's second law

$$d\mathbf{p} = \mathbf{F}(t')dt' \quad (2)$$

from the initial time to t yields

$$\mathbf{p}(t) = \frac{m_0 \mathbf{v}(t)}{(1 - v^2/c^2)^{1/2}} = \mathbf{p}_0 + \int_0^t \mathbf{F}(t')dt' \equiv \mathbf{p}_0 + \mathbf{I}(t) \quad (3)$$

or

$$\mathbf{v}(t) = \frac{\mathbf{p}_0 + \mathbf{I}(t)}{\{m_0^2 + [\mathbf{p}_0 + \mathbf{I}(t)]^2/c^2\}^{1/2}} \quad (4)$$

The Newtonian acceleration of Eq. (1) is then

$$\mathbf{a}(t) = \frac{[m_0^2 + B^2(t)] \mathbf{F}(t) - [\mathbf{B}(t) \cdot \mathbf{F}(t)] \mathbf{B}(t)}{[m_0^2 + B^2(t)]^{3/2}} \quad (5)$$

with

$$\mathbf{B}(t) \equiv [\mathbf{p}_0 + \mathbf{I}(t)]/c = \mathbf{p}(t)/c. \quad (6)$$

The component of $\mathbf{a}$ in the direction of $\mathbf{F}$ is

$$a_F = \frac{\mathbf{a}(t) \cdot \mathbf{F}(t)}{|\mathbf{F}(t)|} = \frac{m_0^2 F^2(t) + B^2(t) F^2(t) - [\mathbf{B}(t) \cdot \mathbf{F}(t)]^2}{|\mathbf{F}(t)| [m_0^2 + B^2(t)]^{3/2}} \geq 0, \quad (7)$$

which is nonnegative for all times. Imposing a three-dimensional coordinate system on the force and acceleration vectors we have

$$\mathbf{a} \cdot \mathbf{F} = a_x F_x + a_y F_y + a_z F_z \geq 0. \quad (8)$$

If all three components of the acceleration were opposite in sign to the components of the force, the left-hand side of Eq. (8) would be negative, which is disallowed by Eq. (7). Thus as a consequence of $\mathbf{a} \cdot \mathbf{F} \geq 0$, all three components of $\mathbf{a}$, independent of reference frame, cannot be directed opposite to the components of $\mathbf{F}$.

To demonstrate that whether zero, one, or two components of $\mathbf{a}$ are directed opposite to those of $\mathbf{F}$ depends on one's reference frame, we consider the initial frame in which $\mathbf{F}$ is directed along the positive z axis and $\mathbf{a}$ is in the xz plane with the components

$$\mathbf{a} = a \sin \Omega \mathbf{u}_x + a \cos \Omega \mathbf{u}_z, \quad (9)$$

with

$$\cos \Omega \equiv \mathbf{a} \cdot \mathbf{F} / |\mathbf{a}| |\mathbf{F}| \quad (10)$$

and

$$0 \leq \Omega \leq \pi/2. \quad (11)$$

Rotating the vectors $\mathbf{F}$ and $\mathbf{a}$ via the Euler angles, i.e., a rotation of $-\gamma$ about the z axis, then a rotation of $-\theta$ with $0 \leq \theta \leq \pi/2$ about the y axis, then a rotation of $-\phi$ with $0 \leq \phi \leq \pi/2$ about the z axis,⁴ we obtain

$$\mathbf{F} = F \sin \theta \cos \phi \mathbf{u}_x + F \sin \theta \sin \phi \mathbf{u}_y + F \cos \theta \mathbf{u}_z \quad (12)$$

with all three components positive and

$$\begin{aligned} \mathbf{a} = & a[(\cos \gamma \cos \theta \cos \phi - \sin \gamma \sin \phi) \sin \Omega \\ & + \sin \theta \cos \phi \cos \Omega] \mathbf{u}_x \\ & + a[(\cos \gamma \cos \theta \sin \phi + \sin \gamma \cos \phi) \sin \Omega \\ & + \sin \theta \sin \phi \cos \Omega] \mathbf{u}_y \\ & + a[\cos \theta \cos \Omega - \cos \gamma \sin \theta \sin \Omega] \mathbf{u}_z. \end{aligned} \quad (13)$$

The choice, for example, of $\gamma = 0$ and $\theta < \pi/2 - \Omega$ gives

$$\mathbf{a} = a \cos \phi \sin(\theta + \Omega) \mathbf{u}_x + a \sin \phi \sin(\theta + \Omega) \mathbf{u}_y + a \cos(\theta + \Omega) \mathbf{u}_z \quad (14)$$

with all three components positive, while $\theta > \pi/2 - \Omega$ gives positive x and y components and a negative z component. On the other hand, $\gamma = \pi$ and $\theta < \Omega$ gives

$$\mathbf{a} = a \cos \phi \sin(\theta - \Omega) \mathbf{u}_x + a \sin \phi \sin(\theta - \Omega) \mathbf{u}_y + a \cos(\theta - \Omega) \mathbf{u}_z \quad (15)$$

with negative x and y components and a positive z component, thus demonstrating the claim that whether zero, one, or two components are negative depends on the choice of reference frame.

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²P. F. González-Díaz, Am. J. Phys. **46**, 932 (1978).

³B. F. Rothenstein and J. Artzner (unpublished).

⁴A. Messiah, *Quantum Mechanics* (Wiley, New York, 1966), Appendix C, Sec. 10.