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Whitehead products in function spaces: Quillen model formulae

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Abstract. We study Whitehead products in the rational homotopy groups of a general component of a function space. For the component of any based map $f : X \to Y$, in either the based or free function space, our main results express the Whitehead product directly in terms of the Quillen minimal model of f. These results follow from a purely algebraic development in the setting of chain complexes of derivations of differential graded Lie algebras, which is of interest in its own right. We apply the results to study the Whitehead length of function space components.

1. Introduction.

Let $f : X \to Y$ be a based map of based, simply connected CW complexes with X a finite complex. Let $\mathrm{map}(X, Y; f)$ denote the path component containing f in the space of basepoint-free continuous functions from X to Y, and $\mathrm{map}_*(X, Y; f)$ the component in the space of basepoint-preserving functions. In this paper, we study the structure of the Whitehead product on the rational homotopy groups of these function spaces.

The paper is organized as follows. In Section 2, we describe the Quillen model of the map

$$\eta \times 1 : S^{p+q-1} \times X \to (S^p \vee S^q) \times X$$

where η is the Whitehead product. Our description is given in the framework of chain complexes of generalized derivations of Quillen models, which was introduced in [**10**] in order to identify the rational homotopy groups of function space components. Section 3 is a purely algebraic development in the setting of chain complexes that arise in the category of differential graded (DG) Lie algebras. Using the form of the Quillen model of $\eta \times 1$ as a guide, we construct a

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"Whitehead product" on the homology of the mapping cone of a map of DG Lie algebras. We extend our construction to the chain complexes of generalized derivations mentioned above. In Section 4, we record a detailed formula useful for applications, and mention briefly some extensions, such as iterated products. In Section 5, we return to the topological setting and prove our main result: we identify Whitehead products in the rational homotopy groups of $\mathrm{map}(X, Y; f)$ and $\mathrm{map}_*(X, Y; f)$ with the "Whitehead products" constructed algebraically from the Quillen model of the map.

We present various applications in Section 6, where we study the rational Whitehead length of function space components. Given a space Z, let $\mathrm{WL}(Z)$, the *Whitehead length of* Z, denote the length of longest, non-zero iterated Whitehead bracket in $\pi_{\geq 2}(Z)$. (We avoid considerations of the fundamental group throughout this paper.) Thus $\mathrm{WL}(Z) = 1$ means all Whitehead products vanish and $\mathrm{WL}(Z) \geq 2$ means that there exists a non-trivial Whitehead product. Let $\mathrm{WL}_{\boldsymbol{Q}}(Z)$, the *rational Whitehead length of* Z, denote the length of longest, non-zero iterated Whitehead bracket in $\pi_{\geq 2}(Z) \otimes \boldsymbol{Q}$. We first observe that, for the null component of a function space, we have $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, Y; 0)) = \mathrm{WL}_{\boldsymbol{Q}}(Y)$ as a consequence of classical ideas (Theorem 6.1). Using our formula, we then prove that, for any map $f : X \to Y$, that is, for a general component, we have

$$\max\{\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, Y; f)), \mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, Y; f))\} \leq \mathrm{WL}_{\boldsymbol{Q}}(Y)$$

provided Y is a coformal space (Theorem 6.2). Focusing on the based function space, we also prove that

$$\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, Y; f)) \leq \mathrm{cl}_0(X),$$

where $\mathrm{cl}_0(X)$ denotes the rational cone-length of X (Theorem 6.4) complementing the corresponding (integral) result at the null component due to Ganea [**6**]. In Theorem 6.5, we apply our formulae to give a complete calculation of the rational Whitehead length of all components of $\mathrm{map}(X, S^n)$ and of $\mathrm{map}_*(X, S^n)$ for X a finite, simply connected CW complex. Finally, we show that the inequality

$$\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, Y; f)) > \mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, Y; 0)) = \mathrm{WL}_{\boldsymbol{Q}}(Y)$$

may hold. Precisely, in Example 6.6, we give a space Y with vanishing rational Whitehead products and a map $f : S^3 \to Y$ such that $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(S^3, Y; f)) \geq 2$.

We assume familiarity with rational homotopy theory from Quillen's point of view. Our main reference for this material is [**5**] (see also [**13**], [**15**]). We introduce notation as we go but recall here that a map $f : X \to Y$ of simply connected CW complexes of finite type has a *Quillen minimal model* which is a map $\mathscr{L}_f : (\mathscr{L}_X, d_X) \to (\mathscr{L}_Y, d_Y)$ of connected DG Lie algebras over $\boldsymbol{Q}$. The Quillen minimal model of f is a complete invariant of the rationalization of f. In particular, there is a natural isomorphism $H_*(\mathscr{L}_X, d_X) \cong \pi_*(\Omega X) \otimes \boldsymbol{Q}$ of graded Lie algebras. The map induced by f on rational homotopy Lie algebras corresponds, with these identifications for X and Y, to the map induced by $\mathscr{L}_f$ on homology. Our main results explain how the Whitehead product in the rational homotopy groups of $\mathrm{map}(X, Y; f)$ and $\mathrm{map}_*(X, Y; f)$ depends on $\mathscr{L}_f$.

REMARK 1.1. Rational Whitehead products for function spaces have been studied by several authors. In [**16**], Vigué-Poirrier gave an elegant formula for Whitehead products in the null-components $\mathrm{map}_*(X, Y; 0)$ and $\mathrm{map}(X, Y; 0)$ (including degree 1) directly in terms of products in the rational homotopy of Y and the cup product in $H^*(X, \boldsymbol{Q})$ under certain restrictions on X and Y. This result was recently extended to full generality by Buijs and Murillo as a special case of their description of the rational homotopy Lie algebra of any component of a function space [**4**]. Also, we mention the recent work of Buijs, Félix and Murillo [**3**] which identifies a Lie model for spaces of sections and, in particular, for components of a function space.

Our work differs from these other results in at least two respects. First, we describe rational Whitehead products for general function space components by means of a construction that proceeds directly from the Quillen model of a map. Because we focus on a description specifically at the level of rational homotopy groups, rather than a more comprehensive description of the rational homotopy type, we are able to give a fairly direct construction: our description lends itself well to the study of specific examples. Second, our construction of topological (rational) Whitehead products is developed from a purely algebraic one on the mapping cone of certain maps of chain complexes (see Section 3). This provides the basis for new developments either in the algebraic settings, or in topological situations other than function spaces that correspond to mapping cones.

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2. The Quillen model of a certain map.

We review the development of ideas in [**10**]. An element $\alpha \in \pi_p(\mathrm{map}(X, Y; f))$ is represented by a map $a : S^p \to \mathrm{map}(X, Y; f)$ whose adjoint is a map $A :$

$S^p \times X \to Y$ that restricts to $f : X \to Y$ on X. By considering the Quillen minimal model of the adjoint A we are led to consider a certain complex of (generalized) derivations of Quillen models, which we denoted by $\mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)$ in [**10**]. The homology groups of this complex may be identified with the homotopy groups of the based mapping space $\mathrm{map}_*(X, Y; f)$, and the homology groups of the mapping cone of the (generalized) adjoint map

$$\mathrm{ad}_{\mathscr{L}_f} : \mathscr{L}_Y \to \mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)$$

may be identified with the homotopy groups of $\mathrm{map}(X, Y; f)$ (see [**10**, Theorem 3.1]). It is in this context that we wish to describe the Whitehead product.

Topologically, a Whitehead product $\gamma = [\alpha, \beta]_w \in \pi_{p+q-1}(\mathrm{map}(X, Y; f))$, for $\alpha \in \pi_p(\mathrm{map}(X, Y; f))$ and $\beta \in \pi_q(\mathrm{map}(X, Y; f))$, is represented by the composition

$$\begin{tikzcd} S^{p+q-1} \arrow[r, "\eta"] \arrow[rr, bend right, "\gamma"'] & S^p \vee S^q \arrow[r, "(a|b)"] & \mathrm{map}(X, Y; f) \end{tikzcd} ,$$

where $\eta = [\iota_1, \iota_2]_w$ is the "universal example" of a Whitehead product. The adjoint C of γ is the composition

$$\begin{tikzcd} S^{p+q-1} \times X \arrow[r, "\eta\times 1"] \arrow[rr, bend right, "C"'] & (S^p \vee S^q) \times X \arrow[r, "(A|B)_f"] & Y \end{tikzcd} .$$

As in the previous paragraph, we will translate this adjoint into the setting of complexes of (generalized) derivations of Quillen models. In order to do so, a description of the Quillen model of $\eta \times 1$ is germane.

We say a graded rational vector space (V, d) with a differential d of degree -1 is a *DG space* or, alternately, a chain complex. By a *DG Lie algebra* (L, d) we will mean a connected, graded Lie algebra L with bilinear product $[\ ,\ ]$ satisfying

(a) $|[x, y]| = |x| + |y|$
(b) $[x, y] = (-1)^{|x||y|+1}[y, x]$ and
(c) $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$

and differential satisfying

$$d([x, y]) = [d(x), y] + (-1)^{|x|}[x, d(y)].$$

We write $\boldsymbol{L}(V)$ for the free graded Lie algebra generated by the graded space V and extend this notation, writing $\boldsymbol{L}(V, W)$ for the free Lie algebra generated by V and W and $\boldsymbol{L}(V, a)$ for the free Lie algebra generated by a and V where a is an element of homogeneous degree. We write $\boldsymbol{L}(V; d)$ for the DG Lie algebra $(\boldsymbol{L}(V), d)$.

We recall that a DG Lie algebra (L, d) has an associated DG Lie algebra of derivations $(\mathrm{Der}(L), D)$. Here $\mathrm{Der}(L)$ denotes the graded space of positive-degree derivations of L with the usual graded commutator product of derivations, that is,

$$[\theta, \phi] = \theta \circ \phi - (-1)^{|\theta||\phi|}\phi \circ \theta$$

for $\theta, \phi \in \mathrm{Der}(L)$, and differential $D(\theta) = [d, \theta] = d\theta - (-1)^{|\theta|}\theta d$. Then the adjoint $\mathrm{ad} : (L, d) \to (\mathrm{Der}(L), D)$, defined by $\mathrm{ad}(l)(l') = [l, l']$, is a map of DG Lie algebras. We are interested in a natural generalization of this set-up. Let $\psi : (L, d_L) \to (K, d_K)$ be a given DG Lie algebra map. Define a ψ-*derivation* of degree n to be a linear map $\theta : L_* \to K_{*+n}$ satisfying

$$\theta([x, y]) = [\theta(x), \psi(y)] + (-1)^{n|x|}[\psi(x), \theta(y)].$$

We write $\mathrm{Der}_n(L, K; \psi)$ for the space of degree-n ψ-derivations. The differential D_ψ defined by

$$D_\psi(\theta) = d_K \circ \theta - (-1)^{|\theta|}\theta \circ d_L$$

makes the pair $(\mathrm{Der}(L, K; \psi), D_\psi)$ a DG space. The ψ-adjoint (or "generalized adjoint") map

$$\mathrm{ad}_\psi : (K, d_K) \to (\mathrm{Der}(L, K; \psi), D_\psi),$$

given by $\mathrm{ad}_\psi(\alpha)(x) = [\alpha, \psi(x)]$ for $x \in L, \alpha \in K$, is a map of DG spaces.

Our description of the Quillen model of $\eta \times 1$ requires a construction featuring these generalized derivations. Let $L = \boldsymbol{L}(V; d)$ be a free DG Lie algebra. Let $p_1, \ldots, p_n$ be given integers > 1 and $a_1, \ldots, a_n$ elements of degree $p_1 - 1, \ldots, p_n - 1$. Write $V^{a_i} = s^{p_i}(V)$ for the p_ith suspension of V and let $S_{a_i} : V \to V^{a_i}$ denote the corresponding degree p_i linear map. We define a new DG Lie algebra $(L(a_1, \ldots, a_n), \partial)$ by setting

$$L(a_1, \ldots, a_n) = \boldsymbol{L}(V, a_1, \ldots, a_n, V^{a_1}, \ldots, V^{a_n}). \tag{1}$$

Observe that the suspension $S_{a_i}: V \to V^{a_i}$ extends as a derivation to an element $S_{a_i} \in \mathrm{Der}_{p_i}(L, L(a_1, \ldots, a_n); \lambda)$ where $\lambda : L \to L(a_1, \ldots, a_n)$ is the inclusion. Using this, we define the differential as follows:

$$\partial(v) = d(v), \partial(a_i) = 0 \text{ and } \partial(S_{a_i}(v)) = (-1)^{p_i - 1}[a_i, v] + (-1)^{p_i} S_{a_i}(dv)$$

for $v \in V$. The definition of ∂ gives the boundary relation

$$D_\lambda(S_{a_i}) = (-1)^{p_i - 1}\mathrm{ad}_\lambda(a_i) \in \mathrm{Der}(L, L(a_1, \ldots, a_n); \lambda). \tag{2}$$

Recall that a simply connected CW complex X of finite type admits a Quillen minimal model $\mathscr{L}_X = \boldsymbol{L}(V; d_X)$ which is a free minimal DG Lie algebra with $V \cong s^{-1}\widetilde{H}_*(X; \boldsymbol{Q})$ and $H_*(\mathscr{L}_X) \cong \pi_*(\Omega X) \otimes \boldsymbol{Q}$. A map $f : X \to Y$ between such spaces induces a DG Lie algebra map

$$\mathscr{L}_f : (\mathscr{L}_X, d_X) \to (\mathscr{L}_Y, d_Y).$$

The connection to the map $\eta \times 1 : S^{p+q-1} \times X \to (S^p \vee S^q) \times X$ is provided by the following result.

THEOREM 2.1. [**10**, Theorem 2.1] *Let X be a simply connected CW complex of finite type. The DG Lie algebra $(\mathscr{L}_X(a_1, \ldots, a_n), \partial)$ defined by* (1) *is the Quillen minimal model for the space $\left(\vee_{i=1}^n S^{p_i}\right) \times X$.* □

By Theorem 2.1, the Quillen model for $\eta \times 1$ is some map of DG Lie algebras

$$\Gamma : (\mathscr{L}_X(c), \partial_c) \to (\mathscr{L}_X(a, b), \partial_{a,b})$$

where $|a| = p - 1$, $|b| = q - 1$ and $|c| = p + q - 2$. It is easy to check that $\Gamma(\chi) = \chi$ for $\chi \in \mathscr{L}_X$ while $\Gamma(c) = (-1)^{p-1}[a, b]$. (This sign is appropriate per the identifications of [**17**, Chapter X, 7.10].) Let us write $S_{[a,b]}$ for the degree $p + q - 1$ linear map induced by Γ via the rule

$$S_{[a,b]}(v) =_{\mathrm{def}} \Gamma(S_c(v))$$

for $v \in V$. Then $S_{[a,b]}$ extends to a λ-derivation $S_{[a,b]} \in \mathrm{Der}_{p+q-1}(\mathscr{L}_X, \mathscr{L}_X(a, b); \lambda)$ satisfying the boundary relation

$$D_\lambda\bigl(S_{[a,b]}\bigr) = (-1)^{q-1}\mathrm{ad}_\lambda([a, b]) \in \mathrm{Der}_{p+q-2}(\mathscr{L}_X, \mathscr{L}_X(a, b); \lambda). \tag{3}$$

Working backward, we see the identification of the derivation $S_{[a,b]}$ satisfying (3) completely determines the Quillen minimal model of Γ.

In the next section, we find a formula for $S_{[a,b]}$. In fact, we identify this derivation as the "universal example" for Whitehead products constructed in the category of DG Lie algebras. (See Remark 3.3, below.) To explain this further, we introduce the *mapping cone* of a DG vector space map $\psi\colon (V, d_V) \to (W, d_W)$, which we denote by $(\mathrm{Rel}(\psi), \delta_\psi)$. This is the DG space with $\mathrm{Rel}_n(\psi) = V_{n-1} \oplus W_n$ and differential δ_ψ defined as $\delta_\psi(v, w) = (-d_V(v), \psi(v) + d_W(w))$. The construction yields a short exact sequence of DG spaces $(W, d_W) \to (\mathrm{Rel}(\psi), \delta_\psi) \to (V, d_V)$ giving rise to a long exact homology sequence whose connecting homomorphism is $H(\psi)$. Applying this to the adjoint $\mathrm{ad}_\lambda\colon (L(a,b), d) \to (\mathrm{Der}(L, L(a,b); \lambda), D_\lambda)$ we see the boundary conditions (2) and (3) are equivalent to the elements

$$\zeta_a = ((-1)^p a, S_a), \quad \zeta_b = ((-1)^q b, S_b) \quad \text{and} \quad \zeta_{[a,b]} = \big((-1)^q [a,b], S_{[a,b]}\big)$$

being three D_λ-cycles in $\mathrm{Rel}(\mathrm{ad}_\lambda)$ of degree p, q and $p+q-1$, respectively. In the next section, we construct a Whitehead product $[\,,\,]_w$ on $H_*(\mathrm{Rel}(\mathrm{ad}_\lambda))$ satisfying

$$[\langle\zeta_a\rangle, \langle\zeta_b\rangle]_w = \langle\zeta_{[a,b]}\rangle$$

thereby completing the description of Γ, the Quillen model of $\eta \times 1$, above.

3. Whitehead products in the category of DG Lie algebras.

In this section, we describe the construction of Whitehead products on the homology of chain complexes of derivations arising from a given DG Lie algebra map $\psi\colon (L, d_L) \to (K, d_K)$. We will approach our final construction in several steps. First we give the definition of a Whitehead product, referring to the classical correspondence between Whitehead products and Samelson products. Let sL denote the suspension of L. Given $x, y \in L$ define a bilinear pairing on sL by the rule

$$[sx, sy]_w =_{\mathrm{def}} (-1)^{|x|} s[x,y].$$

The pairing $[\,,\,]_w$ then satisfies the identities

(i) $\big|[\alpha,\beta]_w\big| = |\alpha| + |\beta| - 1$

(ii) $[\alpha,\beta]_w = (-1)^{|\alpha||\beta|}[\beta,\alpha]_w$ and

(iii) $[\alpha,[\beta,\gamma]_w]_w = (-1)^{|\alpha|+1}[[\alpha,\beta]_w,\gamma]_w + (-1)^{(|\alpha|+1)(|\beta|+1)}[\beta,[\alpha,\gamma]_w]_w$

for $\alpha, \beta, \gamma \in sL$. These identities correspond, of course, to those satisfied by the higher homotopy groups of a space with the Whitehead product [**17**, Chapter X, 7]. We denote a bilinear pairing satisfying (i)–(iii) by $[\ ,\ ]_w$ and call it a *Whitehead product*.

As a preliminary, we next observe that a kind of "pre-Whitehead product" may be defined on any DG Lie algebra (L, d_L). Specifically, define a bilinear pairing on L by setting

$$\{x, y\} =_{\mathrm{def}} (-1)^{|x|+1}[x, d_L(y)]. \tag{4}$$

The pairing $\{\ ,\ \}$ clearly satisfies (i). Further, we have the following:

PROPOSITION 3.1. *The bilinear pairing* $\{\ ,\ \}$ *defined on* L *by* (4) *satisfies the identities* (ii) *and* (iii) *up to boundaries in* (L, d_L).

PROOF. Write $\sim$ for the homologous relation in (L, d_L) and let $d = d_L$. Let $p = |x|, q = |y|$ and $r = |z|$. Use the boundary

$$d([x, y]) = [d(x), y] + (-1)^p[x, d(y)]$$

to obtain

$$\{x, y\} = (-1)^{p+1}[x, d(y)] \sim [d(x), y] = (-1)^{(p-1)q+1}[y, d(x)] = (-1)^{pq}\{y, x\}.$$

For (iii), observe

$$\begin{aligned} \{x, \{y, z\}\} &= (-1)^{p+q}[x, d([y, d(z)])] \\ &= (-1)^{p+q}[x, [d(y), d(z)]] \\ &= (-1)^{p+q}[[x, d(y)], d(z)] + (-1)^{q(p+1)}[d(y), [x, d(z)]]. \end{aligned}$$

Then note that

$$(-1)^{p+q}[[x, d(y)], d(z)] = (-1)^{q+1}[\{x, y\}, d(z)] = (-1)^{p+1}\{\{x, y\}, z\}.$$

Finally, the boundary

$$d([y, [x, d(z)]]) = [d(y), [x, d(z)]] + (-1)^q[y, d([x, d(z)])]$$

implies

$$\begin{aligned}(-1)^{q(p+1)}[d(y),[x,d(z)]] &\sim (-1)^{pq+1}[y,d([x,d(z)])] \\ &= (-1)^{(p+1)(q+1)}\{y,\{x,z\}\}. \end{aligned} \qquad \square$$

Next we consider the case of a DG Lie algebra map $\psi : (L, d_L) \to (K, d_K)$ and its mapping cone $(\mathrm{Rel}(\psi), \delta_\psi)$. We will construct a Whitehead product on the homology of $(\mathrm{Rel}(\psi), \delta_\psi)$. Notice that this is a chain complex, not a DG Lie algebra; it is not immediately evident that such a product may be defined. Our construction here refers to the two previous steps.

Let $(a,\alpha) \in \mathrm{Rel}_p(\psi)$ and $(b,\beta) \in \mathrm{Rel}_q(\psi)$ be given. Recall that this means $a \in L_{p-1}, b \in L_{q-1}$ while $\alpha \in K_p, \beta \in K_q$. Define a bilinear pairing $[\![\ ,\ ]\!]$, using the ordinary bracket in L but the pairing defined by (4) in K, by setting

$$[\![(a,\alpha),(b,\beta)]\!] =_{\mathrm{def}} ((-1)^p[a,b],\{\alpha,\beta\}) = \Big((-1)^p[a,b],(-1)^{p+1}[\alpha,d_K(\beta)]\Big). \quad (5)$$

We then have the following:

Proposition 3.2. *Let $\psi : (L, d_L) \to (K, d_K)$ be a DG Lie algebra map with mapping cone $(\mathrm{Rel}(\psi), \delta_\psi)$. The bilinear pairing $[\![\ ,\ ]\!]$ on $\mathrm{Rel}(\psi)$ defined by (5) induces a Whitehead product $[\ ,\ ]_w$ on $H_*(\mathrm{Rel}(\psi))$.*

Proof. For suppose that (a,α) and (b,β) are δ_ψ-cycles. Then $d_L(a) = d_L(b) = 0$ while $d_K(\alpha) = -\psi(a)$ and $d_K(\beta) = -\psi(b)$. Observe that

$$d_K(\{\alpha,\beta\}) = (-1)^{p+1}d_K([\alpha,d_K(\beta)]) = (-1)^{p+1}[d_K(\alpha),d_K(\beta)] = -\psi((-1)^p[a,b]).$$

Thus the product $[\![(a,\alpha),(b,\beta)]\!]$ is a δ_ψ-cycle, as well.

Next suppose $(a,\alpha) = \delta_\psi(c,\gamma)$ is a δ_ψ-boundary and (b,β) is again a δ_ψ-cycle. Then

$$[\![(a,\alpha),(b,\beta)]\!] = \delta_\psi((-1)^p[c,b],-\{\gamma,\beta\})$$

is a δ_ψ-boundary, as well. To verify this in the second variable observe that

$$\begin{aligned} d_K(-\{\gamma,\beta\}) + (-1)^p\psi([c,b]) &= (-1)^{p+1}[d_K(\gamma),d_K(\beta)] + (-1)^p\psi([c,b]) \\ &= (-1)^{p+1}[\alpha-\psi(c),d_K(\beta)] + (-1)^p\psi([c,b]) \\ &= \{\alpha,\beta\}, \end{aligned}$$

since $d_K(\beta) = -\psi(b)$.

The pairing $[\![\,,\,]\!]$ thus induces a bilinear pairing $[\,,\,]_w$ on $H_*(\mathrm{Rel}(\psi))$ satisfying the degree condition (i) by construction. The induced product satisfies (ii) and (iii) in the second variable by Proposition 3.1. In the first variable, $[\,,\,]_w$ corresponds to the classical Whitehead product (except with grading reduced one instead of increased one). □

Our final step is to consider the (generalized) adjoint

$$\mathrm{ad}_\psi : (K, d_K) \to \big(\mathrm{Der}(L, K; \psi), D_\psi\big)$$

and its mapping cone $\big(\mathrm{Rel}(\mathrm{ad}_\psi), \delta_{\mathrm{ad}_\psi}\big)$. We define Whitehead products on the homology of the latter two complexes. Notice that, once again, neither of these complexes is a DG Lie algebra.

As at the start of Section 2, the Whitehead product of two elements $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$ involves the "universal example" of such Whitehead products, namely $\eta \in \pi_{p+q-1}(S^p \vee S^q)$. This is then mapped into $\pi_{p+q-1}(X)$ by $(\alpha \mid \beta)$, a map induced by the given homotopy elements. Here, we take a similar approach. Given two elements of $H_*(\mathrm{Rel}(\mathrm{ad}_\psi))$, of degree p and q we first describe a "universal example" of the Whitehead product in $H_{p+q-1}(\mathrm{Rel}(\mathrm{ad}_\lambda))$ (see (7) below). This is then mapped to $H_{p+q-1}(\mathrm{Rel}(\mathrm{ad}_\psi))$ using the elements whose product we are forming (see (10) below).

For our universal example, we define a particular product in the mapping cone of the generalized adjoint $\mathrm{ad}_\lambda : L(a,b) \to \mathrm{Der}(L, L(a,b); \lambda)$ defined above (1). So assume now that $(L, d_L) = \boldsymbol{L}(V; d_L)$ is a free DG Lie algebra. Let a and b be of degrees $p-1$ and $q-1$, respectively. Then recall $L(a,b) = \boldsymbol{L}(V, a, b, V^a, V^b; \partial_{a,b})$ and the suspensions $S_a : V \to V^a$ and $S_b : V \to V^b$ extend to elements of $(\mathrm{Der}(L, L(a,b); \lambda), D_\lambda)$ of degree p and q, respectively, satisfying $D_\lambda(S_a) = (-1)^{p-1}\mathrm{ad}_\lambda(a)$ and $D_\lambda(S_b) = (-1)^{q-1}\mathrm{ad}_\lambda(b)$. So $((-1)^p a, S_a)$ and $((-1)^q b, S_b)$ are cycles in degrees p and q of $(\mathrm{Rel}(\mathrm{ad}_\lambda), \delta_{\mathrm{ad}_\lambda})$. Define elements Θ_a, Θ_b of degrees p and q in the DG Lie algebra $(\mathrm{Der}\big(L(a,b)\big), D)$ by setting

$$\Theta_x(v) = S_x(v) \quad \text{and} \quad \Theta_x(a) = \Theta_x(b) = \Theta_x(V^a) = \Theta_x(V^b) = 0 \tag{6}$$

for $x = a, b$ and $v \in V$. Note that $\Theta_x \circ \lambda = S_x \in \mathrm{Der}(L, L(a,b); \lambda)$. From the previous step, we set

$$\{\Theta_a, \Theta_b\} = (-1)^{p+1}[\Theta_a, D(\Theta_b)] \in \mathrm{Der}_{p+q-1}(L(a,b))$$

and observe that $\{\Theta_a, \Theta_b\} \circ \lambda \in \mathrm{Der}_{p+q-1}(L, L(a,b); \lambda)$. Now define

$$[\![((-1)^p a, S_a), ((-1)^q b, S_b)]\!] =_{\mathrm{def}} ((-1)^q [a,b], \{\Theta_a, \Theta_b\} \circ \lambda). \tag{7}$$

Observe that the right-hand side is a cycle of $(\mathrm{Rel}(\mathrm{ad}_\lambda), \delta_{\mathrm{ad}_\lambda})$ of degree $p+q-1$. This is the universal example of a Whitehead product mentioned above.

REMARK 3.3. Taking $(L, d_L) = (\mathscr{L}_X, d_X)$ to be the Quillen model of a simply connected complex X, we see that

$$\{\Theta_a, \Theta_b\} \circ \lambda \in \mathrm{Der}_{p+q-1}(\mathscr{L}_X, \mathscr{L}_X(a,b); \lambda)$$

satisfies the boundary condition (3). Setting $S_{[a,b]} = \{\Theta_a, \Theta_b\} \circ \lambda$ completes the description of the Quillen minimal model of $\eta \times 1 : S^{p+q-1} \times X \to (S^p \vee S^q) \times X$.

Finally, we turn to the mapping cone $(\mathrm{Rel}(\mathrm{ad}_\psi), \delta_{\mathrm{ad}_\psi})$ of the generalized adjoint corresponding to a DG Lie algebra map $\psi : (L, d_L) \to (K, d_K)$ with $L = \boldsymbol{L}(V)$ free. Suppose given two $\delta_{\mathrm{ad}_\psi}$-cycles,

$$\zeta_a = (\chi_a, \theta_a) \in \mathrm{Rel}_p(\mathrm{ad}_\psi) \quad \text{and} \quad \zeta_b = (\chi_b, \theta_b) \in \mathrm{Rel}_q(\mathrm{ad}_\psi).$$

The pair ζ_a, ζ_b induce a DG Lie algebra map

$$(\zeta_a \mid \zeta_b)_\psi : (L(a,b), d_{L(a,b)}) \to (K, d_K) \tag{8}$$

defined, on the basis of $L(a,b)$, as:

$$(\zeta_a \mid \zeta_b)_\psi(x) = (-1)^{|x|+1}\chi_x, \quad (\zeta_a \mid \zeta_b)_\psi(v) = \psi(v) \quad \text{and} \quad (\zeta_a \mid \zeta_b)_\psi(S_x(v)) = \theta_x(v)$$

for $x = a, b$ and $v \in V$. Note that this map commutes with differentials on generators of the form $S_x(v)$ since $(\zeta_a \mid \zeta_b)_\psi \circ S_x$ and θ_x agree on L as ψ-derivations. Define

$$\{\zeta_a, \zeta_b\} =_{\mathrm{def}} (\zeta_a \mid \zeta_b)_\psi \circ \{\Theta_a, \Theta_b\} \circ \lambda \in \mathrm{Der}_{p+q-1}(L, K; \psi). \tag{9}$$

We have the following result concerning the iteration of this pairing.

PROPOSITION 3.4. *Let $\zeta_a, \zeta_b, \zeta_c \in \mathrm{Rel}(\mathrm{ad}_\psi)$ be $\delta_{\mathrm{ad}_\lambda}$-cycles of degree p, q and r, respectively. Then*

$$\{\{\zeta_a, \zeta_b\}, \zeta_c\} = (\zeta_a \mid \zeta_b \mid \zeta_c)_\psi \circ \{\{\Theta_a, \Theta_b\}, \Theta_c\} \circ \lambda \in \mathrm{Der}_{p+q+r-2}(L, K; \psi).$$

Here $(\zeta_a \mid \zeta_b \mid \zeta_c)_\psi : (L(a,b,c), d_{L(a,b,c)}) \to (K, d_K)$ *is defined as the obvious extension of the definition of* $(\zeta_a \mid \zeta_b)_\psi$ *given by* (8).

PROOF. Let $\zeta_z = \{\zeta_a, \zeta_b\} \in \mathrm{Rel}_{p+q-1}(L, K; \psi)$. Let $\gamma_z = ((-1)^q[a,b], \{\Theta_a, \Theta_b\} \circ\lambda) \in \mathrm{Rel}_{p+q-1}(L, L(a,b,c); \lambda)$ be the $\delta_{\mathrm{ad}_\lambda}$-cycle as in (7). Define a DG Lie algebra map

$$\phi_z : L(z,c) \to L(a,b,c)$$

by setting $\phi_z(v) = v$, $\phi_z(c) = c$, $\phi_z(S_c(v)) = S_c(v)$, $\phi_z(z) = (-1)^{q-1}[a,b]$, and $\phi_z(S_z(v)) = \{\Theta_a, \Theta_b\} \circ \lambda(v)$. This is readily checked to define a DG map: use the fact that γ_z is a cycle to check on generators $S_z(v)$. Then we have a commutative diagram

$$\begin{tikzcd} L(z,c) \arrow[rr, "\phi_z"] \arrow[dr, "(\zeta_z|\zeta_c)_\psi"'] & & L(a,b,c) \arrow[dl, "(\zeta_a|\zeta_b|\zeta_c)_\psi"] \\ & K & \end{tikzcd} .$$

The needed identity now follows directly from the observation that

$$\phi_z \circ \{\Theta_z, \Theta_c\} \circ \lambda = \{\{\Theta_a, \Theta_b\}, \Theta_c\} \circ \lambda \in \mathrm{Der}_{p+q+r-2}(L, L(a,b,c); \lambda). \qquad \square$$

We obtain a bilinear pairing $[\![\ ,\ ]\!]$ on pairs of cycles of $\mathrm{Rel}(\mathrm{ad}_\psi)$:

$$[\![\zeta_a, \zeta_b]\!] = [\![(\chi_a, \theta_a), (\chi_b, \theta_b)]\!] =_{\mathrm{def}} ((-1)^p[\chi_a, \chi_b], \{\zeta_a, \zeta_b\}). \tag{10}$$

REMARK 3.5. The diagram of chain maps

$$\begin{tikzcd} L(a,b) \arrow[r, "\mathrm{ad}_\lambda"] \arrow[d, "(\zeta_a|\zeta_b)_\psi"'] & \mathrm{Der}(L, L(a,b); \lambda) \arrow[d, "(\zeta_a|\zeta_b)_{\psi *}"] \\ K \arrow[r, "\mathrm{ad}_\psi"] & \mathrm{Der}(L, K; \psi) \end{tikzcd}$$

commutes, inducing a chain map of mapping cones

$$\zeta : (\mathrm{Rel}(\mathrm{ad}_\lambda), \delta_{\mathrm{ad}_\lambda}) \to (\mathrm{Rel}(\mathrm{ad}_\psi), \delta_{\mathrm{ad}_\psi}).$$

The map ζ carries $\llbracket ((-1)^p a, S_a), ((-1)^q b, S_b) \rrbracket$, the universal example of the Whitehead product defined by (7), to the cycle $\llbracket \zeta_a, \zeta_b \rrbracket$ defined by (10).

Our main result in this section is:

THEOREM 3.6. *Let $\psi : (L, d_L) \to (K, d_K)$ be a DG Lie algebra map with L free and with adjoint map $\mathrm{ad}_\psi : (K, d_K) \to (\mathrm{Der}(L, K; \psi), D_\psi)$. The bilinear pairing $\llbracket\ ,\ \rrbracket$ defined on cycles of $(\mathrm{Rel}(\mathrm{ad}_\psi), \delta_{\mathrm{ad}_\psi})$ by (10) induces a Whitehead product $[\ ,\ ]_w$ on $H_*(\mathrm{Rel}(\mathrm{ad}_\psi))$.*

PROOF. The fact that $\llbracket\ ,\ \rrbracket$ induces a pairing on cycles of $(\mathrm{Rel}(\mathrm{ad}_\psi), \delta_{\mathrm{ad}_\psi})$ follows from Remark 3.5 and the observation immediately following (7). Now we check that the Whitehead identities (i)–(iii) are satisfied up to boundaries. First, (i) is evident. For (ii), we return to (7) and write

$$\begin{aligned}
\llbracket ((-1)^p a, S_a), ((-1)^q b, S_b) \rrbracket &= ((-1)^q [a,b], \{\Theta_a, \Theta_b\} \circ \lambda) \\
&= \Big((-1)^{q+1+(p-1)(q-1)} [b,a], ((-1)^{pq} \{\Theta_b, \Theta_a\} - D[\Theta_a, \Theta_b]) \circ \lambda \Big) \\
&= (-1)^{pq} ((-1)^p [b,a], \{\Theta_b, \Theta_a\}) - \big(0, \big(D[\Theta_a, \Theta_b]\big) \circ \lambda\big).
\end{aligned}$$

In this last term, D denotes the differential in $\mathrm{Der}\big(L(a,b)\big)$; the identity is obtained from the first part of the proof of Proposition 3.1. Now observe that, in $\big(\mathrm{Der}(L, L(a,b); \lambda), D_\lambda\big)$, we have

$$\big(D[\Theta_a, \Theta_b]\big) \circ \lambda = D_\lambda\big([\Theta_a, \Theta_b] \circ \lambda\big),$$

since each expression agrees on every generator v of L. Consequently, in $\mathrm{Rel}(\mathrm{ad}_\lambda)$, we have

$$\delta_{\mathrm{ad}_\lambda}\big(0, [\Theta_a, \Theta_b] \circ \lambda\big) = \big(0, (D[\Theta_a, \Theta_b]) \circ \lambda\big).$$

Returning to the above, we may write

$$\begin{aligned}
\llbracket ((-1)^p a, S_a), ((-1)^q b, S_b) \rrbracket = (-1)^{pq} &\llbracket ((-1)^q b, S_b), ((-1)^p a, S_a) \rrbracket \\
&- \delta_{\mathrm{ad}_\lambda}\big(0, [\Theta_a, \Theta_b] \circ \lambda\big).
\end{aligned}$$

From the observation of Remark 3.5, it now follows that the pairing of (10) of cycles of $\mathrm{Rel}(\mathrm{ad}_\psi)$ satisfies the Whitehead identity (ii) up to boundaries, and in particular induces a pairing $H_*\big(\mathrm{Rel}(\mathrm{ad}_\psi)\big)$ that satisfies (ii).

The proof of (iii) follows the same line of argument making use of Proposition 3.4. Indeed, using it, we may amplify the diagram of Remark 3.5 into the following commutative diagram of chain maps:

$$\begin{tikzcd}
L(z,c) \arrow[rrr, "\mathrm{ad}_\lambda"] \arrow[dr, "(\zeta_z|\zeta_c)_\psi"] \arrow[dd, "\phi_z"'] & & & \mathrm{Der}(L, L(z,c);\lambda) \arrow[dl, "\left((\zeta_z|\zeta_c)_\psi\right)_*"] \arrow[dd, "(\phi_z)_*"] \\
& K \arrow[r, "\mathrm{ad}_\psi"] & \mathrm{Der}(L, K;\psi) & \\
L(a,b,c) \arrow[rrr, "\mathrm{ad}_\lambda"'] \arrow[ur, "(\zeta_a|\zeta_b|\zeta_c)_\psi"'] & & & \mathrm{Der}(L, L(a,b,c);\lambda) \arrow[ul, "\left((\zeta_a|\zeta_b|\zeta_c)_\psi\right)_*"']
\end{tikzcd}$$

which in turn induces chain maps of mapping cones of the horizontal maps. Each term in (iii) may thus be identified as the image of its counterpart in the induced map of mapping cones given by the lower trapezoid of this diagram. Again the corresponding identity holds up to a boundary in this lower $\mathrm{Rel}(\mathrm{ad}_\lambda)$, in which $\lambda : L \to L(a,b,c)$. As in the preceding case, this passes to the needed identity in $\mathrm{Rel}(\mathrm{ad}_\psi)$ by the chain maps induced by the chain map $(\zeta_a \mid \zeta_b \mid \zeta_c)_\psi : L(a,b,c) \to K$.

It remains to show that, if either ζ_a or ζ_b is a boundary, then $[\![\zeta_a, \zeta_b]\!]$ is a boundary also so that the pairing passes to homology. Suppose then that $\zeta_a = \delta_{\mathrm{ad}_\psi}(\zeta_c)$ is a boundary in $\mathrm{Rel}_p(\mathrm{ad}_\psi)$ so that $\zeta_c = (\chi_c, \theta_c) \in \mathrm{Rel}_{p+1}(\mathrm{ad}_\psi)$ satisfies

$$d_K(\chi_c) = -\chi_a \quad \text{and} \quad D_\psi(\theta_c) = \theta_a - \mathrm{ad}_\psi(\chi_c).$$

We show that $[\![\zeta_a, \zeta_b]\!]$ bounds also. To do this, we would like to form the product $[\![\zeta_c, \zeta_b]\!] \in \mathrm{Rel}_{p+q}(\mathrm{ad}_\psi)$ but our construction above requires ζ_c to be a $\delta_{\mathrm{ad}_\psi}$-cycle. To accomodate non-cycles, we modify the construction of $L(a,b)$ as follows. Define a DG Lie algebra $(\widehat{L}(a,b,c), \widehat{d}) = \boldsymbol{L}(V, a, b, c, V^a, V^b, V^c; \widehat{d})$ with $|c| = p$, and with differential given by $\widehat{d}(v) = d_L(v)$, $\widehat{d}(a) = \widehat{d}(b) = 0$, $\widehat{d}(c) = -a$ and

$$\begin{aligned}
\widehat{d}(S_a(v)) &= (-1)^{p-1}[a, v] + (-1)^p S_a(d_L(v)) \\
\widehat{d}(S_b(v)) &= (-1)^{q-1}[b, v] + (-1)^q S_b(d_L(v)) \\
\widehat{d}(S_c(v)) &= (-1)^{p-1}[c, v] + S_a(v) + (-1)^{p+1} S_c(d_L(v))
\end{aligned}$$

for $v \in V$. The formula for the boundary of S_c gives the relation

$$D_\lambda(S_c) = (-1)^{p-1}\mathrm{ad}_\lambda(c) + S_a \in \mathrm{Der}(L, \hat{L}(a,b,c); \lambda)$$

where $\lambda : L \to \hat{L}(a,b,c)$ is the inclusion. Define Θ_b, Θ_c in $\mathrm{Der}(\hat{L}(a,b,c))$ as above:

$$\Theta_x(v) = S_x(v) \quad \text{and} \quad \Theta_x(y) = \Theta_x(V^y) = 0$$

for $x = b, c$, $y = a, b, c$ and $v \in V$. The classes ζ_a, ζ_b and ζ_c induce, as above, a DG Lie algebra map

$$\varphi : (\widehat{L}(a,b,c), d) \to (K, d_K) \quad \text{with}$$
$$\varphi(x) = (-1)^{|x|+1}\chi_x,\ \varphi(c) = (-1)^p\chi_c,\ \varphi(v) = \psi(v) \quad \text{and} \quad \varphi(S_y(v)) = \theta_y(v)$$

for $x = a, b$, $y = a, b, c$ and $v \in V$. Writing $\widehat{\lambda} : L \to \widehat{L}(a,b,c)$ for the inclusion, a straightforward computation in $(\mathrm{Rel}\Big(\mathrm{ad}_{\widehat{\lambda}}\Big), \delta_{\mathrm{ad}_{\widehat{\lambda}}})$ shows that

$$\delta_{\mathrm{ad}_{\widehat{\lambda}}}\Big((-1)^q[c,b], -\{\Theta_c, \Theta_b\} \circ \widehat{\lambda}\Big) = \Big((-1)^q[a,b], \{\Theta_a, \Theta_b\} \circ \widehat{\lambda}\Big).$$

That is, the universal example of a Whitehead product, constructed now in the complex $(\mathrm{Rel}\Big(\mathrm{ad}_{\widehat{\lambda}}\Big), \delta_{\mathrm{ad}_{\widehat{\lambda}}})$, is a boundary there. As in Remark 3.5, the map φ induces a chain map $\phi : (\mathrm{Rel}\Big(\mathrm{ad}_{\widehat{\lambda}}\Big), \delta_{\mathrm{ad}_{\widehat{\lambda}}}) \to (\mathrm{Rel}(\mathrm{ad}_\psi), \delta_{\mathrm{ad}_\psi})$. As $\phi\Big((-1)^q[a,b], \{\Theta_a, \Theta_b\} \circ \widehat{\lambda}\Big) = [\![\zeta_a, \zeta_b]\!]$, it follows that the latter is a boundary. □

Finally, we obtain a Whitehead product on $H_*(\mathrm{Der}(L, K; \psi))$ when L is free as a direct consequence of the above. Suppose $\theta_a \in \mathrm{Der}_p(L, K; \psi)$ and $\theta_b \in \mathrm{Der}_q(L, K; \psi)$ are D_ψ-cycles. Set $\zeta_a^* = (0, \theta_a) \in \mathrm{Rel}_p(\mathrm{ad}_\psi)$ and $\zeta_b^* = (0, \theta_b) \in \mathrm{Rel}_q(\mathrm{ad}_\psi)$. Both are $\delta_{\mathrm{ad}_\psi}$-cycles. Thus we can write

$$[\![\zeta_a^*, \zeta_b^*]\!] = \big(0, \{\zeta_a^*, \zeta_b^*\}\big) \in \mathrm{Rel}_{p+q-1}(\mathrm{ad}_\psi).$$

We define

$$[\![\theta_a, \theta_b]\!] =_{\mathrm{def}} \{\zeta_a^*, \zeta_b^*\} \ \in \mathrm{Der}_{p+q-1}(L, K; \psi). \tag{11}$$

We then obtain:

COROLLARY 3.7. *Let $\psi : (L, d_L) \to (K, d_K)$ be a DG Lie algebra map with L free. The bilinear pairing $[\![\ ,\ ]\!]$ defined on cycles of $(\mathrm{Der}(L, K; \psi), \delta_\psi)$ by (11) induces a Whitehead product $[\ ,\ ]_w$ on $H_*(\mathrm{Der}(L, K; \psi))$.* □

4. Iterated products of the universal example.

In this section, we continue with our algebraic development and record some formulas and results that will be useful for our applications. Let (L,d) be a given free DG Lie algebra and $(L(a,b),\partial_{a,b})$ the associated DG Lie algebra for $|a|=p-1$, $|b|=q-1$. We look in detail at the universal example of a Whitehead product

$$\{\Theta_a,\Theta_b\}\circ\lambda\in \mathrm{Der}_{p+q-1}(L,L(a,b);\lambda).$$

Begin in the DG Lie algebra $\big(\mathrm{Der}\big(L(a,b)\big),D\big)$. We are interested in the restriction of derivations $\Theta\in\mathrm{Der}(L(a,b))$ to $L\subseteq L(a,b)$, i.e., $\Theta\circ\lambda\in\mathrm{Der}(L,L(a,b);\lambda)$ where $\lambda: L\to L(a,b)$ is the inclusion.

Recall from (6) the definitions of Θ_a,Θ_b of degree p and q in $\big(\mathrm{Der}\big(L(a,b)\big),D\big)$. From the definitions, we have that

$$D_\lambda(\Theta_a\circ\lambda)=(-1)^{p-1}\mathrm{ad}_\lambda(a)\qquad\text{and}\qquad D_\lambda(\Theta_b\circ\lambda)=(-1)^{q-1}\mathrm{ad}_\lambda(b).$$

Amongst the terms that occur in $\{\Theta_a,\Theta_b\}\circ\lambda(v)$, we note that $\partial_{a,b}\circ\Theta_b\circ\Theta_a\circ\lambda(v)=0$, whereas, e.g. $\Theta_b\circ\partial_{a,b}\circ\Theta_a\circ\lambda(v)$ is generally non-zero. Using these facts, we obtain that

$$\begin{aligned}\{\Theta_a,\Theta_b\}\circ\lambda(v)&=(-1)^{p+1}[\Theta_a,D(\Theta_b)]\circ\lambda(v)\\&=(-1)^{p+q}\Theta_a\circ\mathrm{ad}_\lambda(b)(v)+(-1)^{pq}D(\Theta_b)\circ\Theta_a\circ\lambda(v)\\&=(-1)^{p+q}\Theta_a\circ\mathrm{ad}_\lambda(b)(v)+(-1)^{pq+q+1}\Theta_b\circ\partial_{a,b}\circ\Theta_a\circ\lambda(v)\\&=(-1)^{p+q}\Theta_a\circ\mathrm{ad}_\lambda(b)(v)+(-1)^{pq+p+q}\Theta_b\circ\mathrm{ad}_\lambda(a)(v)\\&\quad+(-1)^{(p+1)(q+1)}\Theta_b\circ\Theta_a\circ\lambda(dv),\end{aligned}$$

yielding finally

$$\begin{aligned}\{\Theta_a,\Theta_b\}\circ\lambda(v)=(-1)^{q(p+1)}[b,S_a(v)]+(-1)^p[a,S_b(v)]\\+(-1)^{(p+1)(q+1)}\Theta_b\circ\Theta_a\circ\lambda(dv).\end{aligned}\tag{12}$$

The formulae of the previous section may be extended to iterated Whitehead products. We sketch this here. Suppose given $\delta_{\mathrm{ad}_\psi}$-cycles $\zeta_1,\ldots,\zeta_n\in\mathrm{Rel}_*(\mathrm{ad}_\psi)$, with each $\zeta_i=(\chi_{a_i},\theta_{a_i})\in\mathrm{Rel}_{p_i}(\mathrm{ad}_\psi)$, with $n\geq 2$ and each $p_i\geq 2$. Let $a_1,\ldots,a_n$ be of degrees $p_1-1,\ldots,p_n-1$, and define elements Θ_{a_i} of degree p_i in the DG Lie algebra $\big(\mathrm{Der}\big(L(a_1,\ldots,a_n)\big),D\big)$ as at (6) above. Thus, we have derivations $\Theta_{a_i}\circ\lambda\in\mathrm{Der}(L,L(a_1,\ldots,a_n);\lambda)$ that satisfy $D_\lambda(\Theta_{a_i}\circ\lambda)=(-1)^{p_i-1}\mathrm{ad}_\lambda(a_i)$. Write

$$w(a_1,\ldots,a_n)=[[\ldots[[a_1,a_2],a_3]\ldots a_{n-1}],a_n]$$

for the "left-justified" iterated bracket in $L(a_1,\ldots,a_n)$ and, similarly,

$$w(\Theta_{a_1},\ldots,\Theta_{a_n})=\{\{\ldots\{\{\Theta_{a_1},\Theta_{a_2}\},\Theta_{a_3}\}\ldots\Theta_{a_{n-1}}\},\Theta_{a_n}\}$$

for the iterated product of derivations, using the pairing of (4) in $\mathrm{Der}(L(a_1,\ldots,a_n))$. Then

$$(\pm w(a_1,\ldots,a_n),\pm w(\Theta_{a_1},\ldots,\Theta_{a_n})\circ\lambda) \tag{13}$$

is a cycle of $(\mathrm{Rel}(\mathrm{ad}_\lambda),\delta_{\mathrm{ad}_\lambda})$ that is the universal example of iterated Whitehead products of this form.

The ζ_i induce a DG Lie algebra map as in (8) $(\zeta_I)_\psi : L(a_1,\ldots,a_n)\to(K,d_K)$. We define the iterated Whitehead product as

$$\begin{aligned}&[\![[\![\ldots[\![[\![\zeta_1,\zeta_2]\!],\zeta_3]\!]\ldots\zeta_{n-1}]\!],\zeta_n]\!]\\&\qquad=\Big(\pm(\zeta_I)_\psi\big(w(a_1,\ldots,a_n)\big),\pm((\zeta_I)_\psi)_*w(\Theta_{a_1},\ldots,\Theta_{a_n})\circ\lambda\Big).\end{aligned} \tag{14}$$

We conclude this section by observing that the Whitehead products we have constructed for a DG Lie algebra map are invariant under quasi-isomorphisms in the second variable.

THEOREM 4.1. *Let $\psi:(L,d_L)\to(K,d_K)$ be a map between connected DG Lie algebras with (L,d_L) finitely generated and minimal. Suppose $\phi:(K,d_K)\to(K',d_{K'})$ is a surjective DG Lie algebra map such that $H(\phi):H_*(K)\to H_*(K')$ is an isomorphism. Then composition with ϕ induces isomorphisms*

$$H_*(\mathrm{Der}(L,K;\psi))\cong H_*(\mathrm{Der}(L,K';\phi\circ\psi))\quad\textit{and}\quad H_*(\mathrm{Rel}(\mathrm{ad}_\psi))\cong H_*(\mathrm{Rel}(\mathrm{ad}_{\phi\circ\psi})).$$

Further, these are isomorphisms of Whitehead algebras with all spaces equipped with the Whitehead products constructed in Theorem 3.6 *and Corollary* 3.7.

PROOF. Write $\psi'=\phi\circ\psi$; for a cycle $\zeta_a=(\chi_a,\theta_a)\in\mathrm{Rel}(\mathrm{ad}_\psi)$, use ζ_a' to denote the corresponding cycle $(\phi(\chi_a),\phi_*\circ\theta_a)\in\mathrm{Rel}(\mathrm{ad}_{\psi'})$. Then $\phi\circ(\zeta_a\mid\zeta_b)_\psi=(\zeta_a'\mid\zeta_b')_{\psi'}:L(a,b)\to K'$, and we have a commutative diagram as follows:

$$\begin{tikzcd}
K \arrow[rrr, "\mathrm{ad}_\psi"] \arrow[dd, "\phi"'] & & & \mathrm{Der}(L,K;\psi) \arrow[dd, "\phi_*"] \\
& L(a,b) \arrow[ul, "(\zeta_a|\zeta_b)_\psi"'] \arrow[dl, "(\zeta'_a|\zeta'_b)_{\psi'}"] \arrow[r, "\mathrm{ad}_\lambda"] & \mathrm{Der}(L,L(a,b);\lambda) \arrow[ur, "\left((\zeta_a|\zeta_b)_\psi\right)_*"] \arrow[dr, "\left((\zeta'_a|\zeta'_b)_{\psi'}\right)_*"] & \\
K' \arrow[rrr, "\mathrm{ad}_{\psi'}"'] & & & \mathrm{Der}(L,K';\psi')
\end{tikzcd}$$

From this, it is clear that the map ϕ induces a commutative diagram of long exact homology sequences of the respective adjoints ad_ψ and $\mathrm{ad}_{\psi'}$, and also that the construction of our Whitehead product is natural with respect to the maps induced by composition with ϕ.

Thus by the Five-Lemma it suffices to prove composition with ϕ induces an isomorphism $H_*(\mathrm{Der}(L,K;\psi)) \cong H_*(\mathrm{Der}(L,K';\phi\circ\psi))$. The proof of this fact is an adaptation of a standard result for lifting maps with domain a minimal DG Lie algebra (see [**5**, Proposition 22.11]). We give the full details to show composition by ϕ induces a surjection on homology. The proof of injectivity is similar and so we omit it.

Write $L = \boldsymbol{L}(V;d_L)$ where $V = \boldsymbol{Q}(x_1,\ldots,x_n)$ and the x_i are homogeneous of nondecreasing degree. Let $\theta' \in \mathrm{Der}_p(L,K';\phi\circ\psi)$ be a D-cycle where we write $D = D_{\phi\circ\psi}$. We define $\theta \in \mathrm{Der}_p(L,K;\psi)$ and a derivation $\theta'' \in \mathrm{Der}_{p+1}(L,K';\phi\circ\psi)$ so that

$$D_\psi(\theta) = 0 \quad \text{and} \quad \phi\circ\theta = \theta' + D(\theta''). \tag{15}$$

We define θ and θ'' on our basis for V by induction.

Observe that, since $d_L(x_1) = 0$, $d_{K'}(\theta'(x_1)) = D(\theta')(x_1) = 0$. Since $\phi : K \to K'$ induces a homology isomorphism we can choose a d_K-cycle $\chi \in K$ such that $\phi(\chi) = \theta'(x_1) + d_{K'}(\alpha)$ for some $\alpha \in K'$. We set $\theta(x_1) = \chi$ and $\theta''(x_1) = \alpha$.

Now suppose $\theta(x_j)$ and $\theta''(x_j)$ are defined for $j < r$, such that (15) holds on the Lie subalgebra $\boldsymbol{L}(x_1,\ldots,x_{r-1})$ of L. Set $y = (-1)^p\theta(d_L(x_r)) \in K$. Applying our induction hypothesis, we see $d_K(y) = 0$. Furthermore, since $D(\theta') = 0$, we have

$$\begin{aligned} d_{K'}(\theta'(x_r)) &= (-1)^p\theta'(d_L(x_r)) \\ &= \phi(y) + (-1)^{p+1}D(\theta'')(d_L(x_r)) = \phi(y) + (-1)^{p+1}d_{K'}(\theta''(d_L(x_r))). \end{aligned}$$

Thus $\phi(y)$ is a boundary in $(K',d_{K'})$ and so we can choose $z \in K$ with $d_K(z) = y$. Next note that $\theta'(x_r) + (-1)^p\theta''(d_L(x_r)) - \phi(z)$ is a $d_{K'}$-cycle. Thus, as above, we can find a d_K-cycle $\overline{z}$ and $\alpha \in K'$ such that

$$\phi(\overline{z}) = \theta'(x_r) + (-1)^p\theta''(d_L(x_r)) - \phi(z) + d_{K'}(\alpha).$$

We put $\theta(x_r) = \overline{z} + z$ and $\theta''(x_r) = \alpha$ and (15) is satisfied on $\boldsymbol{L}(x_1, \ldots, x_r)$. $\square$

5. Whitehead product formulae for function space components.

In this section we return to the topological setting and prove our main result, the identification of the Whitehead product in the rational homotopy groups of a function space component. Let $f : X \to Y$ be a map between simply connected CW complexes of finite type with X now a finite complex. Let $\mathscr{L}_f : (\mathscr{L}_X, d_X) \to (\mathscr{L}_Y, d_Y)$ be the Quillen minimal model for f. We first recall the identifications

$$\pi_p(\mathrm{map}(X, Y; f)) \otimes \boldsymbol{Q} \cong H_p(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f}))$$

for $p > 1$ given in [**10**, Theorem 3.1].

The adjoint of a representative $a : S^p \to \mathrm{map}(X, Y; f)$ of a homotopy class $\alpha \in \pi_p(\mathrm{map}(X, Y; f))$ is a map $A : S^p \times X \to Y$. By Theorem 2.1, the Quillen minimal model for A is a map

$$\mathscr{L}_A : \big(\mathscr{L}_X(a), \partial_a\big) \to (\mathscr{L}_Y, d_Y).$$

Define $\theta_a \in \mathrm{Der}_p(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)$ by setting $\theta_a(v) = \mathscr{L}_A(S_a(v))$ for $v \in V$ and extending as an $\mathscr{L}_f$-derivation. Then $\chi_a = (-1)^p\mathscr{L}_A(a)$ is a cycle of degree $p - 1$ in $\mathscr{L}_Y$, and $\zeta_a = (\chi_a, \theta_a) \in \mathrm{Rel}_p(\mathrm{ad}_{\mathscr{L}_f})$ is a $\delta_{\mathrm{ad}_{\mathscr{L}_f}}$-cycle. Set

$$\Phi'(\alpha) = \langle \zeta_a \rangle \in H_p(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f})). \tag{16}$$

The map Φ' is then a homomorphism whose rationalization $\Phi : \pi_p(\mathrm{map}(X, Y; f)) \otimes \boldsymbol{Q} \to H_p(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f}))$ is an isomorphism for $p \geq 2$.

Given two homotopy classes $\alpha \in \pi_p(\mathrm{map}(X, Y; f))$ and $\beta \in \pi_q(\mathrm{map}(X, Y; f))$, their Whitehead product $\gamma = [\alpha, \beta]_w$ has adjoint C given by

$$\begin{tikzcd} S^{p+q-1} \times X \arrow[r, "\eta\times 1"] \arrow[rr, bend right, "C"'] & (S^p \vee S^q) \times X \arrow[r, "(A|B)_f"] & Y \end{tikzcd}$$

where η is the universal example of the Whitehead product. Let $\zeta_a = (\chi_a, \theta_a) \in \mathrm{Rel}_p(\mathrm{ad}_{\mathscr{L}_f})$ and $\zeta_b = (\chi_b, \theta_b) \in \mathrm{Rel}_q(\mathrm{ad}_{\mathscr{L}_f})$ satisfy $\langle \zeta_a \rangle = \Phi'(\alpha)$ and $\langle \zeta_b \rangle = \Phi'(\beta)$.

LEMMA 5.1. *The map*

$$(\zeta_a \mid \zeta_b)_{\mathscr{L}_f} : \big(\mathscr{L}_X(a,b), \partial_{a,b}\big) \to (\mathscr{L}_Y, d_Y)$$

defined by (8) *is the Quillen minimal model for*

$$(A \mid B)_f : (S^p \vee S^q) \times X \to Y.$$

PROOF. Denote by $\mathscr{L}$ the composite of the Sullivan and the Quillen functors. That is, we write $\mathscr{L}(Z)$ to denote the DG Lie algebra obtained by applying the Quillen functor to the coalgebra dual of $A^*(Z)$, which is the Sullivan functor applied to Z. (See [**5**, Section 22(e)] or [**15**, I.1(7)].) For a space Z, denote by $\eta_Z : \mathscr{L}_Z \to \mathscr{L}(Z)$ the Quillen minimal model of Z. To establish the Lemma, we want the diagram

$$\begin{tikzcd}
\mathscr{L}_X(a,b) \arrow[r, "(\zeta_a|\zeta_b)_{\mathscr{L}_f}"] \arrow[d, "\eta_{(S^p\vee S^q)\times X}"'] & \mathscr{L}_Y \arrow[d, "\eta_Y"] \\
\mathfrak{L}\big((S^p \vee S^q) \times X\big) \arrow[r, "\mathfrak{L}\left((A|B)_f\right)"'] & \mathfrak{L}(Y).
\end{tikzcd}$$

to be homotopy commutative, in the DG Lie algebra sense. Following [**15**, II.5.(20)], this means we seek a DG Lie algebra map $\mathscr{H} : \mathscr{L}_X(a,b) \to (t, dt) \otimes \mathscr{L}(Y)$ such that $p_0 \circ \mathscr{H} = \eta_Y \circ (\zeta_a \mid \zeta_b)_{\mathscr{L}_f}$ and $p_1 \circ \mathscr{H} = \mathscr{L}\big((A \mid B)_f\big) \circ \eta_{(S^p \vee S^q)\times X}$. Now we have a pushout of DG Lie algebras

$$\begin{tikzcd}
\mathscr{L}_X \arrow[r, "\lambda_a"] \arrow[d, "\lambda_b"'] & \mathscr{L}_X(a) \arrow[d, "\overline{\lambda_b}"] \\
\mathscr{L}_X(b) \arrow[r, "\overline{\lambda_a}"'] & \mathscr{L}_X(a,b),
\end{tikzcd}$$

where the maps $\lambda_a, \lambda_b, \overline{\lambda_a}, \overline{\lambda_b}$ are the appropriate inclusions. Notice that our desired minimal model $(\zeta_a \mid \zeta_b)_{\mathscr{L}_f}$ is exactly the pushout of the minimal models $\mathscr{L}_A : \mathscr{L}_X(a) \to \mathscr{L}_Y$ and $\mathscr{L}_B : \mathscr{L}_X(b) \to \mathscr{L}_Y$. We will obtain our homotopy $\mathscr{H}$ by pushing out homotopies from $\mathscr{L}_X(a)$ and $\mathscr{L}_X(b)$. To this end, suppose that we have our chosen minimal model $\mathscr{L}_f : \mathscr{L}_X \to \mathscr{L}_Y$, and a DG Lie algebra homotopy $\mathscr{H}_f : \mathscr{L}_X \to (t, dt) \otimes \mathscr{L}(Y)$ that satisfies $p_0 \circ \mathscr{H}_f = \eta_Y \circ \mathscr{L}_f$ and $p_1 \circ \mathscr{H}_f = \mathscr{L}(f) \circ \eta_X$.

As in the proof of [**10**, Proposition A.3], we may assume that the following cube is (strictly) commutative:

$$\begin{tikzcd}
& \mathscr{L}_X(a) \arrow[rr, "\overline{\lambda_b}"] \arrow[dd, "\eta_{S^p\times X}" near start] & & \mathscr{L}_X(a,b) \arrow[dd, "\eta_{(S^p\vee S^q)\times X}"] \\
\mathscr{L}_X \arrow[ur, "\lambda_a"] \arrow[rr, "\lambda_b" near start] \arrow[dd, "\eta_X"'] & & \mathscr{L}_X(b) \arrow[ur, "\overline{\lambda_a}"'] \arrow[dd, "\eta_{S^q\times X}" near start] & \\
& \mathfrak{L}(S^p\times X) \arrow[rr, "\mathfrak{L}_{(i_1\times 1)}" near start] & & \mathfrak{L}((S^p\vee S^q)\times X) \\
\mathfrak{L}(X) \arrow[ur, "\mathfrak{L}_{(i_2)}"] \arrow[rr, "\mathfrak{L}_{(i_2)}"'] & & \mathfrak{L}(S^q\times X) \arrow[ur, "\mathfrak{L}_{(i_2\times 1)}"'] &
\end{tikzcd}$$

Next, in the diagram

$$\begin{tikzcd}
\mathscr{L}_X \arrow[r, "\lambda_a"] \arrow[d, "\eta_X"'] & \mathscr{L}_X(a) \arrow[r, "\mathscr{L}_A"] \arrow[d, "\eta_{S^p\times X}"'] & \mathscr{L}_Y \arrow[d, "\eta_Y"] \\
\mathfrak{L}(X) \arrow[r, "\mathfrak{L}_{(i_2)}"'] & \mathfrak{L}(S^p\times X) \arrow[r, "\mathfrak{L}_{(A)}"'] & \mathfrak{L}(Y),
\end{tikzcd}$$

the left-hand square commutes and furthermore, the right-hand square commutes up to homotopy, but we may assume that the homotopy $\mathscr{H}_a : \mathscr{L}_X(a) \to (t,dt)\otimes \mathscr{L}(Y)$ extends the homotopy $\mathscr{H}_f$, that is, that we have $\mathscr{H}_a \circ \lambda_a = \mathscr{H}_f$. This last assertion is easily justified by adapting the usual lifting lemma: rather than lift $\mathscr{L}(A)\circ\eta_{S^p\times X}$ through the quasi-isomorphism η_Y starting with the elements of lowest degree in $\mathscr{L}_X(a)$, we may start with the lift already defined on $\mathscr{L}_X$ as $\mathscr{L}_f$, with $\eta_Y\circ\mathscr{L}_A = \eta_Y\circ\mathscr{L}_f$ and $\mathscr{L}(A)\circ\eta_{S^p\times X} = \mathscr{L}(f)\circ\eta_X$ homotopic by $\mathscr{H}_f$ when resticted to $\mathscr{L}_X$. (See [**7**, Proposition 10.4] for the corresponding result in the DG algebra setting.) We argue similarly with b and B replacing a and A respectively. This gives us the pushout which defines $\mathscr{H}$. We check that $\mathscr{H}$ has the desired properties.

$$\begin{tikzcd}
\mathscr{L}_X \arrow[r, "\lambda_a"] \arrow[d, "\lambda_b"'] & \mathscr{L}_X(a) \arrow[d, "\overline{\lambda_a}"'] \arrow[ddr, bend left, "\mathscr{H}_a"] & \\
\mathscr{L}_X(b) \arrow[r, "\overline{\lambda_b}"] \arrow[drr, bend right, "\mathscr{H}_b"'] & \mathscr{L}_X(a,b) \arrow[dr, dotted, "\mathscr{H}"] & \\
 & & (t, dt) \otimes \mathbf{L}(Y)
\end{tikzcd}$$

□

We next give an official statement of our work in Section 2. Define

$$\Gamma : \big(\mathscr{L}_X(c), \partial_c\big) \to \big(\mathscr{L}_X(a,b), \partial_{a,b}\big)$$

by setting $\Gamma(\chi) = \chi$ for $\chi \in \mathscr{L}_X, \Gamma(c) = (-1)^{p-1}[a,b]$ and

$$\Gamma(S_c(v)) = \{\Theta_a, \Theta_b\} \circ \lambda(v)$$

where $\Theta_a, \Theta_b \in \mathrm{Der}(\mathscr{L}_X(a,b))$ are as defined in (6) and $\lambda : \mathscr{L}_X \to \mathscr{L}_X(a,b)$ is the inclusion.

LEMMA 5.2. *The map*

$$\Gamma : \big(\mathscr{L}_X(c), \partial_c\big) \to \big(\mathscr{L}_X(a,b), \partial_{a,b}\big)$$

is the Quillen model for

$$\eta \times 1 : S^{p+q-1} \times X \to (S^p \vee S^q) \times X.$$

PROOF. We have a commutative diagram

$$\begin{tikzcd}
(\mathscr{L}_X(c), \partial_c) \arrow[r, "\Gamma"] \arrow[d] & (\mathscr{L}_X(a,b), \partial_{a,b}) \arrow[d] \\
\boldsymbol{L}(c; 0) \oplus (\mathscr{L}_X, d_X) \arrow[r, "\phi"'] & \boldsymbol{L}(a, b; 0) \oplus (\mathscr{L}_X, d_X)
\end{tikzcd}$$

where the vertical maps are the projections and $\phi(c) = (-1)^{p-1}[a,b]$ and $\phi(\chi) = \chi$ for $\chi \in \mathscr{L}_X$. Since ϕ is evidently a (non-minimal) Quillen model for $\eta \times 1$, the result follows from uniqueness of the Quillen model of a map. □

Combining these facts we obtain our identification.

THEOREM 5.3. *Let $f : X \to Y$ be a map between simply connected CW complexes of finite type with X finite. The map*

$$\Phi' : \pi_p(\mathrm{map}(X, Y; f)) \to H_p(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_X}))$$

defined for $p > 1$ by (16) *preserves Whitehead products where the latter space has the Whitehead product given by Theorem* 3.6*. Thus Φ' induces an isomorphism*

$$\pi_*(\mathrm{map}(X, Y; f)) \otimes \boldsymbol{Q},\ [\,,\,]_w \cong H_*(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_X})),\ [\,,\,]_w$$

of rational Whitehead algebras in degrees > 1.

PROOF. With notation as above and Lemmas 5.1 and 5.2,

$$(\zeta_a \mid \zeta_b)_{\mathscr{L}_f} \circ \Gamma : (\mathscr{L}_X(c), \partial_c) \to (\mathscr{L}_Y, d_Y)$$

is the Quillen minimal model for the adjoint C of $\gamma = [\alpha, \beta] \in \pi_{p+q-1}(\mathrm{map}(X, Y; f))$. Thus $\Phi'(\gamma)$ is represented by the $\delta_{\mathrm{ad}_{\mathscr{L}_f}}$-cycle $\zeta_c = (\chi_c, \theta_c) \in \mathrm{Rel}_{p+q-1}(\mathrm{ad}_{\mathscr{L}_f})$ with

$$\chi_c = (-1)^{p+q-1}(\zeta_a \mid \zeta_b)_{\mathscr{L}_f} \circ \Gamma(c) = (-1)^p[\chi_a, \chi_b] \in (\mathscr{L}_Y)_{p+q-2} \quad \text{while}$$
$$\theta_c(v) = (\zeta_a \mid \zeta_b)_{\mathscr{L}_f} \circ \Gamma(S_c(v)) = (\zeta_a \mid \zeta_b)_{\mathscr{L}_f}\{\Theta_a, \Theta_b\} \circ \lambda(v) \in \mathrm{Der}_{p+q-1}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f).$$

Thus

$$\Phi'(\gamma) = \left\langle [\![\zeta_a, \zeta_b]\!] \right\rangle = [\langle \zeta_a \rangle, \langle \zeta_b \rangle]_w$$

by Theorem 3.6 and the definition of the pairing $[\![\,,\,]\!]$ at (10). □

We now apply the same line of reasoning to the case of the based function space. We first recall the homomorphism,

$$\Psi' : \pi_p(\mathrm{map}_*(X, Y; f)) \to \pi_p(\mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)) \tag{17}$$

from [**10**, Theorem 3.1] inducing an isomorphism after rationalization for $p > 1$. Given $\alpha \in \pi_p(\mathrm{map}_*(X, Y; f))$ we have $\Psi'(\alpha) = \langle \theta_a \rangle$ where $\theta_a \in \mathrm{Der}_p(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)$ is the $D_{\mathscr{L}_f}$-cycle given by $\theta_a = \mathscr{L}_A \circ S_a$ where $\mathscr{L}_A : (\mathscr{L}_X(a), \partial_a) \to (\mathscr{L}_Y, d_Y)$ is the Quillen minimal model for the adjoint $A : S^p \times X \to Y$ of α. We prove

THEOREM 5.4. *Let $f : X \to Y$ be a map between simply connected CW complexes of finite type with X finite. The map*

$$\Psi' : \pi_p(\mathrm{map}_*(X, Y; f)) \to H_p(\mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f))$$

defined for $p > 1$ by (17) *preserves Whitehead products where the latter space has the Whitehead product given by Corollary* 3.7. *Thus Ψ' induces an isomorphism*

$$\pi_*(\mathrm{map}_*(X, Y; f)) \otimes \boldsymbol{Q},\ [\,,\,]_w \cong H_*(\mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)),\ [\,,\,]_w$$

of rational Whitehead algebras in degrees > 1.

PROOF. Given $\alpha, \beta \in \pi_*(\mathrm{map}_*(X, Y; f))$ of degrees p and q with Whitehead product $\gamma = [\alpha, \beta] \in \pi_{p+q-1}(\mathrm{map}_*(X, Y; f))$, the class $\Psi'(\gamma)$ is represented by

$$(\zeta_a^* \mid \zeta_b^*)_{\mathscr{L}_f} \circ \Gamma \circ S_c = [\![\theta_a, \theta_b]\!] \in \mathrm{Der}_{p+q-1}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)$$

by Lemmas 5.1 and 5.2 and definition of the bilinear pairing $[\![\,,\,]\!]$ in (11). The result now follows from Corollary 3.7. $\square$

REMARK 5.5. For $\alpha_1, \ldots, \alpha_n \in \pi_*(\mathrm{map}(X, Y; f))$, write

$$w(\alpha_1, \ldots, \alpha_n) = \Big[\Big[\ldots\big[[\alpha_1, \alpha_2]_w, \alpha_3\big]_w, \ldots, \alpha_{n-1}\Big]_w, \alpha_n\Big]_w$$

for their "left-justified" iterated Whitehead product. The argument above may easily be extended, using the algebraic universal Whitehead product indicated in (13), and the topological universal example for such Whitehead products, namely

$$w(\iota_1, \ldots, \iota_n) : S^{p_1+\cdots+p_n-n+1} \to S^{p_1} \vee \cdots \vee S^{p_n}.$$

Using Proposition 3.4 and the above arguments, we may show that the algebraic iterated Whitehead product indicated in (14) in Section 4 corresponds with $w(\alpha_1, \ldots, \alpha_n)$ under the map Φ'. We have no immediate need for this, and so we omit details.

6. Whitehead length of function space components.

We apply our formulae to study the Whitehead length of function space components. To begin, we make some remarks concerning the sensitivity of the invariants

$$\mathrm{WL}(\mathrm{map}(X,Y;f)) \quad \text{and} \quad \mathrm{WL}(\mathrm{map}_*(X,Y;f))$$

to the (homotopy class of the) map $f : X \to Y$. For example, in the case $X = Y$ and $f = 1$, the space $\mathrm{map}(X,X;1)$ is a topological monoid, and so $\mathrm{WL}(\mathrm{map}(X,X;1)) = 1$. When Y is an H-space, so too is $\mathrm{map}(X,Y)$, and hence $\mathrm{WL}(\mathrm{map}(X,Y;f) = 1$ for any component. Dually, if X is a co-H-space, then $\mathrm{map}_*(X,Y)$ is an H-space, and hence $\mathrm{WL}(\mathrm{map}_*(X,Y;f)) = 1$ for any component of the based mapping space. On the other hand, we have the following fact concerning the null-component which shows that we may easily have an abundance of non-zero Whitehead products in the free function space. Recall that we are only considering $\pi_{\geq 2}\big(\mathrm{map}(X,Y;f)\big)$ here.

THEOREM 6.1. *Let Y be any space. Then*

$$\max\{\mathrm{WL}(Y), \mathrm{WL}(\mathrm{map}_*(X,Y;0))\} \leq \mathrm{WL}(\mathrm{map}(X,Y;0)).$$

If the universal cover of Y has finite rational type then

$$\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X,Y;0)) = \mathrm{WL}_{\boldsymbol{Q}}(Y).$$

PROOF. The first inequality follows from the evaluation fibration $\mathrm{map}_*(X,Y;0) \to \mathrm{map}(X,Y;0) \to Y$. On the one hand, the obvious section $s : Y \to \mathrm{map}(X,Y;0)$ implies that Y is a retract of $\mathrm{map}(X,Y;0)$. On the other hand, s implies that the fibre inclusion $\mathrm{map}_*(X,Y;0) \to \mathrm{map}(X,Y;0)$ induces an injection on homotopy groups.

For the rational result, we start with a nice observation of Brown-Szczarba [**2**]: writing $\Omega_0 Y$ for the connected component of the constant loop in ΩY, we have $\Omega_0(\mathrm{map}(X,Y;0)) \approx \mathrm{map}(X,\Omega_0 Y;0)$. Next, by [**8**, Theorem 4.10]

$$\mathrm{Hnil}(\mathrm{map}(X,\Omega_0 Y;0)) = \mathrm{Hnil}(\Omega_0 Y)$$

where $\mathrm{Hnil}(G)$ of a loop-space G denotes the *homotopical nilpotency* of G in the sense of Berstein-Ganea [**1**]. Taking $Y = Y_{\boldsymbol{Q}}$ the result follows from the identity $\mathrm{Hnil}(\Omega_0 Y_{\boldsymbol{Q}}) = \mathrm{WL}_{\boldsymbol{Q}}(Y)$ [**14**, Theorem 3]. □

Recall that a simply connected space Y is *coformal* if there is a DG Lie algebra map $\rho : (\mathscr{L}_Y, d_Y) \to (\pi_*(\Omega Y) \otimes \boldsymbol{Q}, 0)$ inducing an isomorphism on homology (see [**5**, p. 334, Example 7]).

THEOREM 6.2. *Let X be a finite simply connected CW complex and Y a simply connected coformal complex of finite type. Then for all $f : X \to Y$ we have*

$$\max\{\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X,Y;f)),\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X,Y;f))\} \leq \mathrm{WL}_{\boldsymbol{Q}}(Y).$$

PROOF. By Theorem 4.1, we may replace $(\mathscr{L}_Y, d_Y)$ by $(H(\mathscr{L}_Y), 0) = (\pi_*(\Omega Y) \otimes \boldsymbol{Q}, 0)$ when we apply Theorem 5.3. We say that a cycle $\zeta = (\chi, \theta) \in \mathrm{Rel}_*(\mathrm{ad}_{\mathscr{L}_f})$ is *of length $\geq r$ in $H(\mathscr{L}_Y)$* if $\chi \in H(\mathscr{L}_Y)$ is of bracket length $\geq r$ and, when applied to a generator $v \in \boldsymbol{L}(V) = \mathscr{L}_X$, $\theta(v)$ is also of bracket length $\geq r$ in $H(\mathscr{L}_Y)$. The result is proved by arguing that iterated Whitehead products in $H_*(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f}))$ of length r are represented by cycles of length $\geq r$ in $H(\mathscr{L}_Y)$.

To see this, consider two cycles ζ_a and ζ_b, and suppose that ζ_a is of length $\geq r$ in $H(\mathscr{L}_Y)$. According to (12), $\{\Theta_a, \Theta_b\} \circ \lambda(v)$ is contained in the ideal of $\mathscr{L}_X(a,b)$ generated by a and $S_a(v)$, and also is of length ≥ 2. Therefore, when the map $(\zeta_a \mid \zeta_b)_{\mathscr{L}_f}$ is applied to it, we obtain an element of bracket length $\geq (r+1)$ in $H(\mathscr{L}_Y)$. Likewise for the bracket $[a,b]$. It follows that a cocycle representative of $[\![\zeta_a, \zeta_b]\!]$ is of length $\geq (r+1)$ in $H(\mathscr{L}_Y)$. An easy induction using this completes the proof. $\square$

We say a simply connected CW complex X is a *rational co-H-space* if $X_{\boldsymbol{Q}}$ is homotopy equivalent to a wedge of spheres. We remarked above that, if X is a co-H-space, then Whitehead products vanish in any component of the based mapping space. The following result provides a large class of examples of free function spaces with vanishing rational Whitehead products.

THEOREM 6.3. *Let X be a finite rational co-H-space and Y a simply connected, coformal complex of finite type. Suppose f induces a surjection on rational homotopy groups. Then* $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X,Y;f)) = 1$.

PROOF. Since X is a rational co-H-space, the differential d_X in the Quillen minimal model for X vanishes. Since Y is coformal, we may replace $(\mathscr{L}_Y, d_Y)$ by $(H(\mathscr{L}_Y), 0)$, and view $\mathscr{L}_f$ as a map $\mathscr{L}_X \to H(\mathscr{L}_Y)$, when we apply Theorem 5.3. Suppose given a pair $\zeta_a = (\chi_a, \theta_a) \in \mathrm{Rel}_p(\mathrm{ad}_{\mathscr{L}_f})$ and $\zeta_b = (\chi_b, \theta_b) \in \mathrm{Rel}_q(\mathrm{ad}_{\mathscr{L}_f})$ of $\delta_{\mathrm{ad}_{\mathscr{L}_f}}$-cycles; with $\chi_a, \chi_b \in H(\mathscr{L}_Y)$ and $\theta_a, \theta_b \in \mathrm{Der}_*(\mathscr{L}_X, H(\mathscr{L}_Y); \mathscr{L}_f)$. Using the fact that

$$(\zeta_a \mid \zeta_b)_{\mathscr{L}_f} : \mathscr{L}_X(a,b) \to H(\mathscr{L}_Y)$$

is a DG Lie algebra map, we see that $[\chi_a, \mathscr{L}_f(v)] = \pm(\zeta_a \mid \zeta_b)_{\mathscr{L}_f}\big(\partial_{a,b} S_a(v)\big) = 0$, for any $v \in \mathscr{L}_X$. By assumption, $\mathscr{L}_f$ is surjective and it follows that the bracket of χ_a with any element of $H(\mathscr{L}_Y)$ is zero. A similar argument yields the same conclusion for χ_b. Finally, we obtain from (12) that $\{\Theta_a, \Theta_b\} \circ \lambda(v) = \pm[b, S_a(v)]$ $\pm[a, S_b(v)]$. It follows that in $H_*\big(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f})\big)$, we have

$$[\![\zeta_a, \zeta_b]\!] = \big(\pm [\chi_a, \chi_b], \pm[\chi_a, \theta_b(v)] \pm [\chi_b, \theta_a(v)]\big) = (0, 0).$$

The result follows from Theorem 5.3. □

In [**6**], Ganea proved $\mathrm{WL}(\mathrm{map}_*(X, Y; 0)) \leq \mathrm{cat}(X)$. We give a rational version of this inequality which applies to all components. Recall the *rational cone length* $\mathrm{cl}_0(X)$ of a space X is the least integer n such that X has the rational homotopy type of an n-cone (see [**5**, p. 359]). Spaces of rational cone length 1 then correspond to rational co-H-spaces.

THEOREM 6.4. *Let X be a finite CW complex and Y a simply connected complex of finite type. Then $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, Y; f)) \leq \mathrm{cl}_0(X)$ for all maps $f : X \to Y$.*

PROOF. Let $n = \mathrm{cl}_0(X)$. By [**5**, Theorem 29.1], the underlying vector space V of the Quillen minimal model of X admits a filtration $\{0\} \subset V(1) \subset V(2) \subset \cdots \subset V(n) = V$ where $d_X(V(i)) \subseteq \boldsymbol{L}(V(i-1))$. The result is proved by arguing that iterated Whitehead products in $H_*(\mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f))$ of length r are represented by cycles that vanish on $V(r-1)$. We argue in a similar fashion to the proof of Theorem 6.2.

Consider two cycles $\theta_a, \theta_b \in \mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f)$. Recall that their Whitehead product is represented by the image of the universal example

$$\{\Theta_a, \Theta_b\} \circ \lambda \in \mathrm{Der}(\mathscr{L}_X, \mathscr{L}_X(a, b); \lambda)$$

under the map

$$\Big((\zeta_a^* \mid \zeta_b^*)_{\mathscr{L}_f}\Big)_* : \mathrm{Der}(\mathscr{L}_X, \mathscr{L}_X(a, b); \lambda) \to \mathrm{Der}(\mathscr{L}_X, \mathscr{L}_Y; \mathscr{L}_f).$$

Here $\zeta_a^* = (0, \theta_a)$ and $\zeta_b^* = (0, \theta_b)$ in $\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f})$. In particular, $(\zeta_a^* \mid \zeta_b^*)_{\mathscr{L}_f}$ maps a and b to zero.

From (12), we see that $\{\Theta_a, \Theta_b\} \circ \lambda(v) \equiv \pm\Theta_b \circ \Theta_a(d_X(v))$ modulo terms in the ideal generated by a and b. Now assume that Θ_a vanishes on $V(r)$. Since $d_X(V(r+1)) \subseteq \boldsymbol{L}(V(r))$, we have that $\{\Theta_a, \Theta_b\} \circ \lambda$ vanishes on $V(r+1)$. An easy induction using this completes the proof. □

We next give a complete calculation of the rational Whitehead length of function spaces in a special case. Let X be a simply connected, finite complex and S^n a sphere with $n \geq 2$. When n is odd, S^n is a rational H-space and hence so too is $\mathrm{map}(X, S^n)$. It follows that, after rationalization, each component of $\mathrm{map}(X, S^n)$

is homotopy equivalent to the null component, which itself is an H-space; in particular we have $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, S^n; f)) = 1$. Identical remarks apply to the based function space.

When n is even, the rational homotopy types of components $\mathrm{map}(X, S^n; f)$ are more complicated. A complete description for X rationally $(2n+1)$-co-connected is given by Møller-Raussen [**12**, Theorem 1]. We compute the rational Whitehead length of all components in both the based and free setting without dimension restriction on X.

Since S^n is a coformal space, $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, S^n; f))$ and $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, S^n; f))$ are each equal to either 1 or 2—but not *a priori* equal to each other—by Theorem 6.2. Suppose first that $H(f; \boldsymbol{Q}) = 0 : H^*(S^n; \boldsymbol{Q}) \to H^*(X; \boldsymbol{Q})$. Then the rationalization of f factors through the fibre $K(\boldsymbol{Q}, 2n-1)$ of the Postnikov decomposition $K(\boldsymbol{Q}, 2n-1) \to (S^n)_{\boldsymbol{Q}} \to K(\boldsymbol{Q}, n)$. This implies f is a *rationally cyclic map* (see [**11**, Definition 2.4 and Example 4.4]). By [**9**, Theorem 3.7], the evaluation fibration $\omega : \mathrm{map}(X, S^n; f) \to S^n$ is then rationally fibre-homotopically trivial and, from the long exact homotopy sequence, we have an isomorphism of Whitehead algebras:

$$\pi_*(\mathrm{map}(X, S^n; f)) \otimes \boldsymbol{Q}, [\,,\,]_w \cong \big(\pi_*(\mathrm{map}_*(X, S^n; f)) \otimes \boldsymbol{Q}, [\,,\,]_w\big) \otimes \big(\pi_*(S^n) \otimes \boldsymbol{Q}, [\,,\,]_w\big).$$

Since $\mathrm{WL}_{\boldsymbol{Q}}(S^n) = 2$, we have $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, S^n; f)) = 2$ and $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, S^n; f))$ is equal to either 1 or 2 in this case.

Suppose $H(f; \boldsymbol{Q}) \neq 0$. Write $(\mathscr{L}_{S^n}, d_{S^n}) = \boldsymbol{L}(u; 0)$ with $|u| = n-1$ and $\mathscr{L}_X = \boldsymbol{L}(V; d_X)$. The condition $H(f; \boldsymbol{Q}) \neq 0$ translated to Quillen models implies there exists $v \in V_{n-1}$ with $\mathscr{L}_f(v) = u$. This implies there are no cycles $(u, \theta) \in \mathrm{Rel}_n(\mathrm{ad}_{\mathscr{L}_f})$ because $\mathrm{ad}_{\mathscr{L}_f}(u) + D_{\mathscr{L}_f}(\theta)$ cannot equal zero: note that $\theta(d_X(v)) = 0$ for degree reasons, and then applied to v we obtain that

$$\mathrm{ad}_{\mathscr{L}_f}(u)(v) + D_{\mathscr{L}_f}(\theta)(v) = [u, \mathscr{L}_f(v)] + d_{S^n}\theta(v) \pm \theta(d_X(v)) = [u, u] \neq 0 \in (\mathscr{L}_{S^n})_{2n-2}.$$

By the formula for the Whitehead product in $H_*(\mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f}))$ [Theorem 3.6], we see directly that the cycle $([u, u], 0) \in \mathrm{Rel}_{2n-1}(\mathrm{ad}_{\mathscr{L}_f})$ does not represent a Whitehead product. Translating back, this means

$$\pi_*(\mathrm{map}(X, S^n; f)) \otimes \boldsymbol{Q}, [\,,\,]_w \cong \big(\pi_*(\mathrm{map}_*(X, S^n; f)) \otimes \boldsymbol{Q}, [\,,\,]_w\big) \oplus \big(\boldsymbol{Q}([\iota, \iota]_w), 0\big)$$

where $\iota \in \pi_n(S^n)$ is nontrivial and $(\boldsymbol{Q}([\iota, \iota]_w), 0)$ denotes the abelian Whitehead algebra generated in degree $2n-1$. Thus in this case

$$\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, S^n; f)) = \mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, S^n; f)) = 1 \text{ or } 2.$$

In both cases, the relevant question is the rational Whitehead length of the based function space. We address this question as an application of our formula:

THEOREM 6.5. *Let X be a finite, simply connected CW complex and $f : X \to S^n$ a based map with n even. Then $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, S^n; f)) = 2$ if and only if there exists a pair $x, y \in H^{\leq n-2}(X; \boldsymbol{Q})$ satisfying*:

(i) $xy \neq 0$,
(ii) $xz \neq 0$ *or* $yz \neq 0$ *for* $z \in H^n(X; \boldsymbol{Q}) \Longrightarrow H(f; \boldsymbol{Q})(z) = 0$ *and*
(iii) $xy = wz$ *for some* $z \in H^n(X; \boldsymbol{Q})$ *and any* $w \Longrightarrow H(f; \boldsymbol{Q})(z) = 0$

Otherwise, $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, S^n; f)) = 1$.

As for the free function space, if $H(f; \boldsymbol{Q}) = 0$ *then* $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, S^n; f)) = 2$. *Otherwise,* $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, S^n; f)) = \mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}_*(X, S^n; f))$*, as given above.*

PROOF. The results for the free function space follow from the discussion preceding the statement of the theorem. Thus we focus on the based function space and so the space $H_*(\mathrm{Der}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f))$ with Whitehead product given by Corollary 3.7. Write $\mathscr{L}_X = \boldsymbol{L}(V; d_X)$ and $\mathscr{L}_{S^n} = \boldsymbol{L}(u; 0)$. Given a homogeneous basis $\{v_1, \ldots, v_s\}$ for $V = s^{-1}\widetilde{H}^*(X; \boldsymbol{Q})$, we will assume the vectors are in nondecreasing order of degree. If $H(f; \boldsymbol{Q}) \neq 0$, we will further assume that there is some basis vector $v_k \in V_{n-1}$ such that $\mathscr{L}_f(v_k) = c_k u$ while $\mathscr{L}_f(v_i) = 0$ for any other basis element v_i of degree $n-1$. Here $c_k \neq 0$. For convenience, we allow the case $c_k = 0$ so that $H(f; \boldsymbol{Q}) = 0$ if and only if $c_k = 0$.

We recall the quadratic part of the differential d_X is dual to the cup product in $H^*(X; \boldsymbol{Q})$ (see [**15**, Section I.1.(10)] and [**5**, Section 22e]). Let $\{x_1, \ldots, x_s\}$ be the corresponding additive basis of $\widetilde{H}^*(X; \boldsymbol{Q})$, that is, $x_i = s(v_i)$. Given any $v \in V$ we may write

$$d_X(v) = \sum_{i \leq j} c_{ij}(v)[v_i, v_j] + \text{longer length terms} \tag{18}$$

with $c_{ij}(v) \in \boldsymbol{Q}$ and $c_{ii}(v) = 0$ for v_i of even degree. As a direct consequence of this duality we have that the cup product $x_i x_j = 0$ if and only if $c_{ij}(v) = 0$ for all $v \in V$. We make use of this repeatedly below.

Let $\theta \in \mathrm{Der}_p(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$. Using (18), we have

$$D_{\mathscr{L}_f}(\theta)(v) = \pm\theta(d_X v) = \pm\sum_{i \leq k} c_{ik}(v)[\theta(v_i), \mathscr{L}_f(v_k)] = \pm\sum_{i \leq k} c_k c_{ik}(v)[\theta(v_i), u].$$

It follows that θ is a cycle if $\theta(v_i) = 0$ for all v_i in V_{n-1-p}. If $\theta(v_i) \neq 0$ for some $v_i \in V_{n-1-p}$ we may alter our basis in this degree so that $\theta(v_j) = \delta_{ij} u$ for $v_j \in V_{n-1-p}$

where δ_{ij} is the Kronecker delta function. Then we see θ is a cycle if and only if $c_k c_{ik}(v) = 0$ for all $v \in V$. Translating, we have shown a derivation $\theta \in \mathrm{Der}_p(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ not vanishing on V_{n-1-p} is a cycle if and only if there exists $x \in H^{\leq n-2}(X; \boldsymbol{Q})$ such that $xz \neq 0$ implies $H(f; \boldsymbol{Q})(z) = 0$ for all $z \in H^n(X; \boldsymbol{Q})$.

Next let $\theta_a, \theta_b \in \mathrm{Der}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ be cycles of degree p and q, respectively. As in the discussion preceding Corollary 3.7, let $\zeta_x^* = (0, \theta_x) \in \mathrm{Rel}(\mathrm{ad}_{\mathscr{L}_f})$, $x = a, b$, be the corresponding cycles so that, by (11), the derivation cycle

$$[\![\theta_a, \theta_b]\!] = (\zeta_a^* \mid \zeta_b^*)_{\mathscr{L}_f} \circ \{\Theta_a, \Theta_b\} \circ \lambda \in \mathrm{Der}_{p+q-1}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$$

represents $[\langle\theta_a\rangle, \langle\theta_b\rangle]_w$. Using (18) again, (12) and the fact that $(\zeta_a^* \mid \zeta_b^*)_{\mathscr{L}_f}(x) = 0$ for $x = a, b$, we obtain

$$\begin{aligned} [\![\theta_a, \theta_b]\!](v) &= \pm(\zeta_a^* \mid \zeta_b^*)_{\mathscr{L}_f} \circ \Theta_b \circ \Theta_a \circ \lambda(d_X v) \\ &= \sum_{i<j} \pm\, c_{ij}(v)\big([\theta_a(v_i), \theta_b(v_j)] \pm [\theta_b(v_i), \theta_a(v_j)]\big) \pm \sum_i 2c_{ii}(v)[\theta_a(v_i), \theta_b(v_i)]. \end{aligned}$$

From this we conclude that $[\![\theta_a, \theta_b]\!]$ nonvanishing for derivation cycles θ_a, θ_b implies $\theta_a(v_i) \neq 0$ for some $v_i \in V_{n-1-p}$ and $\theta_b(v_j) \neq 0$ for some $v_j \in V_{n-1-q}$ and for this i, j we have $c_{ij}(v) \neq 0$ for some $v \in V$. By the above computation, the fact that θ_a and θ_b are cycles implies $c_k c_{ik}(w) = c_k c_{jk}(w) = 0$ for all $w \in V$. Conversely, suppose there exist indices i, j such that $c_{ij}(v) \neq 0$ and $c_k c_{ik}(w) = c_k c_{jk}(w) = 0$ for all $w \in V$. Then define θ_a, θ_b by setting $\theta_a(v_l) = \delta_{li} u$ and $\theta_b(v_l) = \delta_{lj} u$ and extend by the $\mathscr{L}_f$-derivation law. By the preceding paragraph, the θ_a and θ_b are derivation cycles. Computing as above $[\![\theta_a, \theta_b]\!](v) = \pm c_{ij}[u, u]$ is non-vanishing. Combining and translating to cohomology, we have shown that there exists a nontrivial pairing $[\![\theta_a, \theta_b]\!] \in \mathrm{Der}_{p+q-1}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ for cycles $\theta_a \in \mathrm{Der}_p(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ and $\theta_b \in \mathrm{Der}_q(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ if and only if there exists a pair $x, y \in H^{\leq n-2}(X; \boldsymbol{Q})$ satisfying (i) and (ii).

Finally, suppose $\theta_a, \theta_b \in \mathrm{Der}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ are cycles of degree p and q with $[\![\theta_a, \theta_b]\!] \neq 0$. As above, let $v_i \in V_{n-1-p}$ and $v_j \in V_{n-1-q}$ be basis elements so that $\theta_a(v_i) \neq 0$ and $\theta_b(v_j) \neq 0$. If $p \neq q$ we arrange our basis in degree $n-1-p$ and $n-1-q$ so that $\theta_a(v_l) = \delta_{il} u$ and $\theta_b(v_m) = \delta_{jm} u$ for $v_l \in V_{n-1-p}$ and $v_m \in V_{n-1-q}$. If $p = q$, we must allow for the case $v_i = v_j$. In this case, we may arrange the basis in degree $n-1-p$ so that $\theta_a(v_l) = c_a \delta_{il} u$ and $\theta_b(v_m) = c_b \delta_{jm} u$ for $v_l, v_m \in V_{n-1-p}$. Here $c_a, c_b \neq 0$ and can be taken to be 1 when $i \neq j$. We use this identification in all cases by taking $c_a = 1$ and $c_b = 1$ except, perhaps, when $i = j$.

Let $x = s(v_i) \in H^{n-p}(X; \boldsymbol{Q})$ and $y = s(v_j) \in H^{n-q}(X; \boldsymbol{Q})$ be the corresponding cohomology elements. Then the pair x, y satisfy (i) and (ii). We show $[\![\theta_a, \theta_b]\!] \in$

$\mathrm{Der}_{p+q-1}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ bounds if and only if the pair x, y violates (iii). Since $[\![\theta_a, \theta_b]\!]$ is nonvanishing, the preceding discussion shows there is a vector $v \in V_{2n-1-p-q}$ with

$$[\![\theta_a, \theta_b]\!](v) = \pm c_a c_b c_{ij}(v)[u, u] \neq 0.$$

Suppose $[\![\theta_a, \theta_b]\!] = D_{\mathscr{L}_f}(\theta)$ for some $\theta \in \mathrm{Der}_{p+q}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$. Applying this to $v \in V$ using (18) we obtain

$$\pm c_a c_b c_{ij}(v)[u, u] = [\![\theta_a, \theta_b]\!](v) = D_{\mathscr{L}_f}(\theta)(v) = \pm\theta(d_X v) = \pm \sum_{r<k} c_k c_{rk}(v)[\theta(v_r), u].$$

We conclude that if $[\![\theta_a, \theta_b]\!]$ is a nonvanishing boundary then there is $v \in V$ with $c_{ij}(v) \neq 0$ and $c_k c_{rk}(v) \neq 0$ for some r which directly translates to imply the pair x, y violates (iii) with $w = s(v_r)$ and $z = s(v_k)$. Conversely, if x, y violate (iii) then $c_k \neq 0$ and there exists some $v \in V_{2n-1-p-q}$ such that $c_{ij}(v) \neq 0$ and $c_{rk}(v) \neq 0$ for some index r. Notice that $v_r \in V_{n-1-p-q}$. Define a derivation $\theta \in \mathrm{Der}_{p+q}(\mathscr{L}_X, \mathscr{L}_{S^n}; \mathscr{L}_f)$ by setting $\theta(v_l) = \delta_{lr} u$ and extending. We then see $D_{\mathscr{L}_f}(\theta)(v) = \pm c_k c_{rk}(v)[u, u] \neq 0$ while $[\![\theta_a, \theta_b]\!](v) = \pm c_a c_b c_{ij}(v)[u, u] \neq 0$. To complete the proof, we show the derivations $D_{\mathscr{L}_f}(\theta)$ and $[\![\theta_a, \theta_b]\!]$ differ by a constant. Note that both derivations increase bracket length. This implies they both vanish on V except in degree $2n-1-p-q$. In this degree, they are linear maps $V_{2n-1-p-q} \to \boldsymbol{Q}([u, u])$. We have shown both are nonzero on a particular vector $v \in V_{2n-1-p-q}$. Since the target is one-dimensional, there is a constant $c \neq 0$ such that $[\![\theta_a, \theta_b]\!] = cD_{\mathscr{L}_f}(\theta) = D_{\mathscr{L}_f}(c\theta)$, as needed. □

We conclude with an example realizing the inequality

$$\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, Y; f)) > \mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, Y; 0)) = \mathrm{WL}_{\boldsymbol{Q}}(Y)$$

for some map $f : X \to Y$.

EXAMPLE 6.6. Let $X = S^3$, and let Y be a space with Sullivan minimal model $\Lambda(x_1, x_2, x_3, y; d)$, the free DG algebra with generators of degrees $|x_1| = 2, |x_2| = |x_3| = 3$, and $|y| = 7$. Define the (degree $+1$) differential here by setting $d(x_i) = 0$ and $d(y_7) = x_1 x_2 x_3$. Then Y has vanishing Whitehead products since d has no quadratic term [**5**, Proposition 13.16]. Consequently, $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(X, Y; 0)) = 1$ by Theorem 6.1. The Quillen minimal model $(\mathscr{L}_Y, d_Y)$ for Y is of the form $\boldsymbol{L}(W; d_Y)$ where $W = s^{-1}\widetilde{H}_*(Y, \boldsymbol{Q})$. We use that the quadratic part of the differential d_Y is dual to the cup-product in $H^*(Y, \boldsymbol{Q})$ (see [**5**, Section

22(e)]). In low degrees, we see W contains elements $w_1, w_2, w_3, w_{1,1}, w_{1,2}, w_{1,3}, w_{2,3}$ with $|w_1| = 1, |w_2| = |w_3| = 2, |w_{1,1}| = 3, |w_{1,2}| = |w_{1,3}| = 4$ and $|w_{2,3}| = 5$. Here w_i corresponds to x_i and $w_{i,j}$ to the cup-product $x_i \cdot x_j$. We may write the differential as

$$d_Y(w_1) = d_Y(w_2) = d_Y(w_3) = 0 \quad \text{with} \quad d_Y(w_{1,1}) = \frac{1}{2}[w_1, w_1],$$
$$d_Y(w_{1,2}) = [w_1, w_2],\ d_Y(w_{1,3}) = [w_1, w_3] \quad \text{and} \quad d_Y(w_{2,3}) = [w_2, w_3]$$

on these generators. Write the Quillen minimal model for S^3 as $\boldsymbol{L}(v; 0)$ with v in degree 2 and let $f: S^3 \to Y$ correspond, after rationalization, to the class $w_3 \in H_2(\mathscr{L}_Y)$. That is, $\mathscr{L}_f(v) = w_3$. We show that $\mathrm{WL}_{\boldsymbol{Q}}(\mathrm{map}(S^3, Y; f)) \geq 2$. Observe that an element $\zeta_a = (\chi_a, \theta_a) \in \mathrm{Rel}_p(\mathrm{ad}_{\mathscr{L}_f})$ is a $\delta_{\mathrm{ad}_{\mathscr{L}_f}}$-cycle if $d_Y(\chi_a) = 0$ and $d_Y(\theta_a(v)) = -[\chi_a, w_3]$. Thus $\zeta_a = (w_1, \theta_a)$ and $\zeta_b = (w_2, \theta_b)$ are $\delta_{\mathrm{ad}_{\mathscr{L}_f}}$-cycles of degree 2 and 3 respectively where $\theta_a(v) = -w_{1,3}$ and $\theta_b(v) = -w_{2,3}$. Write $\alpha \in \pi_2(\mathrm{map}(X, Y; f)) \otimes \boldsymbol{Q}$ and $\beta \in \pi_3(\mathrm{map}(X, Y; f)) \otimes \boldsymbol{Q}$ for the corresponding homotopy elements as in Section 5. Applying Theorem 5.3, their Whitehead product $[\alpha, \beta]_w \in \pi_4(\mathrm{map}(X, Y; f)) \otimes \boldsymbol{Q}$ corresponds to the class represented by the $\delta_{\mathrm{ad}_{\mathscr{L}_f}}$-cycle $[\![\zeta_a, \zeta_b]\!] = ([w_1, w_2], \{\zeta_a, \zeta_b\}) \in \mathrm{Rel}_4(\mathrm{ad}_{\mathscr{L}_f})$; where, by using formula (12), we have that $\{\zeta_a, \zeta_b\}(v) = -[w_2, w_{1,3}] - [w_1, w_{2,3}]$. This cannot be a boundary. For if $\delta_{\mathrm{ad}_{\mathscr{L}_f}}(\eta, \theta) = ([w_1, w_2], \{\zeta_a, \zeta_b\})$, then we have $\eta = -w_{1,2} + \chi$ for some χ a cycle in $\mathscr{L}_Y$ of degree 4. Since Y has no rational homotopy of degree 5 (direct from the Sullivan model), we see that $\chi = d_Y(\xi)$ for some $\xi \in \mathscr{L}_Y$. Further, we then obtain that

$$\mathrm{ad}_{\mathscr{L}_f}(-w_{1,2} + d_Y(\xi))(v) + D_{\mathscr{L}_f}(\theta)(v) = \{\zeta_a, \zeta_b\}(v),$$

which implies that

$$d_Y([\xi, w_3] + \theta(v)) = [w_{1,2}, w_3] - [w_1, w_{2,3}] - [w_2, w_{1,3}].$$

However, when d_Y is applied to this latter term, it yields $2[[w_1, w_2], w_3]$ and not zero, so it cannot be a cycle (boundary). We conclude that $[\alpha, \beta]_w \neq 0$.

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