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EFFECTIVE AREA AND EFFECTIVE VOLUME CALCULATIONS FOR
CERAMIC TEST SPECIMENS

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EFFECTIVE AREA AND EFFECTIVE VOLUME CALCULATIONS FOR CERAMIC TEST SPECIMENS

RAHUL JAIN

ABSTRACT

Calculation of effective volume and/or effective area is a key step in estimating reliability of ceramic component life cycle. Most common tests performed to assess the strength and reliability of components made from ceramics are bend bar specimens tested in three-point and four-point flexure, C-ring and O-ring specimen under diametral compressive or tensile loads and biaxial ring-on-ring specimens. ASTM closed form solutions for the effective volume and area exists for these specimen geometries which are based on classical theories with underlying assumptions. In general the closed form expressions are valid for limited specimen geometry bounds. An alternative numerical approach has been utilized to calculate the effective volume and area for the ceramic test specimens. The results obtained through the use of the numerical approach are compared with the closed form solutions and these comparisons point to the need for revisiting the underlying assumptions used in developing the closed form expressions.

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NOMENCLATURE

K_{IC} = Fracture toughness

m = Weibull modulus

m_A = Weibull modulus associated with surface area

m_V = Weibull modulus associated with volume of specimen

$\tilde{m}$ = Estimated Weibull modulus

$\tilde{\sigma}_\theta$ = Estimated Weibull characteristic strength

γ = Threshold parameter

β = Scale parameter, also parameter based on load configuration and type for fracture calculations

σ_θ = Weibull characteristic strength

$(\sigma_\theta)_A$ = Weibull characteristic strength associated with surface area of specimen

$(\sigma_\theta)_V$ = Weibull characteristic strength associated with volume of specimen

σ_o = Weibull material scale parameter

$(\sigma_o)_A$ = Weibull material scale parameter associated with surface area of specimen

$(\sigma_o)_V$ = Weibull material scale parameter associated with volume of specimen

σ = Stress at a point

$\sigma_{\max}$ = The maximum stress in the test specimen given the failure load

$\sigma_{failure}$ = Failure stress for the component

A_{eff} = Effective area (kA)

V_{eff} = Effective volume (kV)

P_f = Probability of failure

P_i = Ranked fracture probability

i = Rank of the individual specimen in the strength set

N = Total number of specimens in the strength set

σ_i = Ranked failure stress

V = Volume of specimen

V_{gage} = Volume of the gage section

A = Surface area of specimen

A_{gage} = Surface area of gage section

L_{gage} = Length of the gage section

d_{gage} = Diameter of the gage section

L_i = Length of the inner span

L_o = Length of the outer span

V_B = Volume of the gage section

b = Width of the specimen, also parameter for linear regression

d = Depth of the specimen

r = Radial distance

r_i = Inner radius

r_o = Outer radius

t = Thickness of the ring

D = Test specimen diameter

D_S = Support ring diameter

D_L = Load ring diameter

h = Thickness of the test specimen

ν = Poisson's ratio of the material

E = Modulus of elasticity of material

a = Parameter for linear regression , also half crack length

x_i = Parameter for linear regression

y_i = Parameter for linear regression

n = Total number of data sets

$\mathcal{L}$ = Likelihood function

$f()$ = Probability density function

$F()$ = Cumulative density function

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CHAPTER I

INTRODUCTION

Common ceramic materials are fabricated from various types of alumina and silicon-based powders that are hardened through a firing process. Ceramics include commonly used items such as pottery and china. However, from an engineering perspective, ceramics are utilized for their high strength in compression and superior wear resistance. For the most part, advanced ceramics used for structural purposes retain their high strength at elevated temperatures, and some also offer attractive features that include resistance to slow crack growth and creep. Ceramics are damage tolerant to a degree, and they can be resistant to corrosion and oxidation. Typically, ceramics offer a significant weight savings in applications where this issue is important, e.g., aerospace applications. More general commercial applications include heat exchangers, automotive engine components such as turbocharger rotors and cam roller followers.

Given the widespread use of ceramics noted above, it is important to realize that structural ceramics are inherently brittle with low fracture toughness. Ceramics fail in tension due to flaws that are introduced during processing, and/or machining. Critical flaws introduced during these operations are so small that they are beyond detection by current non-destructive evaluation techniques. This issue of relatively minute critical flaw size is a result of the combination of high strength and relatively low fracture toughness. Consider a generalized Griffith crack relationship, i.e.,

$$K_{IC} = \sigma \cdot \beta \cdot \sqrt{\Pi \cdot a} \quad (1.1)$$

where, σ is the applied far field stress, β is a parameter based on load configuration and load type, and a is half the crack length. Solving this expression for the critical crack size yields

$$a_{critical} = \left(\frac{1}{\beta} \right)^2 \left(\frac{1}{\Pi} \right) \left(\frac{K_{IC}}{\sigma_{failure}} \right)^2 \quad (1.2)$$

With this last expression, it is quite evident that a low fracture toughness (K_{IC}) along with a high strength material ($\sigma_{failure}$) can lead to a very small critical flaw size. Thus, the design engineer is faced with the situation where components fabricated from ceramic materials contain flaws that cannot be detected. The size of these flaws as well as their orientation differs from part to part. As a consequence, the basic tensile strength can vary considerably. Due to this variation of tensile strength, ***statistical design approaches are mandated.***

For the reasons discussed above tensile strength for ceramics must be considered a random variable. This consideration is underscored by data from fundamental tensile strength test specimens such as bend bars. This type of test specimen is often referred to as modulus of rupture, or MOR bars. The most common types of MOR bar configurations include three-point and four-point flexure tests (see Figures 1.1 and 1.2). The four-point flexure test introduces a uniaxial state of stress within the gage section of the specimen, whereas configurations such as ball-on-ring and pressurized ring-on-ring tests (see Figure 1.3) produce biaxial states of tensile stress. The three-point flexure specimen introduces shear between the load span and is considered an inferior test specimen geometry when interrogating the tensile strength of the material.

Figure 1.1 Three-point flexure test arrangement

Figure 1.2 Four-point flexure test arrangement

Figure 1.3 Pressurized ring-on-ring biaxial flexure test arrangement

Regardless of the type of tensile test specimen configuration utilized, the tensile strength of ceramics can easily vary by more than 50%. This variation is highlighted in the following figure that depicts the maximum tensile stress at failure for an alumina ceramic tested using a four-point flexure bar specimen (ASTM C1239^[3], 2005). The variation is a direct result of a distribution of flaws with random orientations as well as random lengths, and the variation is typical of commonly used ceramic materials.

Figure 1.4 Four-point flexure strengths (ASTM C 1161 Size B^[1]) for a hot pressed alumina with 25 wt% SiC whiskers

In the above figure a flaw key has been provided and it identifies eight categories of flaws. All the flaw types are volume flaws except the “machine damage” and “uncertain” category. This “machine damage” flaw would occur along the surface of the test specimens. For this data, fractography was performed on all the specimens. As stipulated above for the most part, the flaws identified were spatially distributed through the volume of the test specimens as indicated by the flaw key “large grain.” The exception is the flaw category identified as uncertain. Each category of flaw represents a “flaw population” and will be represented by separate and distinct statistical distributions. Thus, the data indicates that one must consider the spatial distribution of flaws in a test specimen or a component and many types of strength limiting defects can be introduced into the tensile regions of the test specimen. Consider the classic case involving test specimens machined from a ceramic billet that contains defects distributed through the volume from processing. Each specimen is machined from this billet, and as a result, machining defects are introduced on the surface of each test specimen. In this work only two basic types of defect populations are assumed present in the material, volume defects and surface defects. These defect populations form the most general situation relative to spatial defects. This issue is important later when determination of distribution parameters is discussed.

TENSILE STRENGTH AS A RANDOM VARIABLE

The random variable representing uniaxial tensile strength of an advanced ceramic assumes only positive values (tensile), and the distribution is asymmetrical about the mean. This asymmetry is depicted in Figure 1.5 where the average tensile strength is 508 MPa.

Figure 1.5 Histogram of the alumina data in Figure 1.4

The non-negative aspect and asymmetry of ceramic tensile strength can be captured by several probability density function, e.g., lognormal, Weibull, etc. The

argument is made that the density function chosen should match the physics of the problem. For the tensile strength of ceramics, this argument points to the use of a minimum extreme value probability density function.

Extreme value statistics is an outgrowth of ordered statistics. Consider the case where the maximum three-second wind velocity is recorded at a site on a daily basis. In a given year if the data is ordered from minimum to maximum value, one can focus on the maximum annual extreme three-second wind velocity. If this type of data is recorded year after year, then the yearly maximum three-second wind velocity will vary every year. This variation indicates that the maximum three-second wind velocity is a random variable and may be characterized by some appropriate probability density function. This probability density function, if derived properly, is referred to as a maximum extreme value probability density function.

In the case of the tensile strength of a ceramic material, the approach outlined above could be easily applied. The focus would be placed on the minimum tensile strength in a given group of test specimens. Here we would need a minimum extreme value probability density function. There are three classifications of extreme value probability density functions, i.e., type I, type II and type III. This discussion focuses on the Weibull distribution, which is a type III extreme value probability density function. Moreover, we wish to focus on the minimum value probability density distribution.

If the random variable σ representing uniaxial tensile strength of an advanced ceramic is characterized by a three-parameter Weibull distribution (a type III minimum value distribution, Nelson^[20] 1982), then the probability density function is given by the expression

$$f(\sigma) = \left(\frac{m}{\beta}\right) \left(\frac{\sigma - \gamma}{\beta}\right)^{m-1} \exp\left[-\left(\frac{\sigma - \gamma}{\beta}\right)^m\right] \quad \sigma > \gamma \quad (1.3)$$

and

$$f(\sigma) = 0 \quad \sigma \leq \gamma \quad (1.4)$$

where m is the Weibull modulus (and is always positive), β is the scale parameter, γ is a threshold parameter. The cumulative distribution function for a three-parameter Weibull distribution is given by the expression

$$F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma - \gamma}{\beta}\right)^m\right] \quad \sigma > \gamma \quad (1.5)$$

and

$$F(\sigma) = 0 \quad \sigma \leq \gamma \quad (1.6)$$

Often the value of the threshold parameter is taken to be zero. In component design this consideration represents a conservative assumption and yields the more widely used two-parameter Weibull formulation. Here the expression for the probability density function simplifies to

$$f(\sigma) = \left(\frac{m}{\beta}\right) \left(\frac{\sigma}{\beta}\right)^{m-1} \exp\left[-\left(\frac{\sigma}{\beta}\right)^m\right] \quad \sigma > 0 \quad (1.7)$$

and

$$f(\sigma) = 0 \quad \sigma \leq 0 \quad (1.8)$$

The cumulative distribution function for a two-parameter Weibull distribution is given by

$$F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma}{\beta}\right)^m\right] \quad \sigma > 0 \quad (1.9)$$

and

$$F(\sigma) = 0 \quad \sigma \leq 0 \quad (1.10)$$

Figure 1.6 shows a plot of Weibull PDF for the alumina ceramic fracture data of Figure 1.4; also plotted is a normal PDF, which highlights the skewness of Weibull PDF.

Figure 1.6 Weibull PDF and normal PDF of the alumina data in Figure 1.4

OBJECTIVE

The tensile strength of ceramics as discussed earlier can be treated as a random variable. The chapters that follow highlight how the size of a ceramic component affects its overall tensile strength. The phenomenon of size effect emphasizes the need for using a system reliability theory, thus requiring in this work the extension of the Weibull distribution to Weibull theory. Further, Weibull theory correlates the strength distribution parameters based on specimen geometry to a strength distribution parameter based on the material property. This correlation depends on the concepts of effective volume and effective area. Calculation of effective volume and/or effective area is a key step in

estimating the reliability of ceramic components. Most common tests performed to assess the strength distribution parameters are bend bar specimens tested in three-point and four-point flexure, C-ring and O-ring specimen under diametral compressive or tensile loads and biaxial ring-on-ring specimens. ASTM closed form solutions for the effective volume and area exists for these specimen geometries, with attending underlying assumptions. In general the closed form expressions are valid for limited specimen geometries. An alternative numerical approach can be utilized to calculate the effective volume and area for the ceramic test specimens^[11]. The objective of this research work is comparison of effective volume and effective area values determined from the numerical approach with the values obtained by currently accepted closed form solutions available in the literature. The intent of this effort is to observe the difference in the effective volume and area values between the two approaches and the impact thereof on the reliability of the ceramic components.

CHAPTER II

TENSILE STRENGTH AND SIZE EFFECT

The term “size effect” as it is used here is a description of a behavior wherein the dimensions of a structural component or a test specimen affect the value of the mechanical properties measured. Size effect relative to tensile strength is quite prominent in ceramics. As the size of a component or test specimen geometry is increased, then, on average, the tensile strength of the component decreases. The reason for this is that as the volume (or surface area) of the component is increased, the likelihood of encountering a critical flaw with deleterious orientations to the load applied increases.

To account for size effects and to deal with the probability of failure for a component in a general manner, the component or test specimen should be treated as a system. The typical approach to designing structural components with varying stress fields involves discretizing the component in order to characterize the stress field using

finite element methods. Because component failure may initiate in any of the discrete elements, it is convenient to consider a component as a system and utilize system reliability theories. A component is a series system when failure in one discrete element leads to a sudden catastrophic failure of the component. This type of failure can be modeled using the weakest-link theory. Thus, one needs a distribution that describes this extreme phenomenon (minimum extreme value) of the failure of the whole component due to the failure of a small link (component). In addition, a system model is necessary to capture observed size effects. The combination of a type III minimum value Weibull distribution with a weakest-link system approach is known as Weibull theory in the ceramics community, and will be utilized herein.

Relative to the discussions above, a *series system* will yield an upper bound on component probability of failure whereas a *parallel system*, a system with active redundancies, will yield a lower bound on the probability of failure. A parallel system in a reliability model can consist of ductile or brittle elements. In a parallel system, the failure of all subcomponents is required for the overall system to fail, whether the subcomponents are ductile or brittle. An obvious example of a parallel system would be the ultimate tensile strength of a unidirectional fiber bundle. In certain material systems such as a composite system, hybrid reliability models can be utilized which essentially consist of both the series and the parallel system arranged in some specific combination that captures the relevant failure behavior of the material. These hybrid system models can be highly complex and will not be considered here.

The probability that an advanced ceramic will fail under an applied uniaxial tensile stress σ is given by the cumulative distribution function expressed as follows

$$P_f = 1 - \exp \left[- \left(\frac{\sigma}{\sigma_\theta} \right)^m \right] \quad (2.1)$$

where P_f is the probability of failure and σ_θ is the Weibull characteristic strength. The exponent m is the Weibull modulus introduced earlier. Here the notation σ_θ is used for β . This notation is consistent with the nomenclature used in the ceramics industry. The characteristic strength σ_θ is representative of the test specimen or component and is dependent on the uniaxial tensile test, flexural test, or pressurized ring test specimen. When there is a change in specimen geometry, the magnitude of the characteristic distribution parameters strength also changes. This change in magnitude is directly related to the size effect. Due to its dependence on specimen geometry, the characteristic strength distribution parameters and the expression above proves awkward in the analysis of failure probability for a given component. To utilize the expression above, one would have to test the component under use conditions in order to estimate the cumulative distribution parameters.

The discussion above focuses on the probability of failure of the test specimens. This discussion carries over to the component design using probabilistic methods. It is noted here that it is always far easier to test simple specimen geometries and infer the probability of the failure of a component from these simpler test specimen geometries. Detailed derivations as described by Duffy and Hu^[14] show that an infinitesimal element,

such as volume (dV) or surface area (dA), can be treated as link in a chain. In this approach an alternative expression for the probability of failure (originally derived by Weibull, 1939) can be expressed as

$$P_f = 1 - \exp \left[- \int_V \left(\frac{\sigma}{(\sigma_o)_V} \right)^m dV \right] \quad (2.2)$$

The integration within the exponential function is performed over all tensile regions of the specimen volume if the strength-controlling flaws are randomly distributed through the volume of the material, or over all tensile regions of the specimen surface if flaws are restricted to the specimen surface. The Weibull material scale parameter $(\sigma_o)_V$ appearing in the integral above for volume defects has units of (stress) $\cdot$ (volume) $^{(1/m_V)}$. If the strength controlling flaws are restricted to the surface of the specimens in a sample, then the Weibull material scale parameter has units of (stress) $\cdot$ (area) $^{(1/m_A)}$. The Weibull material scale parameter can be described as the Weibull characteristic strength of a specimen with unit volume loaded in uniform uniaxial tension.

Since the characteristic strength σ_θ reflects specimen geometry and stress gradients, the Weibull material scale parameter σ_o represents a fundamental material property. This parameter along with the previous integral within which it appears allows one to compute the probability of failure of a component based on strength test from simple test components. Conceptually one determines σ_θ from a simple test configuration and uses the integral expression applied to the test specimen to compute σ_o .

Once σ_o is determined for the material, this value along with the integral expression is applied to the component under analysis. This second application of the integral yields the probability of the failure of the component.

As a simple example, Equations (2.1) and (2.2) can be equated for a uniaxially loaded tensile specimen^[4] that results in the following expression for σ_o ,

$$(\sigma_o)_V = (V_{gage})^{1/m} (\sigma_\theta)_V \quad (2.3)$$

Here V_{gage} is the volume of the uniform gage section of the tensile specimen, and the fracture origins are spatially distributed strictly within this volume. This type of fundamental relationship between σ_o and σ_θ will be explored throughout this work.

Unlike a simple uniaxially loaded test specimen, for the general case, stress is a function of position in the component. Thus, performing the integration given in Equation (2.2) can lead to a generic relationship suggested by Johnson and Tucker^[17] that is representative of various test specimen geometries. Identifying $(\sigma_o)_V$ as a parameter and rearranging the terms of Equation (2.2) yields

$$P_f = 1 - \exp \left[- \left(\frac{\sigma_{max}}{(\sigma_o)_V} \right)^m \int_V \left(\frac{\sigma}{\sigma_{max}} \right)^m dV \right] \quad (2.4)$$

where σ_{max} is the maximum stress in the test specimen given the failure load. Now let

$$V_{eff} = \int_V \left(\frac{\sigma}{\sigma_{max}} \right)^m dV \quad (2.5)$$

Then,

$$P_f = 1 - \exp \left[-V_{eff} \left(\frac{\sigma_{\max}}{\sigma_o} \right)^m \right] \quad (2.6)$$

where V_{eff} is effective volume that accounts for specimen geometry and stress gradients. In general, V_{eff} is a function of the estimated Weibull modulus and is always less than or equal to V_{gage} . The effective volume can be interpreted as the size of an equivalent uniaxial tensile specimen that has the same probability of failure as the test specimen or component. As the term implies, the product represents the volume of material subjected to a uniform uniaxial tensile stress. Thus, Equation (2.3) can be expressed as

$$(\sigma_o)_V = (V_{eff})^{1/m_V} (\sigma_\theta)_V \quad (2.7)$$

Thus, for an arbitrary test specimen, evaluating the integral identified in Equation (2.5) for the effective volume (V_{eff}), and utilizing Equation (2.7), one can obtain the estimated Weibull material scale parameter σ_o . Some researchers use notation kV for the effective volume (V_{eff}) where k is considered a coefficient based upon the geometry of test specimen, e.g., $k = 1$ for uniaxial tensile specimen. The simpler V_{eff} (and correspondingly, the effective area, A_{eff}) notation is used here.

CHARACTERIZING DISTRIBUTION PARAMETERS

In order to determine the probability of failure (or reliability) of any structural component, one needs to characterize the Weibull distribution parameters discussed above. These parameters are generally calculated based on data obtained from experimental observations made on simple test specimens.

Most of the tests performed to evaluate the mechanical reliability of ceramic materials are conducted on small laboratory specimens. In general these small laboratory specimen do not simulate the size and stress field of actual structural components. This disparity occurs because the testing of full scale components subjected to complex stress fields at high temperatures is cost prohibitive. Hence, a sound test methodology is needed whereby the mechanical reliability of standard ceramic components can be evaluated with a high degree of confidence when simple laboratory specimens are used.

As discussed in the previous section, the following expression is a key relationship for structural reliability analyses

$$(\sigma_o)_V = (V_{eff})^{1/m} (\sigma_\theta)_V \quad (2.7)$$

The Weibull parameters m & σ_θ must be estimated from failure data, and σ_o is computed from the expression above. It is important to keep in mind that σ_o is a derived quantity, one does not test for this parameter. Different techniques can be employed to estimate parameters m & σ_θ (or β) from the experimental results such as ranking the measured strength and using linear regression (LR), or Maximum likelihood estimation (MLE)

techniques (considered more appropriate). Both the estimation techniques will be discussed here in detail for completeness. However, the ceramic community has adopted MLE techniques due to the efficiency (quantified by confidence bounds) of the MLE estimator.

LINEAR REGRESSION – TWO PARAMETER WEIBULL DISTRIBUTION

Linear regression requires the application of a ranking rule to the strength data to estimate a failure probability for each data. Once an estimate of individual probabilities is made, the failure data is regressed to determine the distribution parameters. This procedure involves use of the ranking schemes such as

$$P_i = \left(\frac{i - 0.5}{N} \right) \quad (2.8)$$

where P_i is the ranked fracture probability, i is the rank of the individual specimen in the strength set and N is the total number of specimens in the strength set. The ranking scheme used in Equation (2.8) has been adopted by ASTM (C 1239)^[3].

Finding distribution parameters via linear regression involves utilizing the ranked probability of failure along with the associated failure stresses in the following expression. The estimated distribution parameters are obtained by taking the logarithm of

$$(1 - P_i) = \exp \left[- \left(\frac{\sigma_i}{\sigma_\theta} \right)^m \right] \quad (2.9)$$

where σ_i is the ranked failure stress, σ_θ is the Weibull scale parameter and m is the Weibull shape parameter. Taking the double log of both sides of this expression yields a straight line, i.e.,

$$\ln \left[\ln \left(\frac{1}{(1-P_i)} \right) \right] = m \ln(\sigma_i) - m \ln(\sigma_\theta) \quad (2.10)$$

where the quantities in this expression have been previously identified. Rearranging the terms of Equation (2.10) results in the following expression

$$\ln(\sigma_i) = \frac{1}{m} \ln \left[\ln \left(\frac{1}{(1-P_i)} \right) \right] + \ln(\sigma_\theta) \quad (2.11)$$

Equation (2.11) is in the form of equation of a line, i.e.,

$$y_i = ax_i + b \quad (2.12)$$

where

$$\begin{aligned} y_i &= \ln(\sigma_i) \\ x_i &= \ln \left[\ln \left(\frac{1}{1-P_i} \right) \right] \\ b &= \ln(\sigma_\theta) \\ a &= \frac{1}{m} \end{aligned} \quad (2.13)$$

Using the traditional linear regression, the following expressions for a and b are obtained

$$\begin{aligned}
a &= \frac{n \sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n \sum_{i=1}^n (x_i)^2 - \left(\sum_{i=1}^n x_i \right)^2} \\
b &= \frac{\sum_{i=1}^n (x_i)^2 \sum_{i=1}^n y_i - \sum_{i=1}^n x_i \sum_{i=1}^n x_i y_i}{n \sum_{i=1}^n (x_i)^2 - \left(\sum_{i=1}^n x_i \right)^2}
\end{aligned} \tag{2.14}$$

Once a and b are determined, the Weibull parameters $\tilde{m}$ and $\tilde{\sigma}_\theta$ (where the tilde designates the value as an estimate) can be extracted from the expressions in Equation (2.13). Graphically the value of m is the slope of Equation (2.11), and σ_θ is the value corresponding to the 63.2% probability of failure (refer to Figure 1.4).

MAXIMUM LIKELIHOOD ESTIMATE – TWO PARAMETER WEIBULL DISTRIBUTION

Maximum likelihood methods are superior (International energy agency^[15], 1993) to linear regression methods for determining the value of m from a set of fracture strengths. The maximum likelihood estimators (MLE) yield unique solutions for the distribution parameters for a two-parameter Weibull distribution. As the sample size increases, these estimators asymptotically converge to the true distribution parameters and the confidence interval narrows as well relative to other estimators. This latter facet makes this the estimation scheme of choice.

Let $\sigma_1, \sigma_2, \dots, \sigma_N$ represent realizations of the ultimate tensile strength (a random variable) in a given sample, where it is assumed that the ultimate tensile strength is characterized by the two-parameter Weibull distribution. The likelihood function associated with this sample is the joint probability density evaluated at each of the N sample values. This likelihood function is dependent on the two unknown Weibull distribution parameters (m, σ_θ) . The likelihood function for an uncensored sample under these assumptions is given by the expression

$$\mathcal{L} = \prod_{i=1}^N \left(\frac{\tilde{m}}{\tilde{\sigma}_\theta} \right) \left(\frac{\sigma_i}{\tilde{\sigma}_\theta} \right)^{\tilde{m}-1} \exp \left[- \left(\frac{\sigma_i}{\tilde{\sigma}_\theta} \right)^{\tilde{m}} \right] \quad (2.15)$$

For an uncensored sample the system of equations obtained by differentiating the log likelihood function with respect to m and σ_θ and setting the derivatives equal to zero is given by

$$\frac{\sum_{i=1}^N (\sigma_i)^{\tilde{m}} \ln(\sigma_i)}{\sum_{i=1}^N (\sigma_i)^{\tilde{m}}} - \frac{1}{N} \sum_{i=1}^N \ln(\sigma_i) - \frac{1}{\tilde{m}} = 0 \quad (2.16)$$

and

$$\tilde{\sigma}_\theta = \left[\left(\sum_{i=1}^N (\sigma_i)^{\tilde{m}} \right) \frac{1}{N} \right]^{1/\tilde{m}} \quad (2.17)$$

The Equation (2.15) is solved in iterative fashion first for $\tilde{m}$, which satisfies the equality. Subsequently $\tilde{\sigma}_\theta$ is computed from the Equation (2.16) using the value of $\tilde{m}$ calculated from Equation (2.15). A closed form solution for the Equation (2.15) is not available,

thus this expression must be solved numerically. Similar expressions for censored data can be found elsewhere (see ASTM C1239^[3]).

Once the distribution parameters are estimated by utilizing one of the estimation techniques discussed above, the next step in deriving the Weibull material scale parameter σ_0 is to determine effective volume and effective area for the test component. The following chapter highlights the use of a numerical technique for determining the effective volume and area for some of the standard test specimens widely utilized by testing communities. The numerical results thus obtained are then compared to the well-accepted closed form solutions included in the ASTM standards. This comparison is the core component of this research effort.

CHAPTER III

EFFECTIVE VOLUME AND EFFECTIVE AREA

To account for strength-size-scaling effects introduced and discussed in the previous chapter, one needs to compute the effective volume and/or the effective area. As discussed earlier, the Weibull material scale parameter σ_o , which is a material property, is requisite in computing the probability of failure of a component using Equation (2.2). Thus from the following expression

$$(\sigma_o)_V = (V_{eff})^{1/m} (\sigma_\theta)_V \quad (2.7)$$

one can evaluate σ_o given knowledge of σ_θ and m , as well as the effective volume (V_{eff}) which is also a function of m and load conditions. So as a key to an accurate value of σ_o , one requires accurate values of V_{eff} (discussed in this chapter) and accurate estimates of the Weibull distribution parameters σ_θ and m . Closed form expressions for effective volume (V_{eff}) as well as effective area (A_{eff}) are available for a number of test specimens used in the ceramics community. These expressions are presented in this chapter and an

outline of an alternate numerical approach in computing effective volumes and effective areas is given.

As a simple example, consider a uniaxial test specimen. The following figure depicts the geometry of the uniaxial test specimen developed by Oak Ridge National Laboratory (ORNL)^[15].

Figure 3.1 ORNL Button Head (Round) uniaxial tensile test specimen

As discussed in the previous chapter, the effective volume for a test specimen is given by the following expression

$$V_{eff} = \int_V \left(\frac{\sigma}{\sigma_{max}} \right)^m dV \quad (2.5)$$

where the integration is performed over all tensile regions of the specimen volume. A uniaxially loaded tensile specimen is subject to a uniform stress state throughout the gage volume. Thus

$$\sigma = \sigma_{\max} \quad (3.1)$$

and performing the integral in Equation (2.5) over the gage volume of the uniaxial specimen yields the following relationship

$$V_{eff} = V_{gage} \quad (3.2)$$

where

$$V_{gage} = \frac{\pi}{4} (d_{gage}^2) L_{gage} \quad (3.3)$$

Here V_{gage} is volume of the gage section, L_{gage} is length of the gage section and d_{gage} is diameter of the gage section as shown in Figure 3.1. Similarly, the effective area equation for a uniaxially loaded tensile specimen can be obtained as

$$A_{eff} = A_{gage} \quad (3.4)$$

$$A_{gage} = (\pi d_{gage}) L_{gage} \quad (3.5)$$

Here A_{gage} is surface area of the gage section and all other terms are as described above. This uniaxial specimen is included here for completeness as it easily explains the concept of effective volume and area. Again, the thrust of this work is the comparison of well accepted closed form expressions for V_{eff} (and A_{eff}) with numerically obtained values for a given test specimen geometry.

EFFECTIVE VOLUME/AREA USING A NUMERICAL APPROACH

Duffy et al.^[9] have outlined a numerical approach that utilizes Weibull analysis along with finite element analysis to compute σ_o for a given test specimen boundary conditions. Given that

$$P_f = 1 - \exp \left[- \int_V \left(\frac{\sigma}{(\sigma_o)_V} \right)^m dV \right] \quad (2.2)$$

the numerical procedure assumes the Weibull modulus is known, i.e.,

$$m = \tilde{m}_V \quad (3.6)$$

where a tilde ($\sim$) indicates that a parameter has been estimated from failure data. Also identifying

$$(\sigma_o)_V = \sigma^* \quad (3.7)$$

as a parameter, then,

$$P_f = 1 - \exp \left[- \int_V \left(\frac{\sigma}{\sigma^*} \right)^{\tilde{m}_V} dV \right] \quad (3.8)$$

Rearranging yields

$$P_f = 1 - \exp \left[- \left(\frac{\sigma_{\max}}{\sigma^*} \right)^{\tilde{m}_V} \int_V \left(\frac{\sigma}{\sigma_{\max}} \right)^{\tilde{m}_V} dV \right] \quad (3.9)$$

Now let

$$V_{eff} = \int_V \left(\frac{\sigma}{\sigma_{\max}} \right)^{\tilde{m}_V} dV \quad (3.10)$$

where V_{eff} captures both the geometry and a normalized stress gradient (stress being normalized with respect to the maximum stress). Then,

$$P_f = 1 - \exp \left[-V_{eff} \left(\frac{\sigma_{max}}{\sigma^*} \right)^{\tilde{m}_V} \right] \quad (3.11)$$

Solving the above equation for V_{eff} (treating σ^* as a parameter) yields

$$V_{eff} = \frac{-\ln(1 - P_f)}{\left(\frac{\sigma_{max}}{\sigma^*} \right)^{\tilde{m}_V}} \quad (3.12)$$

At this step the numerical approach which utilizes finite element analysis along with Weibull analysis can be performed to evaluate the effective volume V_{eff} . The numerical approach is as follows:

- Conduct a finite element analysis to obtain the stress field throughout the test specimen. This analysis will also provide σ_{max} .
- Estimate $\tilde{m}_V$ from failure data.
- Numerically evaluate Equation (3.8) for P_f using the CARES^[13] algorithm, here $\tilde{m}_V$ from step 2 is used with an arbitrarily selected value of σ^* .
- With σ^* , P_f and σ_{max} solve the Equation (3.12) for V_{eff} .

For a given Weibull modulus, the parameter V_{eff} depends only on the load configuration and test specimen geometry and does not depend on the magnitude of the applied load, ultimate strength of material, or elastic material properties (assuming the

material response is elastic throughout the material for this work). A similar approach can be followed to compute A_{eff} as well. Utilizing the above procedure, a comparison of values obtained from published closed form expressions for V_{eff} and A_{eff} to numerically generated values can be made. Figure 3.2 shows a flowchart for the numerical procedure explained above.

Figure 3.2 Flow chart for numerical procedure

In the following sections of this chapter closed form expressions for effective volume and effective area associated with the various test specimens considered in this

work are given. This is followed by a discussion of the finite element models for each test specimen. This serves as a prelude for the results obtained using the numerical procedure just outlined, and these results are presented in Chapter 4. The closed form expressions for effective volume and area have traditionally been derived based on certain simplifying assumptions. In general these expressions are based on consideration that the specimen is subjected to uniaxial stress states that are not necessarily uniform and thus neglects multi-axial states of stress.

On the other hand the numerical approach considers the full stress state at each point in the component. The multi-axial stress states for a given test specimen under the applied loads were obtained using the ANSYS (Canonsburg, PA) finite element software. The stress states serve as input for the CARES/Life (Ceramic Analysis and Reliability Evaluation of Structures) and WeibPar (both programs developed by Connecticut Reserve Technologies and NASA-GRC, Cleveland, OH). These software algorithms are utilized to compute P_f in the numerical analysis procedure explained above for calculating σ_o .

Most commercial ceramics have been reported to possess Weibull moduli in the range of 5 through 25, and for this reason the effective volumes and areas were calculated for a range of pre-selected Weibull modulus values of 5, 10, 15, 20 and 25 for the various specimen types assessed in this work. For basic material properties the modulus of elasticity (E) was taken as 370,000 MPa, and Poisson's ratio (ν) was taken as 0.22.

These are values typical for aluminum oxide. The following sections discuss the effective volume and area calculations for:

- The four-point flexure bar – rectangular cross-section
- The three-point flexure bar – rectangular cross-section
- The four-point flexure bar – circular cross-section
- The three-point flexure bar – circular cross-section
- O-ring
- C-ring
- Ring-on-ring biaxial disc
- Burst-tube (Pressurize-tube) specimen

Some of the test specimens and their attending geometries listed above are time tested and are utilized throughout the ceramics community. Others in the list have served as appropriate specimen geometries for specific projects. The more widely used geometries have ASTM standards associated with the particular specimen. Each will be discussed in turn in the following sections.

RECTANGULAR TEST SPECIMEN (ASTM C1161^[1])

Four-point flexure – Effective volume

The basic modulus of rupture (MOR) or bend test (see ASTM C 1161^[1] and ASTM C 1211^[2]) is depicted in the following figure.

Figure 3.3 Arrangement showing four-point flexure of beam specimen

In the four-point flexure test it is important to note that for the regions enclosed between the inner load points (defined by span L_i in Figure 3.3) no shear stresses act over the beam cross-section and hence that region is subjected to pure bending. Shear stresses are present between the inner load span and outer support span, but these stresses are ignored in developing the closed form expressions for V_{eff} and A_{eff} . For fracture origins that are spatially distributed within the volume defined by the outer span supports, the effective volume is obtained by integrating the following expression^[18]

$$V_{eff} = \int_V \left[\frac{\sigma(x, y)}{\sigma_{max}} \right]^{m_f} dV \quad (2.5)$$

where the x , y and z axes are as shown in Figure 3.3. The longitudinal stress $\sigma(x,y)$ at a point in the rectangular bar specimen under the four-point flexure with a total applied load of P (two concentrated load of $P/2$) can be expressed as

$$\begin{aligned}\sigma(x,y) &= \frac{M c}{I} \\ &= \left[\frac{6 P x y}{b d^3} \right]_{x=0}^{x=\frac{L_o-L_i}{2}} + \left[\frac{3 P (L_o-L_i) y}{b d^3} \right]_{x=\frac{L_o-L_i}{2}}^{x=\frac{L}{2}}\end{aligned}\quad (3.13)$$

where M is the bending moment due to the total applied load P , I is the moment of cross-sectional area, L_i is length of the inner span, L_o is length of the outer span, b and d are dimensions identified in Figure 3.3. The maximum longitudinal stress for the specimen will occur at the bottom of the mid-section and can be expressed as

$$\sigma_{\max} = \frac{3 P (L_o - L_i)}{2 b d^2} \quad (3.14)$$

Expanding the volume integral of Equation (2.5) into a triple integral will result in

$$V_{eff} = 2 \int_0^b \int_0^{d/2} \int_0^{L_o/2} \left(\frac{\sigma(x,y)}{\sigma_{\max}} \right)^{m_v} dx dy dz \quad (3.15)$$

Using Equations (3.13) and (3.14) to solve the triple integral results in

$$V_{eff} = \left[\left(\frac{L_i}{L_o} \right)^{m_v+1} + 1 \right] \left[\frac{1}{2(m_v+1)^2} \right] V_B \quad (3.16)$$

where

$$V_B = b d L_o \quad (3.17)$$

where V_B is volume of the gage section, and m_V is the Weibull modulus associated with strength controlling volume flaws.

For conventional rectangular flexure strength specimens, the effective volume is very small, i.e., on the order of about 0.07 to 8.33% of gage volume of the specimen. The lower bound on the above percentage range is controlled by the minimum value of the inner load span ($L_i = 0$) and the maximum value of Weibull modulus used in the study, i.e., 25. Similarly the upper bound on percentage range is controlled by the minimum value of Weibull modulus, i.e., 5, and the largest value of the inner load span (equal to the outer support span). As the maximum stress under the four-point flexure occurs at the outer surface of the specimen this test specimen is very sensitive to the “surface quality” of the tensile side of the specimen in addition to the intrinsic properties of the material itself.

Four-point flexure – Effective area

For strength controlling fracture origins that are spatially distributed along the surface of the test specimen and between the outer support span, the equation for the effective area can be obtained by modifying the Equation (2.5) as

$$A_{eff} = \int_A \left[\frac{\sigma(x, y)}{\sigma_{max}} \right]^{m_A} dA \quad (3.18)$$

where m_A is the Weibull modulus associated with the strength controlling surface flaws, A is the specimen surface area of the gage section and all the other terms are described

earlier. The gage area (section) is defined as the specimen surface area between the outer support span. The integration in the above equation is performed over the specimen gage section. Thus using Equations (3.13) – (3.14) solving Equation 3.18, the effective area can be expressed as

$$A_{eff} = L_o \left[\frac{d}{(m_A + 1)} + b \right] \left[\left(\frac{L_i}{L_o} \right)^{m_A + 1} \right] \left[\frac{1}{m_A + 1} \right] \quad (3.19)$$

where all the terms are as explained earlier. For conventional rectangular flexure strength specimens, the effective area is on the order of about 1.32 to 37.50% of gage section area of the specimen.

Three-point flexure – Effective volume

The three-point flexure specimen can be considered a special case of four-point flexure specimen as the inner load points approach each other. Figure 3.4 shows the arrangement for the typical three-point flexure of a rectangular beam specimen.

Figure 3.4 Arrangement showing three-point flexure of beam specimen

For fracture origins that are spatially distributed along the test specimen between the outer support span, the effective volume, as discussed earlier, can be obtained by analytically integrating the following expression

$$V_{eff} = \int_V \left[\frac{\sigma(x, y)}{\sigma_{max}} \right]^{m_V} dV \quad (2.5)$$

The longitudinal stress, σ , at a point in the cross section of a three-point flexure specimen with an applied central load of P can be expressed as

$$\sigma(x, y) = \frac{6 P x y}{b d^3} \quad (3.20)$$

where the x , y and z axes are as shown in Figure 3.4, and b and d are dimensions identified in Figure 3.4. The maximum longitudinal stress for the specimen will occur at the bottom of the mid-section and can be expressed as

$$\sigma_{max} = \frac{3 P L_o}{2 b d^2} \quad (3.21)$$

Expanding the volume integral of Equation (2.5) into a triple integral will result in

$$V_{eff} = 2 \int_0^b \int_0^{d/2} \int_0^{L_o/2} \left(\frac{\sigma(x, y)}{\sigma_{max}} \right)^{m_V} dx dy dz \quad (3.22)$$

Once again note that shearing stresses are ignored in the above formulation. Using the Equations (3.20) and (3.21) to solve the triple integral results in

$$V_{eff} = \left[\frac{1}{2(m_V + 1)^2} \right] V_B \quad (3.23)$$

where

$$V_B = b d L_o \quad (3.24)$$

where L_o is length of the outer span, V_B is volume of the gage section. Alternatively the closed form equations for effective volume for rectangular beams in three-point flexure can be obtained by simply setting $L_i = 0$ in Equation (3.16).

Three-point flexure – Effective area

Considering Equation (3.18) and using an approach similar to that followed for effective volume calculations [Equations (3.20) – (3.24)] for fracture origins that are spatially distributed along the surface of the test specimen between the outer support span, the equation for effective area can be expressed as

$$A_{eff} = L_0 \left[\frac{d}{(m_A + 1)} + b \right] \left[\frac{1}{m_A + 1} \right] \quad (3.25)$$

where all the terms have been previously identified. Alternatively Equation (3.25) can be obtained by just letting the inner span, $L_i = 0$ in Equation (3.19) for the four-point flexure.

Finite element modeling and analysis

As discussed earlier, finite element modeling has been performed using the commercial software ANSYS to capture the multi-axial stress state of the specimen under consideration. The finite element analysis geometry of the four-point flexure specimen was selected in accordance with ASTM C 1161^[1] configuration B guidelines, which specify overall dimensions of 4 x 3 x 45 mm. The outer support span of 40 mm (with 2.5

mm long overhang on either ends) was chosen for all load configurations. To optimize the computational resources by taking the advantage of symmetry, only one quarter of the test specimens were modeled for the full depth of the beam. This simulates the application of load. Symmetry conditions were enforced by restraining the displacement of nodes present along planes of symmetry in the direction normal to the planes of symmetry. Force boundary conditions were imposed (see red arrows in Figure 3.4) at the inner load supports, which were centered along the top of the specimen 20 mm apart (for $L_i / L_o = 0.5$). Support conditions were simulated by constraining the relevant nodal degree of freedom of the bottom layer nodes at the outer load span.

Figure 3.5 Finite element model of quarter symmetry rectangular bar showing applied forces (red arrows)

One could utilize the ANSYS SOLID45 element or the ANSYS SOLID95 element for the analysis of this test specimen. The SOLID95 element is a higher order version of the three-dimensional eight-node SOLID45 element. The higher order element was used for the models presented here for the numerical efficiency, i.e., fewer elements are needed. The SOLID95 brick element has twenty nodes with three translational degrees of freedom per node.

Meshing options were developed using ANSYS macro language to control the density of the mesh generated in certain regions. Basically the regions of interest for this work are regions with high tensile stresses. The numerical procedure outlined in this work is quite sensitive to the maximum tensile stress value generated by the finite element model. Since the maximum tensile flexural stress occurs along bottom of the specimen at the mid-span, the mesh was very dense in this region. A mesh sensitivity study was performed using six different meshes – from one mesh identified as a very coarse mesh to the last one identified as a fine mesh. The six mesh densities used are summarized in Table 3.1. The very coarse (VC) and the fine (F) meshes are shown in Figures 3.6 and 3.7, respectively.

Table 3.1 Mesh designations and corresponding number of elements for rectangular bars (Four-point flexure – $L_i/L_o = 0.5$)

Mesh Density	Volume Elements	Surface Elements (CARES generated)	Maximum Bending Tensile Stress (MPa)
Very Coarse (VC)	1950	725	163.64
Coarsest (CS)	1950	725	166.44
Coarse (CO)	6150	1585	167.89
Coarse/Medium (CM)	16800	3100	168.21
Medium (M)	28400	4660	168.39
Fine (F)	53750	7075	168.43

Figure 3.6 Finite element model of quarter symmetry rectangular bar with very coarse (VC) mesh

Figure 3.7 Finite element model of quarter symmetry rectangular bar with fine (F) mesh

Figure 3.8 shows the contour plot of the longitudinal stresses (σ_x) along x-axis, for the one-quarter symmetry model shown in Figure 3.7. For the co-ordinate system (shown by triad in Figure 3.8) selected for the model along with the applied load condition, the principal axis direction coincides with the x-direction. Figure 3.8 also shows the maximum (positive) and minimum (negative) longitudinal stresses in the specimen.

Figure 3.8 Longitudinal stress (σ_x) field in quarter symmetry rectangular bar with fine mesh (Four-point flexure – $L_i/L_o = 0.5$)

To validate the stress state obtained in the finite element analysis, the maximum longitudinal tensile stress from the finite element analysis was compared with the classical strength of materials solution. For a four-point flexure beam with a inner load span to outer support span ratio of 0.5 and with a total applied load of 200 N (2 x 100 N concentrated load) the calculated maximum theoretical longitudinal stress is 166.67 MPa. The maximum longitudinal stress value of 168.48 MPa (see Figure 3.8) obtained from the most refined finite element mesh (Fine 'F' - see Table 3.1) has a good agreement with the theoretical maximum stress value.

The procedure described above for maximum stress comparison was also used for comparing the maximum deflection in the specimen. The maximum deflection for the four-point flexure (as well as three-point flexure specimen) occurs in the bottom layer at the mid-span of the specimen. From the finite element analysis, the maximum deflection at neutral axis was found to be 0.0557 mm (see legend in Figure 3.8) and again this deflection is in close agreement with the value of 0.0551 mm calculated from the classical equation. In essence theoretical stress values and deflection values serve as an initial discriminant for the finite element analysis. The quality of the mesh will be discussed further in the next chapter.

To observe the effect of transitioning from four-point flexure to three-point flexure load configuration over the effective volume and area calculations, a range of span ratios of inner load span to outer support span (L_i / L_o) were considered. Keeping the outer support span constant at 40 mm and changing inner load span values results in the different span ratios which are listed in Table 3.2.

Table 3.2 Inner load span to outer support span ratios (L_i/L_o) for rectangular bars

Inner Load Span (L_i) in mm	Outer Support Span (L_o) in mm	Span Ratio (L_i/L_o)	Load Configuration
20	40	0.5	Four-point
16	40	0.4	Four-point
12	40	0.3	Four-point
8	40	0.2	Four-point
4	40	0.1	Four-point
0	40	0.0	Three-point

The finite element modeling, analysis and validation procedures discussed previously for the span ratio of 0.5 were likewise used for the range of span ratios discussed above. The calculated values of maximum stress and deflection in all the specimens were found to be in close agreement between the strength of materials solution and finite element solution. Based on these comparisons the finite element analyses were considered valid for the initial work conducted. The stress contour plots of Figure 3.8 is representative of all the models with the range of span ratios (listed in Table 3.2) for the six mesh types (listed in Table 3.1).

Figure 3.9 shows contour plot for the first principal stress for the four-point flexure specimen shown in Figure 3.7. In the contour plot, the tensile stresses are marked as positive stresses. The region of the specimen between the outer support span below

the neutral axis will be under tensile stresses with the region of high tensile stresses at the bottom layer as highlighted by the red color contour. This region of tensile stresses fades out near the outer support span. The upper half region (above the neutral axis) of the specimen will be under compressive (or neutral-zero) stresses, thus this region do not contribute towards the calculation of the effective volume and area of the specimen as per the underlying assumptions of the Weibull theory. Thus Figure 3.9 highlights the regions of reliability integration performed over the specimen volume by CARES software.

Figure 3.9 First principal stress (σ_1) contour plot in quarter symmetry rectangular bar with fine mesh (Four-point flexure – $L_i/L_o = 0.5$)

CIRCULAR ROD TEST SPECIMEN (*ASTM C1683*^[7])

Four-point flexure - Effective volume

Researchers (Quinn^[21]) have utilized round rods into their testing programs as some commercial applications involve use of circular components. The common tests performed on these round specimens for strength determination are four-point and three-point flexure tests. Figure 3.10 represent the four-point flexure test setup for round rods.

Figure 3.10 Round rod four-point flexure test geometry

Figure 3.11 Cross-section for round rod showing segment used for integration

The effective volume for the rod specimen for fracture origins that are spatially distributed along the test specimen between the outer support span, can be once again obtained by analytically integrating the following expression

$$V_{eff} = \int_V \left[\frac{\sigma(x, y)}{\sigma_{max}} \right]^{m_f} dV \quad (2.5)$$

Looking into the stress state under the flexural load it may be observed that the lower halve of the gage volume will be stressed in tension. In the four-point flexure rod specimen with an applied total load of P (two concentrated loads of $P/2$) the stress at a point will depend upon the radial distance r (along vertical y-axis see Figure 3.11) and position along the x-axis (see Figure 3.10). The normal stress can be expressed as

$$\sigma = \frac{r}{R} \left(1 + \frac{2x}{(L_o - L_i)} \right) \sigma_{max} \quad (3.26)$$

where R is the radius of the rod, L_i is length of the inner span, L_o is length of the outer span as defined in Figures 3.10 and 3.11. Once again note that shearing stresses are ignored in the above formulation. The maximum longitudinal stress for the specimen will occur at the bottom of the mid-section and can be expressed as

$$\sigma_{\max} = \frac{P(L_o - L_i)}{\pi R^3} \quad (3.27)$$

Also noting

$$dA = 2l dr \quad (3.28)$$

and

$$l = [R^2 - r^2]^{1/2} \quad (3.29)$$

and

$$dA = 2[R^2 - r^2]^{1/2} dr \quad (3.30)$$

Expanding the volume integral of Equation (2.5) will result in

$$V_{eff} = 2 \int_0^{L_o/2} \int_A \left(\frac{\frac{r}{R} \left(1 + \frac{2x}{(L_o - L_i)} \right) \sigma_{\max}}{\sigma_{\max}} \right)^{m_V} dA dx \quad (3.31)$$

As shown in Figure 3.11, the shaded segment $2l dr$ is integrated over the cross-section of the rod specimen. Using expressions (3.26) through (3.30) the solution of the integral in Equation (3.31) yields the closed form equation for effective volume as

$$V_{eff} = \left[\left(\frac{L_i}{L_o} \right) m_V + 1 \right] \left[\frac{1}{(m_V + 1)} \right] \left(\frac{G}{\pi} \right) V_B \quad (3.32)$$

where V_B is the gage volume within the outer support span expressed as

$$V_B = \pi R^2 L_o \quad (3.33)$$

Note that G in Equation (3.32) is a combined gamma function given by

$$G = \left[\frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{m+4}{2}\right)} \right] \quad (3.34)$$

where Γ is the gamma function.

For conventional circular rod flexure specimens, similar to the rectangular bar specimens, the effective volume is very small, i.e., on the order of about 0.2 to 4.9% of gage volume of the specimen. Since the maximum stress under four-point flexure occurs at the outer surface of the specimen, this test is very sensitive to the “surface quality” of the tensile side of the specimen in addition to the intrinsic properties of the material itself.

Four-point flexure - Effective Area

For fracture origins that are spatially distributed along the surface of the gage section between the outer support span, the effective area for the rod specimen can be obtained by utilizing the following equation we saw earlier

$$A_{eff} = \int_A \left[\frac{\sigma(x,y)}{\sigma_{max}} \right]^{m_A} dA \quad (3.18)$$

In the above equation solving the integral over the gage area following an approach similar to the one followed for effective volume calculations and using Equations (3.26) – (3.30), the effective area can be expressed as

$$A_{eff} = \left[\left(\frac{L_i}{L_o} \right) m_A + 1 \right] \left[\frac{(m_A + 2)}{2} \right] \left[\frac{1}{(m_A + 1)} \right] \left(\frac{G}{\pi} \right) A_B \quad (3.35)$$

where

$$A_B = L_o (\pi D) \quad (3.36)$$

where all the terms are explained earlier. For conventional circular rod flexure specimens the effective area is on the order of about 0.3 to 17% of gage section of the specimen.

Three-point flexure - Effective Volume

The three-point flexure specimen can be considered a special case of four-point flexure specimen as the inner load points approach each other. Figure 3.12 shows the arrangement for the typical three-point flexure of a circular rod specimen.

Figure 3.12 Round rod three-point flexure test geometry

For fracture origins that are spatially distributed along the test specimen between the outer support span, the effective volume, as discussed earlier, can be obtained by analytically integrating the following expression

$$V_{eff} = \int_V \left[\frac{\sigma(x, y)}{\sigma_{max}} \right]^{m_f} dV \quad (2.5)$$

Again only the lower halve of the gage volume will be stressed in tension under the three-point flexure condition. The stress at a point in the three-point flexure specimen with an applied concentrated load of P will depend upon the radial distance (depth) r (see Figure 3.11) and with position along the x-axis (see Figure 3.12) and can be expressed as

$$\sigma = \left(\frac{r}{R} \right) \left(\frac{2x}{L_o} \right) \sigma_{max} \quad (3.37)$$

where R is the radius of the rod, L_o is length of the outer span as defined in Figure 3.12.

The maximum longitudinal stress for the specimen will occur at the bottom of the mid-span and can be expressed as

$$\sigma_{\max} = \frac{P L_o}{\pi R^3} \quad (3.38)$$

Also noting

$$dA = 2l \, dr \quad (3.39)$$

and

$$l = [R^2 - r^2]^{1/2} \quad (3.40)$$

and

$$dA = 2 [R^2 - r^2]^{1/2} dr \quad (3.41)$$

Expanding the volume integral of Equation (2.5) will result in

$$V_{eff} = 2 \int_0^{L_o/2} \int_A \left(\frac{\left(\frac{r}{R}\right) \left(\frac{2x}{L_o}\right) \sigma_{\max}}{\sigma_{\max}} \right)^{m_v} dA \, dx \quad (3.42)$$

Using the above Equations (3.37) through (3.41), solution of the integral in Equation (3.42) yields the closed form equation for effective volume as

$$V_{eff} = \left[\frac{1}{(m_v + 1)} \right] \left(\frac{G}{\pi} \right) V_B \quad (3.43)$$

where V_B is the gage volume within the outer support span and is expressed as

$$V_B = \pi R^2 L_o \quad (3.44)$$

where all the other terms are as explained earlier. For the circular cross section flexure rods the closed form equation for effective volume in three-point flexure can be obtained simply by setting $L_i = 0$ in Equation (3.32).

Three-point flexure - Effective Area

Using similar approach as explained above for fracture origins that are spatially distributed along the surface of the test specimen between the outer support span, the equation for effective area is

$$A_{eff} = \left[\frac{(m_A + 2)}{2} \right] \left[\frac{1}{(m_A + 1)} \right] \left(\frac{G}{\pi} \right) A_B \quad (3.45)$$

where the total gage area within the outer supporting points A_B is given by

$$A_B = L_o (\pi D) \quad (3.46)$$

where all the terms are explained earlier.

Finite element modeling and analysis

As discussed earlier, finite element modeling has been performed using the commercial software ANSYS to capture the stress state of the specimen under consideration. The dimensions selected for the circular rod were 6.0 mm diameter and an overall length of 115.0 mm (Quinn^[21]). The outer support span was chosen as 110.0 mm (with 2.5 mm long overhang on either ends) for all the load configurations. To take the

advantage of symmetry, only one quarter of the rods were modeled for the full depth of the rods. Symmetry conditions were enforced by restraining the displacement of nodes present along planes of symmetry in the direction normal to the planes of symmetry. Force boundary conditions were imposed (see red arrows in Figure 3.13) at the inner load supports, which were centered along the top of the specimen 55 mm apart (for $L_i/L_o = 0.5$). Support conditions were simulated by constraining the relevant nodal degree of freedom of bottom layer nodes at the outer load span.

Figure 3.13 Finite element model of quarter symmetry circular rod specimen showing applied forces (red arrows)

The 20-node SOLID95 brick elements selected for the rectangular bars were also used to model the circular rods. A mesh sensitivity study was performed using six different meshes, from one mesh identified as a very coarse mesh to the last one identified as a fine mesh. The six mesh densities used are summarized in Table 3.3. The very coarse (VC) and fine (F) meshes are shown in Figures 3.14 and 3.15, respectively. The modeling of denser mesh in the bottom mid-span region is due to sensitivity of the numerical procedure to the maximum tensile stress value similar to the rectangular bars.

Table 3.3 Mesh designations and corresponding number of elements for circular rods (Four-point flexure – $L_i/L_o = 0.5$)

Mesh Density	Volume Elements	Surface Elements (CARES generated)	Maximum Tensile Stress (MPa)
Very Coarse (VC)	2880	576	2.477
Coarsest (CS)	6480	936	2.572
Coarse (CO)	15360	1664	2.588
Coarse/Medium (CM)	27000	2400	2.598
Medium (M)	39930	3164	2.603
Fine (F)	51840	3744	2.604

Figure 3.14 Finite element model of quarter symmetry circular rod specimen with very coarse (VC) mesh

Figure 3.15 Finite element model of quarter symmetry circular rod specimen with fine
(F) mesh

Figure 3.16 shows the contour plot of the longitudinal stresses (σ_x) along x-axis, for the one-quarter symmetry model shown in Figure 3.15. For the co-ordinate system (shown by triad in Figure 3.16) selected and the applied loads, the principal axis direction coincides with the x-direction. Figure 3.16 also shows the maximum (tensile) and minimum (compressive) longitudinal stresses in the specimen.

Figure 3.16 Longitudinal stress (σ_x) field in quarter symmetry circular bar with fine mesh (Four-point flexure – $L_i/L_o = 0.5$)

The initial validation of the finite element model was performed in a manner similar to that followed for the rectangular bars. For a four-point flexure rod with a inner load span to outer support span ratio of 0.5, and a total load of 2.0 N (2 x 1.0 N concentrated load) the calculated maximum theoretical stress is 2.59 MPa. The maximum stress obtained from the most refined finite element mesh (Fine 'F' - see Table 3.3) was 2.60 MPa (see Figure 3.16). The maximum deflection under the load condition occurs at mid-span for the case of four-point (and three-point) flexure specimen. From finite element analysis, the maximum deflection is calculated to be 0.0033 mm (see

legend in Figure 3.16), which is in close agreement with the value of 0.0032 mm calculated from classical equation. So once again there is seemingly good agreement between results from the finite element analysis and theoretical values.

To observe the effect of transitioning from four-point flexure to three-point flexure load configuration over the effective volume and area calculations, a range of span ratios of inner load span to outer support span (L_i / L_o) were considered. By keeping the outer support span constant at 110 mm and changing inner load span values results in the different span ratios tabulated in Table 3.4.

Table 3.4 Inner load span to outer support span ratios (L_i / L_o) for circular rods

Inner Load Span (L_i) in mm	Outer Support Span (L_o) in mm	Span Ratio (L_i / L_o)	Load Configuration
55	110	0.5	Four-point
44	110	0.4	Four-point
33	110	0.3	Four-point
22	110	0.2	Four-point
11	110	0.1	Four-point
0	110	0.0	Three-point

The finite element modeling, analysis and validation procedures discussed previously for the span ratio of 0.5 were likewise used for the range of span ratios discussed above. The calculated values of maximum stress and deflection in all the specimens were found to be in close agreement between the strength of materials solution and finite element solution. Based on these comparisons the finite element analyses were considered valid for the initial work conducted.

Figure 3.17 shows the contour plot of the first principal stresses (σ_x) along x-axis, for the one-quarter symmetry model shown in Figure 3.15. In the contour plot, the tensile stresses are marked as positive stresses. The region of the specimen between the support span and bottom half thickness will be under tensile stresses with the region of high tensile stresses at the bottom layer as highlighted by the red color contour. This region of tensile stresses fades out near the outer support span. The upper half region (above the neutral axis) of the specimen will be under compressive (or neutral-zero) stresses, thus this region do not contribute towards the calculation of the effective volume and area of the specimen as per the underlying assumptions of the Weibull theory. The stress contour plots of Figure 3.16 and Figure 3.17 are representative of all the models with the different span ratios (listed in Table 3.4) for the six mesh types (listed in Table 3.3).

Figure 3.17 First principal stress (σ_I) contour plot for quarter symmetry rectangular bar with fine mesh (Four-point flexure – $L_i/L_o = 0.5$)

C – RING TEST SPECIMEN (*ASTM C1323-2006*^[5])

Consider the fabrication of ceramic tubes, i.e., the tube as the finished product, or the structural component. To examine surface defects spatially distributed around the circumference of the outer diameter of tubular components, as well as surface defects spatially distributed down the outer length of the tubular components two types of test specimen geometries are used. Generally, a diametral compression test of a C-ring geometry interrogates circumferential flaws, and sector flexure MOR bar specimen cut from ceramic tubes down the length of the tube (subjected to four-point flexure) interrogates longitudinally distributed flaws in a component. The sector flexure bar specimen is a highly specialized test specimen for an army tank barrel. A typical sector flexure bar specimen is obtained by cutting a longitudinal section from whole tubes as shown in Figure 3.18. The study on sector flexure bar specimen has already been performed elsewhere (see Duffy et al.^[10]) and it will not be considered in this work. The C-ring geometry as shown in Figure 3.19 is discussed briefly in this section.

Figure 3.18 Arrangement showing sectored flexure MOR bar specimen subjected to four-point flexure

Figure 3.19 Arrangement showing C-ring under diametral compression

Effective volume

Using formulae for bending of curved beams^[19] the distribution of circumferential (hoop) stresses in a C-ring under compressive loads can be expressed as

$$\sigma_{\theta} = \frac{P R}{b t r} \left[\frac{r - r_a}{r_a - R} \right] \cos \theta \quad (3.47)$$

where

$$R = \frac{(r_o - r_i)}{\ln \left(\frac{r_o}{r_i} \right)} \quad (3.48)$$

$$r_a = \frac{(r_o + r_i)}{2} \quad (3.49)$$

and

$$t = r_o - r_i \quad (3.50)$$

where P is the applied compressive load, r_i is the inner radius, r_o is the outer radius, b is the length, t is the thickness, r is the radius at the point of interest in the specimen, θ is the angle measured from the horizontal symmetry (see Figure 3.19), i.e., $\theta = 90^\circ$ is the point under the applied load. Under the compressive load the maximum circumferential (tensile) stress will occur at the horizontal symmetry at the outer radius and can be expressed as

$$(\sigma_\theta)_{\max} = \frac{PR}{bt r_o} \left[\frac{r_o - r_a}{r_a - R} \right] \quad (3.51)$$

where all the terms are explained above. For the C-ring specimens the effective volume (V_{eff}) can be calculated using an alternate form of Equation (2.5)

$$V_{eff} = \int_V \left[\frac{\sigma_\theta}{(\sigma_\theta)_{\max}} \right]^{m_V} dV \quad (3.52)$$

where m_V is the Weibull modulus associated with strength controlling volume flaws and the other terms appearing in the expression are explained earlier. Solving the integral in Equation (3.52) over the gage volume of the ring by utilizing Equations 3.49 through 3.52, the effective volume can be expressed as

$$V_{eff} = b(r_o)^{m_V} f(\theta) f(r) \quad (3.53)$$

where

$$f(\theta) = 2 \int_0^{\pi/2} \cos^m \theta \, d\theta \quad (3.54)$$

$$= \sqrt{\pi} \left[\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}+1\right)} \right]$$

and

$$f(r) = \int_{r_a}^{r_o} \left(\frac{r-r_a}{r_o-r_a} \right)^m r^{1-m} dr \quad (3.55)$$

where all the terms appearing in the expressions are discussed earlier. Closed form solution for the integral (after the substitutions) in Equation (3.52) does not exist and thus the equation needs to be solved numerically. As observed from the expressions above V_{eff} is dependent on the Weibull modulus m , and the geometry of the specimen. For the C-ring specimen the gage volume is the continuum volume between the two load points and can be expressed as

$$V_{gage} = \pi (r_o^2 - r_i^2) b / 2 \quad (3.56)$$

where all the terms appearing in the expression are explained earlier.

Effective area

The effective area for the C-ring specimen can be obtained by utilizing an alternate form of the Equation (3.18)

$$A_{eff} = \int_A \left[\frac{\sigma_\theta}{(\sigma_\theta)_{\max}} \right]^{m_A} dA \quad (3.57)$$

Using an approach similar to that followed for effective volume calculations [Equations (3.49) – (3.53)] for fracture origins that are spatially distributed along the surface of the test specimen, the equation for effective area can be expressed as

$$A_{eff} = b r_o f(\theta) + 2 r_o^{m_A} f(\theta) f(r) \quad (3.58)$$

where all the terms appearing in the equation are described earlier. The gage area (surface) of the specimen can be defined as the continuum area between the two load points including the back and the front flat surfaces along with the inner and the outer curved surfaces and can be expressed as

$$A_{gage} = \pi (r_o^2 - r_i^2) + \pi b (r_o + r_i) \quad (3.59)$$

where all the terms in the expression are explained earlier.

Analysis of C-ring specimens has already been published elsewhere (see Duffy et al.^[9]). The stress distributions for the C-ring is calculated assuming plane stress conditions. For a C-ring under diametral compression the locus of the highest tensile stress is in the horizontal symmetry plane at the outer radius (outer surface) as highlighted in Figure 3.20. The outcomes of the C-ring analysis from the above study are discussed in Chapter 4, where the results from the numerical efforts of this research work are discussed in details.

Figure 3.20 First principal stress contour plot of C-ring under diametral compression^[9]

O-RING TEST SPECIMENS (*ASTM WK88*^[8]) (Duffy, Segall communications)

To interrogate inner surface circumferential flaws along a tubular component one can employ an O-ring test specimen geometry (see Figure 3.21). For an O-ring under diametral compression, the maximum tensile stress occurs in the vertical symmetry plane at the inner radius (inner surface). Hence, the O-ring specimen loaded in compression predominately provides information on the strength distribution and flaw population(s) at the internal surface of a tubular component. A lesser magnitude tensile field exists on the outer surface.

Figure 3.21 Arrangement showing O-ring under diametral compression^[8]

Effective volume

Equations for the circumferential stress in the O-ring specimen are available which are based on the classical theories such as the elasticity approach and the strength of materials approach. Different closed form solutions for the effective volume and area for the O-ring specimen exists which are fundamentally derived based on the above mentioned classical theories. An elasticity solution^[16] utilizing Boussinesq approach gives the circumferential stress in an O-ring as

$$\sigma_{\theta} = \frac{P}{b t} \frac{[-Q(r, \psi)]}{r_o} \quad (3.60)$$

where P is the applied compressive load as shown in Figure 3.21, t is the radial thickness, b is the length, r_o is the outside radius, r_i is the inside radius, and Q is the stress distribution function defined as

$$Q(r, \psi) = 2A_0 - B_0 \left(\frac{r}{r_0} \right)^{-2} + \sum_{n=2,4,6,\dots} \left[\begin{aligned} &2(n+1)(n+2)A_n \left(\frac{r}{r_0} \right)^n + 2n(n-1)B_n \left(\frac{r}{r_0} \right)^{n-2} \\ &+ 2n(n+1)C_n x^n \left(\frac{r}{r_0} \right)^{-n-2} + 2n(n-1)D_n x^{n-2} \left(\frac{r}{r_0} \right)^{-2} \end{aligned} \right] n \cos(n\psi) \quad (3.61)$$

where

$$A_0 = \frac{1}{2(1-x^2)} \quad (3.62)$$

$$B_0 = -\frac{x^2}{(1-x^2)} \quad (3.63)$$

$$A_n = \frac{1}{(2E_n)} \left[\left(\frac{1-x^{2n}}{n} \right) - \left(\frac{1-x^{2n-2}}{n+1} \right) \right] \quad (3.64)$$

$$B_n = \frac{1}{(2E_n)} \left[\left(\frac{1-x^{2n}}{n} \right) - \left(\frac{1-x^{2n+2}}{n-1} \right) \right] \quad (3.65)$$

$$C_n = \frac{x^n}{(2E_n)} \left[- \left(\frac{1-x^{2n}}{n} \right) + \left(\frac{1-x^{2n-2}}{n+1} \right) x^2 \right] \quad (3.66)$$

$$D_n = \frac{x^n}{(2E_n)} \left[- \left(\frac{1-x^{2n}}{n} \right) x^2 + \left(\frac{1-x^{2n+2}}{n-1} \right) \right] \quad (3.67)$$

$$E_n = (1-x^{2n})^2 - n^2 \left[x^{2n-2} (1-x^2)^2 \right] \quad (3.68)$$

$$x = \frac{r_i}{r_o} \quad (3.69)$$

and

$$\psi = 90^\circ - \theta \quad (3.70)$$

where θ is measured from the horizontal (see Figure 3.21), i.e., $\theta = 90^\circ$ is the point under the applied load. For the O-ring specimens the effective volume (V_{eff}) can be calculated using the following equation we saw earlier

$$V_{eff} = \int_V \left[\frac{\sigma_\theta}{(\sigma_\theta)_{\max}} \right]^{m_V} dV \quad (3.52)$$

Solving the integral in the above equation over the gage volume of the ring, the effective volume can be expressed as

$$V_{eff} = \frac{4b}{C^m} \left[2 \int_0^{\theta_1} \int_{r(\theta)}^{r_o} [F(r, \theta)]^m r dr d\theta + 2 \int_{\theta_2}^{\pi/2} \int_{r_i}^{r(\theta)} [F(r, \theta)]^m r dr d\theta \right] \quad (3.71)$$

where

$$F(r, \theta) = -0.637 \left(\frac{r_a}{I} \right) r + \left(\frac{r_a}{I} \right) r \cos \theta - \cos \theta \left(\frac{r_a^2}{I} + \frac{1}{b t} \right) + 0.637 \left(\frac{r_a^2}{I} \right) \quad (3.72)$$

$$r(\theta) = \frac{\cos \theta \left(\frac{r_a^2}{I} + \frac{1}{b t} \right) + 0.637 \frac{r_a^2}{I}}{\frac{r_a}{I} (\cos \theta - 0.637)} \quad (3.73)$$

$$C = 3.82 \left(\frac{r_a}{b t^2} \right) \quad (3.74)$$

$$r_a = \frac{r_o + r_i}{2} \quad (3.75)$$

$$t = r_o - r_i \quad (3.76)$$

and

$$I = \frac{b t^3}{12} \quad (3.77)$$

other terms appearing in the expressions are identified in Figure 3.21. For the O-ring specimen the gage volume can be expressed as

$$V_{gage} = \pi (r_o^2 - r_i^2) b \quad (3.78)$$

where all the terms appearing in the expression are explained earlier.

Effective area

Utilizing Equation (3.57) and Equations (3.60) through (3.70), the effective area (A_{eff}) for the O-ring specimen can be expressed as

$$A_{eff} = \frac{4}{C^m} \left[b r_o \int_0^{\theta_1} [F(r_o, \theta)]^m d\theta + b r_i \int_{\theta_2}^{\pi/2} [F(r_i, \theta)]^m d\theta + \right. \\ \left. 2 \int_0^{\theta_1} \int_{r(\theta)}^{r_o} [F(r, \theta)]^m r dr d\theta + 2 \int_{\theta_2}^{\pi/2} \int_{r_i}^{r(\theta)} [F(r, \theta)]^m r dr d\theta \right] \quad (3.79)$$

where all the terms appearing in the above equation are described earlier. The gage area (surface) of the specimen can be expressed as

$$A_{gage} = 2 \pi (r_o^2 - r_i^2) + 2 \pi b (r_o + r_i) \quad (3.80)$$

where all the terms appearing in the expression are explained earlier. The numerical effective volume and area analysis of O-ring specimens has already been performed elsewhere (Duffy et al.^[12]). As discussed earlier the maximum tensile stress occurs in the vertical symmetry plane at the inner radius (inner surface) which is highlighted in Figure 3.22. The outcomes of the O-ring analysis from the above mentioned reference are discussed in Chapter 4.

Figure 3.22 First principal stress plot of O-ring under diametral compression^[12]

RING-ON-RING BIAXIAL TEST SPECIMEN (*ASTM C1499*^[6])

The ring-on-ring test geometry is employed to examine the components subject to equibiaxial state of stress. The specimen geometry also minimize the effects of specimen edge preparation as compared to uniaxial tests because the generated stresses are lowest at the specimen edges. Two types of test specimens, i.e., circular and rectangular discs can be tested as per the ASTM standard^{[6][7]} with circular being preferable. Thus circular specimen geometry is studied here.

Weibull analysis as it is applied to uniform uniaxial tension can be extended to multi-axial stresses by utilizing a multi-axial failure model. Models widely used in ceramic community that allow for multi-axial stress states are the Principle of Independent Action (PIA) model, the Weibull normal stress averaging model, the Batdorf coplanar strain energy release rate model and the Batdorf-Shetty model. Earlier we noted that

$$P_f = 1 - \exp \left[- \int_V \left(\frac{\sigma}{(\sigma_o)_V} \right)^m dV \right] \quad (2.2)$$

The PIA model is developed based on the alternate form of Equation (2.2), which can be stated as

$$P_f = 1 - \exp \left[- \int_V \psi dV \right] \quad (3.81)$$

where ψ is identified as a failure function per unit volume. For the PIA multi-axial approach considered here the ceramic material behavior is considered isotropic. Failure

from any of the three principal stresses are all assumed to be acting statistically independent of each other. Assuming that the ceramic material for the test specimen is characterized by the weakest link PIA reliability model, Equation (3.81) results in the probability of failure given as

$$P_f = 1 - \exp \left\{ - \int_V \left[\left(\frac{\sigma_1}{\sigma_\theta} \right)^m + \left(\frac{\sigma_2}{\sigma_\theta} \right)^m + \left(\frac{\sigma_3}{\sigma_\theta} \right)^m \right] dV \right\} \quad (3.82)$$

where σ_1 , σ_2 and σ_3 represents the first, second and third principal stress respectively in the specimen under the given load configuration; all the other terms have been discussed earlier. The PIA model incorporates the principal stresses and hence captures the multi-axial state of stress of the specimen. The PIA model does not specify the nature of the defect causing failure mode, and thus referred to as a phenomenological model.

Previously alluded Batdorf-Shetty multi-axial reliability model has also been utilized for calculating the effective volume and area to perform a comparative study between the two reliability models. The Batdorf model is based on the recognition that the brittle fracture in ceramic is governed by linear elastic fracture mechanics (LEFM). This model incorporates the effect of fracture mechanics into the weakest link Weibull theory. As pointed out in earlier discussion that it is not possible to determine the size and orientation of flaw sizes in brittle ceramics, in the Batdorf model, the material flaws distributed throughout the volume (and/or over the surface) are assumed to possess a statistically random orientation. In addition, the flaws are assumed to be non-interacting discontinuities with consistent crack geometry. Failure in the component is assumed to

occur when a far-field effective stress associated with the weakest flaw reaches a critical level. The effective stress is a predefined combination of the far-field normal stress and the far-field shear stress in this model. It is also a function of the assumed crack configuration, the existing stress state, and the fracture criterion employed. Accounting for the presence of a far-field shear stress reduces the far-field normal stress needed for fracture. The model is developed based on the Equation (3.81) we saw earlier

$$P_f = 1 - \exp \left[- \int_V \psi \, dV \right] \quad (3.81)$$

where ψ is identified as a failure function per unit volume. This function is identified by taking

$$\psi = \int_0^{(\sigma_e)_{\max}} \frac{\Omega(\Sigma, \sigma_{cr})}{4\pi} \frac{d[\eta(\sigma_{cr})]}{d\sigma_{cr}} d\sigma_{cr} \quad (3.83)$$

where Ω is a solid angle that is dependent on the fracture criterion selected, the crack configuration, and the applied stress and can be expressed as

$$\Omega = \int_{\beta=0}^{2\pi} \int_{\alpha=0}^{\pi} H(\sigma_e, \sigma_{cr}) \sin \alpha \, d\alpha \, d\beta \quad (3.84)$$

where α and β are radial and azimuthal angles in the principal stress space, and the Heaviside step function is expressed as

$$H(\sigma_e, \sigma_{cr}) = 1 \quad \sigma_e \geq \sigma_{cr} \quad (3.85)$$

and

$$H(\sigma_e, \sigma_{cr}) = 0 \quad \sigma_e < \sigma_{cr} \quad (3.86)$$

Here σ_e is the far-field normal stress, the maximum effective stress $(\sigma_e)_{\max}$ is defined as an equivalent mode I fracture stress for mixed-mode load and σ_{cr} is defined as the critical far-field normal stress for a given crack configuration under mode I load. The Batdorf crack density function $\eta(\sigma_{cr})$ is considered independent of the uniform multi-axial stress state Σ throughout the finite volume. The function is a distribution function characterizing the distribution of crack size and shape and is expressed in power law as

$$\eta(\sigma_{cr}) = k_B (\sigma_{cr})^m \quad (3.87)$$

where k_B is the Batdorf crack density coefficient and it is evaluated from experimental fracture data similar to the Weibull modulus m . The parameter k_B is a material specific parameter in much the same manner that σ_o (the Weibull material scale parameter) is a material specific parameter. Thus, the component failure probability for volume flaws can be expressed as

$$P_f = 1 - \exp \left\{ - \int_V \left[m k_B \int_0^{(\sigma_e)_{\max}} \left(\frac{\Omega(\Sigma, \sigma_{cr})}{4\pi} \sigma_{cr}^{m-1} \right) d\sigma_{cr} \right] dV \right\} \quad (3.88)$$

where all the terms appearing in the expression are explained earlier. Chapter 4 discusses the results of the effective volume and area calculations using the PIA and the Batdorf reliability models and compares the results obtained using the two theories.

Effective volume

Consider a circular disc specimen under the concentrically applied ring load as shown in Figure 3.23. Under this load configuration the resulting stress state is biaxial

tension over the entire gage volume. The gage volume is defined as the volume of the disc specimen inside the outer support ring diameter.

Figure 3.23 Ring-on-ring biaxial flexure test arrangement (ASTM C1499^[6])

Figure 3.24 Circular ring-on-ring test arrangement showing disc dimensions

For fracture origins spatially distributed along the gage volume of the test specimen, the effective volume can be obtained by analytically integrating the following expression

$$V_{eff} = \int_V \left[\left(\frac{\sigma_1}{\sigma_{max}} \right)^m + \left(\frac{\sigma_2}{\sigma_{max}} \right)^m + \left(\frac{\sigma_3}{\sigma_{max}} \right)^m \right] dV \quad (3.89)$$

where σ_{max} is the maximum tensile stress in the specimen and all the other terms have been discussed earlier. For the biaxial flexure specimen utilizing simple plate theory with an applied total load of P (applied through the load ring), the radial and tangential stresses at a point (valid only for thickness over deformation ratio of two or greater) can be expressed as

$$\sigma_r = \sigma_t = \sigma_{max} = \frac{3P}{4\pi h^2} \left[2(1+\nu) \ln \frac{D_s}{D_L} + \frac{(1-\nu)(D_s^2 - D_L^2)}{D^2} \right] \quad (3.90)$$

for $0 < d < D_L$. Alternatively

$$\sigma_r = \frac{2(1+\nu) \ln \frac{D_s}{d} + \frac{(1-\nu)D_L^2(D_s^2 - d^2)}{D^2 d^2}}{2(1+\nu) \ln \frac{D_s}{D_L} + \frac{(1-\nu)(D_s^2 - D_L^2)}{D^2}} \sigma_{max} \quad (3.91)$$

and

$$\sigma_t = \frac{2(1+\nu) \ln \frac{D_s}{d} - \frac{(1-\nu)D_L^2(D_s^2 + d^2)}{D^2 d^2} + 2(1-\nu) \frac{D_s^2}{D^2}}{2(1+\nu) \ln \frac{D_s}{D_L} + \frac{(1-\nu)(D_s^2 - D_L^2)}{D^2}} \sigma_{max} \quad (3.92)$$

for $D_L < d < D$, with $\sigma_z = \sigma_3 = 0$. Here d is the diameter at the point where stress is being calculated, D is the test specimen diameter, D_s is the outer support ring diameter, D_L is

the inner load ring diameter, h is thickness of the test specimen as shown in Figure 3.24, ν is the Poisson's ratio of the material, the radial stress (σ_r) is the first principal stress, and the tangential stress (σ_t) is the second principal stress. For thin plates, the third principal stress is generally considered to be zero. Solving the integral in Equation (3.83) numerically over the gage volume by substituting Equations (3.84) - (3.86), the effective volume for the ring-on-ring biaxial flexure specimen can be approximated as^[22]

$$V_{eff} = \frac{\pi}{2} D_L^2 \left\{ 1 + \left[\frac{44(1+\nu)}{3(1+m)} \right] \left[\frac{5+m}{2+m} \right] \left(\frac{D_s - D_L}{D_s D} \right)^2 \times \right. \\ \left. \left[\frac{2 D^2(1+\nu) + (D_s - D_L)^2(1-\nu)}{(3+\nu)(1+3\nu)} \right] \right\} \left[\frac{h}{2(m+1)} \right] \quad (3.93)$$

where all the terms appearing in the expression are discussed earlier. For conventional ring-on-ring biaxial flexure specimens, the effective volume is very small, i.e., on the order of about 0.22 to 1.37% of gage volume of the specimen. The gage volume for the circular disc specimen can be expressed as

$$V_{gage} = \frac{\pi}{4} D_s^2 h \quad (3.94)$$

where all the terms in the expression are discussed earlier.

Effective area

Using an approach similar to that followed for the effective volume calculations for fracture origins that are spatially distributed along the surface of the test specimen between the outer support ring, the equation for effective area can be expressed as

$$A_{eff} = \int_A \left[\left(\frac{\sigma_1}{\sigma_{max}} \right)^m + \left(\frac{\sigma_2}{\sigma_{max}} \right)^m \right] dA \quad (3.95)$$

Solving the integral in the above expression using numerical method^[22] by utilizing Equation (3.84) – (3.86), the effective area for a ring-on-ring specimen can be approximated as

$$A_{eff} = \frac{\pi}{2} D_L^2 \left\{ 1 + \left[\frac{44(1+\nu)}{3(1+m)} \right] \left[\frac{5+m}{2+m} \right] \left(\frac{D_s - D_L}{D_s D} \right)^2 \times \right. \\ \left. \left[\frac{2D^2(1+\nu) + (D_s - D_L)^2(1-\nu)}{(3+\nu)(1+3\nu)} \right] \right\} \quad (3.96)$$

where all the terms in the expression are described earlier. For conventional ring-on-ring biaxial flexure specimens, the effective volume is on the order of about 5.71 to 8.22% of gage area of the specimen. The gage area for the circular disc specimen can be expressed as

$$A_{gage} = \frac{\pi}{2} D_s^2 \quad (3.97)$$

where all the terms in the expression are discussed earlier.

Finite Element Modeling and Analysis

The circular disc was modeled using the ANSYS PLANE82 element. The element is defined by eight nodes having two degrees of freedom at each node. The disc dimensions were selected in accordance with the ASTM C1499^[6] recommendations to

ensure behavior reasonably described by simple plate theory. The dimensions are as follows:

Disc diameter, $D = 23 \text{ mm}$

Support ring diameter, $D_S = 11 \text{ mm}$

Load ring diameter, $D_L = 5 \text{ mm}$

Disc thickness, $h = 0.4 \text{ mm}$

By taking the advantage of symmetry which optimizes computational resources, the disc was modeled using the PLANE82 element as an axisymmetric element. Force boundary conditions were imposed at the load ring diameter (see Figure 3.25). Support conditions were simulated by constraining the relevant nodal degree of freedom of bottom layer nodes along the support ring diameter. Figure 3.26 shows one quarter of the disc developed from the axisymmetric two-dimensional model shown in Figure 3.25.

Figure 3.25 Finite element model of the disc cross-section (axisymmetric) showing applied boundary conditions (Load - Red arrows, Support - Cyan triangle)

Figure 3.26 One quarter finite element model of the biaxial disc specimen

A mesh sensitivity study was performed to properly capture the stress state in the specimens. Similar to the beam flexure models, six mesh types from very coarse mesh to fine mesh were selected for the test specimens. The six mesh types are summarized in Table 3.5. The very coarse (VC) and the fine (F) meshes are shown in Figures 3.27 and 3.28 respectively.

Table 3.5 Mesh designation and corresponding number of elements for ring-on-ring
biaxial disc specimen (axisymmetric PLANE82 element)

Mesh Density	Axisymmetric Elements	Maximum Tensile Stress (MPa)
Very Coarse (VC)	2034	3.089
Coarsest (CS)	3768	3.102
Coarse (CO)	6030	3.107
Coarse/Medium (CM)	8820	3.110
Medium (M)	12138	3.112
Fine (F)	15984	3.113

Figure 3.27 Biaxial circular disc finite element model with very coarse (VC) mesh
using axisymmetric element

Figure 3.28 Biaxial circular disc finite element model with fine (F) mesh using axisymmetric element

Figure 3.29 shows the contour plot of the radial stresses (σ_x) along x-axis, for the axisymmetric model shown in Figure 3.28. For the co-ordinate system (shown by triad in Figure 3.28) selected for the model along with the applied load condition, the principal axis direction coincides with the x-direction. Figure 3.29 also shows the maximum (positive) and minimum (negative) radial stresses in the specimen.

Figure 3.29 Radial stress contour plot for the biaxial circular disc with fine (F) mesh

To validate the stress state obtained in the finite element analysis, the maximum stress under the applied load from the finite element analysis was compared to the classical theory of plates solution. For the ring line load of 1 N/mm the total concentrated breaking load can be calculated as

$$P = 1 \times (2 \pi r) = 15.71 \text{ N} \quad (3.98)$$

By substituting the above calculated value of P in the Equation (3.84) the equibiaxial strength for the disc specimen can be calculated as

$$\begin{aligned}
\sigma &= \frac{3 P}{4 \pi (0.4)^2} \left\{ 2 (1 + 0.22) \ln \left(\frac{11}{5} \right) + (1 - 0.22) \left[\frac{11^2 - 5^2}{23^2} \right] \right\} \\
&= 0.2032 P \\
&= 3.19 \text{ MPa}
\end{aligned} \tag{3.99}$$

Thus for the circular disc with a total applied load of 15.71 N, the calculated maximum theoretical stress is 3.19 MPa. The maximum stress value obtained from the most refined mesh (Fine ‘F’ - see Table 3.6) finite element model gives a value of 3.11 MPa (see Figure 3.29) which is in good agreement with the calculated theoretical stress value.

The procedure described above for maximum stress comparison was also followed for comparing the maximum deflection in the specimen. Maximum deflection under the applied load occurs in the bottom layer at the center (mid-span) of the disc specimen. The deflection for such a plate is given by

$$\begin{aligned}
\delta &= \frac{3 F (1 + \nu^2) D_L^2}{8 \pi E h^3} \left(\frac{D_s^2}{D_L^2} \left[1 + \frac{(1 - \nu) (D_s^2 - D_L^2)}{2 (1 + \nu) D^2} \right] - \left[1 + \ln \frac{D_s}{D_L} \right] \right) \\
&= 4.401 \times 10^{-4} \text{ mm}
\end{aligned} \tag{3.100}$$

where all the terms appearing in the expression have been discussed earlier. From the finite element analysis, the maximum deflection was found to be 4.53×10^{-4} mm (see legend in Figure 3.29), which is in close agreement with the value of 4.40×10^{-4} mm calculated from the classical equation. Based on these comparisons the finite element analyses were considered valid for the initial work conducted.

Figures 3.30 and 3.31 show the first and second principal stress contour plots for the biaxial circular disc specimen. The region of the specimen between the outer support ring and below the neutral axis is subject to tensile stresses with the region of high tensile stress at the bottom layer as highlighted by the red color contour. This region of tensile stresses fades out near the outer support ring. The upper half region (above the neutral axis) of the specimen will be under compressive (or neutral-zero) stresses, thus this region does not contribute towards the calculation of the effective volume and area of the specimen as per the underlying assumptions of the Weibull theory. For the above mentioned reasons, the finite element mesh was kept coarse in the regions of compressive stresses to optimize on the computational resources. The various models with the six mesh densities discussed above (see Table 3.5) have the similar principal stress contour plots.

Figure 3.30 First principal stress contour plot for the biaxial circular disc with fine (F) mesh

Figure 3.31 Second principal stress contour plot for the biaxial circular disc with fine (F) mesh

PRESSURIZED (BURST) TUBE SPECIMEN (Duffy, Lara-cuzaro, Carter^[11])

Pressurized tube (or burst tube) is a type of specimen that promotes failure in the volume of the material and therefore interrogates volume defect populations that are present inside. Figure 3.32 shows the pressurize tube specimen. The experimental geometry consists of a ceramic tube with an elastomer plug inside it so as to provide pressurization within the tube with certain gage length.

Figure 3.32 Pressurize (burst) tube specimen^[11]

The stress field varies with elastic properties (Young's modulus and Poisson's ratio) of both the elastomer and the ceramic material being tested. There are no closed formed expressions to determine the effective volume for this test geometry but the numerical approach discussed earlier can be exploited. A numerical study of burst tube specimens has already been performed elsewhere (Duffy, Lara-cuzaro, Carter^[11]). Figure 3.33 shows the first principal stress contour plot of burst tube for the axisymmetric finite element model. The complex modeling of the burst tube involves use of hyper-elastic material properties for the elastomer plug and contact analysis for the surface between the

elastomer and tube elements. The results of the burst tube specimen analysis from the above study are also discussed in Chapter 4.

Figure 3.33 First principal stress contour plot for the burst tube specimen using axisymmetric elements (dark blue rectangle – elastomer plug)^[11]

CHAPTER IV

RESULTS

In this chapter the results obtained by utilizing the numerical approach outlined earlier for the test specimens considered in this work are discussed. Comparison of the effective volume and area values obtained utilizing the numerical approach with the values calculated from the ASTM closed form expressions (as presented in Chapter 3) are discussed in details. For example, the effective volume and area curves for both the closed form and the numerical approach are plotted and the observations relative to the discrepancies in the values are discussed.

RECTANGULAR TEST SPECIMEN (*ASTM C1161*^[1])

Four-point flexure – Effective volume

For the rectangular four-point flexure specimen with an inner load span to outer support span ratio of 0.5 the effective volume values obtained utilizing the numerical approach along with the ASTM closed form expressions are plotted as a function of Weibull modulus (m). These results are shown in Figure 4.1 as a function of the mesh density (listed in Table 3.1). The effective volume curves obtained utilizing both the methods exhibit a knee in the curve with the increasing value of Weibull modulus. As a general rule of thumb, as the Weibull modulus for a material increases, the material has a diminished capacity of exhibiting size effects.

Figure 4.1 Effective volume (V_{eff}) as a function of Weibull modulus (m) for rectangular flexure specimen ($L_i / L_o = 0.5$)

The difference between the estimated numerical effective volume values and the ASTM closed form effective volume values is evident from the Figure 4.1. Figure 4.2 shows the effective volume values plotted as a function of number of elements (on logarithmic scale) in a specimen for the Weibull modulus values of 5 and 25. The number of elements represents the range of mesh densities (listed in Table 3.1) utilized in the work. As the degree of fineness (from coarse to fine) of mesh increases the numerical effective volume curves show an asymptotic convergence to a value other than the closed form value. Amongst all the models with different mesh densities, the fine mesh model represents the best numerical approximation of the effective volume values obtained through the numerical approach. The fine mesh model shows the maximum difference in the effective volume values over the closed form solution in comparison to the other coarser mesh models. Figure 4.3 bolsters the above arguments. The figure shows a plot of effective volume (V_{eff}) values plotted as a function of 1/thickness of the bottom layer elements for the rectangular flexure specimen for the span ratio of 0.5. Similar to the above observed behavior the numerical effective volume curves exhibit the asymptotic convergence to a value other than the closed form value.

Figure 4.2 Effective volume (V_{eff}) as a function of number of elements for rectangular flexure specimen ($L_i/L_o = 0.5$)

Figure 4.3 Effective volume (V_{eff}) as a function of 1/thickness for bottom layer elements for rectangular flexure specimen ($L_i/L_o = 0.5$)

The percentage difference between the numerical and the closed form effective volume values are plotted in Figure 4.4. The curves for the calculated numerical effective volume values plot below the ASTM closed form effective volume curve for all the mesh types. It is interesting to note that with the increasing value of Weibull modulus the percentage difference between the numerical and the closed form value also increases. As an example for the fine mesh model with the span ratio of 0.5 and the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 5% and this difference increases to about 21% for the Weibull moduli value of 25.

Figure 4.4 Percentage difference in V_{eff} (numerical over ASTM) as a function of Weibull modulus for rectangular flexure specimen ($L_i/L_o = 0.5$)

The above observed behavior of the difference in the effective volume values for the span ratio of 0.5 is representative of the range of span ratios (listed in Table 3.2) considered for the rectangular four-point flexure specimen. Figures 4.5 and 4.6 highlight the differences in the numerical effective volume values over the closed form effective volume values for the span ratio of 0.1. The two span ratios of 0.5 and 0.1 represents the bounds on the effective volume values for the other span ratios and the behavior for the specimens with other span ratios follows the trends shown in the graphs associated with these two span ratios.

Figure 4.5 Effective volume (V_{eff}) as a function of the Weibull modulus (m) for rectangular flexure specimen ($L_i/L_o = 0.1$)

Figure 4.6 Percentage difference in V_{eff} (numerical over ASTM) as a function of Weibull modulus for rectangular flexure specimen ($L_i/L_o = 0.1$)

For the fine mesh four-point flexure specimen the plot in Figure 4.7 shows the effective volume values on ordinate and the span ratio (L_i/L_o) on abscissa plotted as a function of Weibull modulus for both the numerical and the ASTM closed form approach. The dotted lines in the plot represent the closed form solutions whereas adjacent solid lines (similar colored) represent corresponding numerical solutions. Along with including all the cases of four-point flexure specimens (designated by the range of span ratios), the plot also includes the three-point flexure specimens, i.e., $L_i/L_o = 0$. It can be observed from the figure that for all the cases of four-point flexure, the closed form values plot higher than the numerical values in contrast to the three-point flexure case where the numerical values plot higher than the corresponding closed form values.

Figure 4.7 Effective volume (V_{eff}) as a function of range of span ratios (L_i/L_o) for rectangular flexure specimen

Figure 4.8 shows the percentage difference in the effective volume values for the fine mesh model plotted as a function of the range of span ratios and as a function of the Weibull modulus values. It can be observed from the figure that the difference in the numerical effective volume values over the closed form values for the three-point flexure case shows larger variations (in magnitudes) with opposite sense in contrast to the four-point flexure specimens.

Figure 4.8 Percentage difference in V_{eff} (numerical over ASTM) as a function of span ratio for rectangular flexure specimen

It is interesting to observe the effect of variation in span ratio over the difference in the effective volume values. As the span ratio decreases from 0.5 to 0.1 the difference in the effective volume decreases gradually, on the other hand the difference in the effective volume between the four-point flexure specimens with a span ratio of 0.1 and three-point flexure specimens is comparatively large. This difference in behavior between the three-point and the four-point flexure specimens can be attributed to the effects of shear force in the flexure specimens. Neglecting shear stresses in a three-point bend specimen has a more pronounced effect than ignoring shear stresses in the four-point specimen.

The region between the inner load span in the four-point flexure specimens is under pure bending, i.e., no shear force acts between the inner load span and for the three-point flexure specimens shear is present throughout the outer support span. The increase in the effective volume difference for the three-point flexure case can be attributed to the neglect of shear force in the ASTM closed form formulation as discussed in Chapter 3. On the contrary the numerical approach includes the effects of shear forces by capturing of three-dimensional stress state through the finite element analysis. The presence of smaller regions (between the inner and outer span) subjected to shear in the four-point flexure explains the lower difference in the closed form and the numerical effective volume values as compared to the three-point flexure case.

Four-point flexure – Effective area

For the rectangular four-point flexure specimen with an inner load span to outer support span ratio of 0.5 the effective area values obtained utilizing the numerical approach along with the ASTM closed form expressions are plotted as a function of Weibull modulus (m). These results are shown in Figure 4.9 as a function of the mesh density (listed in Table 3.1). The effective area curves obtained utilizing both approaches exhibit a trend similar to that observed for the effective volume curves with the increasing value of Weibull modulus.

The difference between the estimated numerical values and the closed form values is evident from the Figure 4.9. The numerical curves consistently plot below the

closed form curve maintaining nearly constant difference between the two methods regardless of the mesh density of the specimen. The insensitivity of the mesh density to the effective area values can be attributed to the fact that the integral as observed in Equation (3.18) and in the numerical procedure is performed over the specimen gage area. The coarse mesh model captures the surface stresses within the specimen gage area with reasonable accuracy. Unlike the high stress gradients present thorough the volume of the elements the surface elements has a lower stress gradient and the coarse mesh is enough to capture the essence of the stress state.

Figure 4.9 Effective area (A_{eff}) as a function of the Weibull modulus (m) for rectangular flexure specimen ($L_i/L_o = 0.5$)

The percentage difference between the numerical and the closed form effective area values are plotted in Figure 4.10. The curves for the calculated numerical effective area values plots below the ASTM closed form effective area curve maintaining a constant difference between the two approaches regardless of the mesh densities. It is interesting to note that with the increasing value of Weibull modulus the percentage difference between the numerical and the closed form value also increases. As an example for the fine mesh density with a span ratio of 0.5, for the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 4.4% and this difference increases to about 20.4% for the Weibull moduli value of 25.

Figure 4.10 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for rectangular flexure specimen ($L_i/L_o = 0.5$)

The above observed behavior for the difference in effective area values for the span ratio of 0.5 is representative of the range of span ratios (listed in Table 3.2) considered for the rectangular four-point flexure specimens. Figures 4.11 and 4.12 highlight the differences in the numerical effective area values over the closed form effective area values for the span ratio of 0.1. The two span ratios of 0.5 and 0.1 represents the bounds on the effective volume values for the other span ratios and the behavior for the specimens with other span ratios follows the trends shown by the two ratios.

Figure 4.11 Effective area (A_{eff}) as a function of Weibull modulus (m) for rectangular flexure specimen ($L_i/L_o = 0.1$)

Figure 4.12 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for rectangular flexure specimen ($L_i / L_o = 0.1$)

For the fine mesh four-point flexure specimen the plot in Figure 4.13 shows the effective area values on ordinate and the span ratio (L_i / L_o) on abscissa plotted as a function of Weibull modulus for both the numerical and the ASTM closed form approach. The dotted lines in the plot represent the closed form solutions whereas adjacent solid lines (similar colored) represent corresponding numerical solutions. Along with including all the cases of four-point flexure specimens (designated by the range of span ratios), the plot also includes the three-point flexure specimens, i.e., $L_i / L_o = 0$. It can be observed from the figure that for all the cases of four-point flexure, the closed form values plot higher than the numerical values in contrast to the three-point flexure case where the numerical values plot higher than the closed form values.

Figure 4.13 Effective area (A_{eff}) as a function of the range of span ratios (L_i/L_o) for rectangular flexure specimen

Figure 4.14 shows the percentage difference in the effective area values for the fine mesh model plotted as a function of the range of span ratios and as a function of the Weibull modulus value. It can be observed from the figure that the difference in the numerical effective area values over the closed form values for the three-point flexure case shows larger variations (in magnitudes) with opposite sense in contrast to the four-point flexure specimens.

Figure 4.14 Percentage difference in A_{eff} (numerical over ASTM) as a function of span ratio for rectangular flexure specimen

It is interesting to observe the effect of variation in span ratio over the difference in the effective area values. As the span ratio decreases from 0.5 to 0.1 the difference in the effective area decreases gradually, on the other hand the difference in the effective area between the four-point flexure specimens with a span ratio of 0.1 and three-point flexure specimens is comparatively large. This difference in the values between the three-point and the four-point flexure specimens can be attributed to the difference between the longitudinal and the principal stresses near the surface of the four-point flexure specimen.

Figure 4.15 Mohr's circle showing state of stress at a point in four-point flexure specimen

Consider the state of stress at a point on the surface of flexure specimen as shown in Figure 4.15. An infinitesimal cube with normal (longitudinal) and shear stresses acting over the cube faces as shown in the figure can represent the state of stress at the point under consideration. In the case of four-point flexure specimen the region between the inner load points is under pure bending and the longitudinal stress will be equal to the

principal stress. For the region between the inner load and outer support points the shear stress is present although no shear acts on the outer surface (face) of the element. As discussed previously the closed form expressions does not accommodate contribution from the shear stress and for the point under consideration (between the inner load and outer support point) the principal stress will be larger than the longitudinal stress. These arguments explain the difference in the effective area values between the two approaches. For the particular case of three-point flexure the difference between the longitudinal and the principal stresses when integrated over the gage area will result in the maximum difference in the effective area values in comparison to the four-point flexure specimen.

Three-point flexure - effective volume

For the rectangular three-point flexure specimen the effective volume values obtained utilizing the numerical approach along with the ASTM closed form expressions are plotted as a function of Weibull modulus (m). These results are shown in Figure 4.16 as a function of the mesh density (as listed in Table 3.1). The effective volume curves obtained utilizing both the methods exhibit a knee in the curve with the increasing value of Weibull modulus.

The difference between the estimated numerical effective volume values and the ASTM closed form effective volume values is evident from the Figure 4.16. As discussed earlier for the case of four-point flexure specimens the fine mesh (Fine 'F') model amongst all the mesh density represents the best numerical approximation of the

effective volume values obtained through the numerical approach. Similar to the four-point flexure specimens the three-point fine mesh flexure model shows the maximum difference in the effective volume values in comparison to the other coarser mesh models.

Figure 4.16 Effective volume (V_{eff}) as a function of the Weibull modulus (m) for rectangular flexure specimen ($L_i/L_o = 0.0$)

The percentage difference between the numerical and the closed form effective volume values are plotted in Figure 4.17. The curves for the calculated numerical effective volume values are above the ASTM closed form effective volume curve for the range of mesh densities considered for the three-point flexure specimen. It is interesting to note that with the increasing value of Weibull modulus the percentage difference between the numerical and the closed form value also increases. As an example, for the

fine mesh density case with a span ratio of 0.5 and the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 7.2% and the difference increases to about 42.2% for the Weibull moduli value of 25. The difference of 42.2% is significant considering the contribution of effective volume to the reliability calculations.

Figure 4.17 Percentage difference in V_{eff} (numerical over ASTM) as a function of Weibull modulus for rectangular flexure specimen ($L_i/L_o = 0.0$)

Three-point flexure - effective area

For the rectangular three-point flexure specimen the effective area values obtained utilizing the numerical approach along with the ASTM closed form expressions are

plotted as a function of Weibull modulus (m). These results are shown in Figure 4.18 as a function of the mesh density (listed in Table 3.1). The effective area curves obtained utilizing both the approach exhibit a knee similar to that observed for the effective volume curves with the increasing value of Weibull modulus. The difference between the estimated numerical values and the closed form values is evident from the Figure 4.18. From the figure it can be observed that all the numerical effective area curves consistently plots above the closed form effective area curve maintaining nearly constant difference between the two methods regardless of the mesh density of the specimen for the reasons explained earlier for the four-point flexure case.

Figure 4.18 Effective area (A_{eff}) as a function of Weibull modulus (m) for rectangular flexure specimen ($L_i/L_o = 0.0$)

The percentage difference between the numerical effective area and the closed form effective area values are plotted in Figure 4.19. The curves for the calculated numerical values are above (contrary to the four-point flexure case) the ASTM closed form curve maintaining nearly constant difference between the two approaches regardless of the mesh density of the specimen. It is interesting to note that with the increasing value of Weibull modulus the percentage difference between the numerical and the closed form effective area also increases. As an example, for the fine mesh density case with the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 8% and the difference increases to about 47% for the Weibull moduli value of 25. This difference is significant from the viewpoint of overall design of components failing due to surface flaws as highlighted by the reliability discussions in the next chapter. Note that all curves for the range of mesh densities are coincident and thus pointing to the insensitivity of effective area values to the variation in the mesh density of the specimen.

Figure 4.19 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for rectangular flexure specimen ($L_i/L_o = 0.0$)

CIRCULAR ROD TEST SPECIMEN (*ASTM C1683*^[7])

Four-point flexure - Effective volume

Figure 4.20 shows the effective volume values obtained for the circular four-point flexure specimen with an inner load span to outer support span ratio of 0.5. The effective volume values obtained utilizing the numerical approach along with the ASTM closed form expressions are plotted as a function of the Weibull modulus (m). The effective volume curves exhibit a knee in the curve with the increasing value of Weibull modulus plotted for the various mesh density (listed in Table 3.3) and the ASTM closed form solution.

Figure 4.20 Effective volume (V_{eff}) as a function of Weibull modulus (m) for circular flexure specimen ($L_i/L_o = 0.5$)

Figure 4.20 also shows the difference between the estimated numerical and the ASTM closed form effective volume values. Figure 4.21 shows the effective volume values plotted (on logarithmic scale) as a function of number of elements in the specimen for the Weibull modulus values of 5 and 25. The range of mesh densities (listed in Table 3.3) utilized in the work are represented by the number of elements in a specimen. As the degree of fineness (from coarse to fine) of the mesh increases the numerical effective volume curves show an asymptotic convergence to a value other than the closed form value. The fine mesh model amongst all the models with different mesh densities represents the best numerical approximation of the effective volume values obtained through the numerical approach. The fine mesh model shows the maximum difference in the effective volume values over the closed form solution in comparison to the other coarser mesh models. Figure 4.22 shows a plot of effective volume (V_{eff}) values plotted as a function of 1/thickness of the bottom layer elements for the circular flexure specimen with a span ratio of 0.5. Similar to the above observed behavior the numerical effective volume curves exhibit the asymptotic convergence to a value other than the closed form value.

Figure 4.21 Effective volume (V_{eff}) as a function of number of elements for circular flexure specimen ($L_i/L_o = 0.5$)

Figure 4.22 Effective volume (V_{eff}) as a function of 1/thickness for bottom layer elements for circular flexure specimen ($L_i/L_o = 0.5$)

Figure 4.23 shows a plot of the percentage difference between the numerical and the ASTM closed form effective volume values plotted as a function of the Weibull modulus values for the curves shown in Figure 4.20. For all the mesh types (except for the very coarse ‘VC’ mesh) the curves for the numerical effective volume values plot below the closed form effective volume curve. It shall be noted that the percentage difference between the numerical and the closed form value increases as the value of Weibull modulus increases. Consider the case of the fine mesh density with a span ratio of 0.5, for the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 2.2% and this difference increases to about 9.7% for the Weibull moduli value of 25.

Figure 4.23 Percentage difference in V_{eff} (numerical over ASTM) as a function of Weibull modulus for circular flexure specimen ($L_i/L_o = 0.5$)

For the circular four-point flexure specimen with the span ratio of 0.5 the above discussed behavior of the difference in the effective volume values is representative of the range of span ratios considered (as listed in Table 3.4). For the span ratio of 0.1 the Figures 4.24 and 4.25 highlight the above discussed differences in the numerical effective volume values over the closed form effective volume values. The two span ratios of 0.1 and 0.5 represents the bounds on the effective volume values for the other span ratios and the behavior for the specimens with other span ratios follows the trends shown in the graphs associated with these two span ratios.

Figure 4.24 Effective volume (V_{eff}) as a function of Weibull modulus (m) for circular flexure specimen ($L_i / L_o = 0.1$)

Figure 4.25 Percentage difference in V_{eff} (numerical over ASTM) as a function of Weibull modulus for circular flexure specimen ($L_i/L_o = 0.1$)

The plots in Figure 4.26 show the effective volume values plotted for the circular four-point flexure specimen with fine mesh as a function of Weibull modulus for both the numerical and the ASTM closed form approach. The effective volume values are plotted on ordinate and the span ratios (L_i/L_o) on abscissa. The solid lines in the plot represent the numerical solution and the similar colored dotted lines represent the closed form solution. The plot includes all the cases of four-point flexure specimens (designated by the range of span ratios) along with the three-point flexure specimens, i.e., $L_i/L_o = 0$. From the figure it can be observed that for all the cases of four-point flexure the closed form values plots higher then the numerical values and in contrast for the three-point flexure case the numerical values plots higher then the corresponding closed form values.

Figure 4.26 Effective volume (V_{eff}) as a function of the range of span ratios (L_i/L_o) for circular flexure specimen

For the fine mesh model for circular flexure specimen Figure 4.27 shows the percentage difference in the effective volume values plotted as a function of the range of span ratios. It can be observed from the figure that the difference in the numerical effective volume values over the closed form values for the three-point flexure case shows larger variations (in magnitudes) with opposite sense in contrast to the four-point flexure specimens.

Figure 4.27 Percentage difference in V_{eff} (numerical over ASTM) as a function of span ratio for circular flexure specimen

From the Figure 4.27 the change in the difference in the effective volume values with the change in span ratio can be observed. The effective volume values decreases gradually with the decrease in the span ratio from 0.5 to 0.1. The difference in the effective volume between the three-point and four-point flexure specimens with a span ratio of 0.1 is comparatively large. The effect of shear forces present in the flexure specimen can be contributing to this difference in four-point and three-point flexure specimens. The effect of neglecting shear stresses in the four-point flexure specimen has less pronounced effect as compared to neglecting shear stresses in a three-point flexure specimen.

For the case of three-point flexure specimen shear force is present throughout the outer support span in contrast to the four-point flexure specimen wherein no shear force is present between the inner load span as this region (between the inner load span) is under pure bending. As previously discussed the ASTM closed form formulation does not take into consideration the effect of shear force thus explaining the increase in the effective volume difference for the three-point flexure case. The numerical approach instead includes the effect of shear forces by capturing the three-dimensional stress state through the finite element analysis. The lower difference in the closed form and the numerical effective volume values in comparison to the three-point flexure can be attributed to the presence of smaller regions (between the inner and outer span) subjected to shear in the four-point flexure.

Four-point flexure – Effective area

The effective area values obtained for the circular four-point flexure specimen with inner load span to outer support span ratio of 0.5 for both the numerical and the ASTM closed form solution are plotted in Figure 4.28 as a function of Weibull modulus (m). The effective area curves plotted as a function of mesh density (listed in Table 3.3) shows similar trends to that observed for the case of the effective volume curves with the increasing value of Weibull modulus.

The insensitivity of the mesh density to the effective area values is evident from the Figure 4.28. The numerical curves consistently plot below and maintain constant

difference from the closed form curves for all the different mesh density considered. As explained earlier the insensitivity of the mesh density to the effective area values can be attributed to the fact that the integral as observed in Equation (3.18) and in the numerical procedure is performed over the specimen gage area. The coarse mesh model captures the surface stresses with reasonable accuracy within the specimen gage area. The surface elements have a lower stress gradients in comparison to the high stress gradients present through the volume elements and the coarse mesh captures the essence of the surface stress state as effectively as the finer mesh.

Figure 4.28 Effective area (A_{eff}) as a function of Weibull modulus (m) for circular flexure specimen ($L_i / L_o = 0.5$)

The plot in Figure 4.29 shows the percentage difference between the numerical and the closed form effective area values. The figure reconfirms the insensitivity of the mesh density to the effective area values for the calculated numerical values that consistently plot below the ASTM closed form curve. The percentage difference between the numerical and the closed form value increases with the increasing Weibull modulus. Considering the fine mesh density case as an example the difference between the numerical and the closed form value for the Weibull moduli value of 5 is 1.8% and this difference for the Weibull moduli value of 25 increases to about 9.0%.

Figure 4.29 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for circular flexure specimen ($L_i/L_o = 0.5$)

For the circular four-point flexure specimen the above discussed behavior of the difference in effective area values for the span ratio of 0.5 is representative of the range of span ratios (as listed in Table 3.4) considered in this work. For the span ratio of 0.1 the differences in the numerical effective area values over the closed form effective area values are shown in Figures 4.30 and 4.31. The effective area curves for the span ratios of 0.5 and 0.1 represents the bounds on the other span ratios and the behavior for the specimens with these span ratios follow the trends exhibited by the two span ratios discussed here.

Figure 4.30 Effective area (A_{eff}) as a function of Weibull modulus (m) for circular flexure specimen ($L_i / L_o = 0.1$)

Figure 4.31 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for circular flexure specimen ($L_i / L_o = 0.1$)

Figure 4.32 shows the effective area values plotted against span ratio for the four-point flexure specimen with fine mesh for both the numerical and the closed form solution. The plot covers all the cases of four-point flexure specimens (designated by the range of span ratios) as well as the three-point flexure case. In the plot the closed form solution is represented by the dotted lines and the adjacent solid lines (similar colored) represent corresponding numerical solutions. It is interesting to observe that for all the cases of four-point flexure, the closed form values plot higher than the numerical values as opposed to the three-point flexure case where the numerical values plot higher than the corresponding closed form values.

Figure 4.32 Effective area (A_{eff}) as a function of span ratio (L_i/L_o) for circular flexure specimen

The percentage difference in the effective area values for the fine mesh model are plotted in Figure 4.33 as a function of the range of span ratios and as a function of the Weibull modulus value. The case of three-point flexure shows larger variations (in magnitudes) with opposite sense in comparison with the four-point flexure specimens for the difference in the numerical effective area values over the closed form values.

Figure 4.33 Percentage difference in A_{eff} (numerical over ASTM) as a function of span ratio for circular flexure specimen

From the Figure 4.33 the change in the difference in the effective area values with the change in span ratio can be observed. The effective area values decreases gradually with the decrease in the span ratio from 0.5 to 0.1. The difference in the effective area between the three-point and four-point flexure specimens with a span ratio of 0.1 is comparatively large. This difference in the values between the three-point and the four-point flexure specimens can be attributed to the difference between the longitudinal and the principal stresses near the surface of the four-point flexure specimen.

As discussed earlier for the rectangular bar specimen similar arguments can be made for the circular flexure specimen. For any point on surface lying in the region

between the inner load and outer support spans for the four-point flexure specimen is under pure bending and the longitudinal stress will be equal to the principal stress. No shear acts on the outer surface of the elements in this region between the inner load and outer support spans even though the shear is present through the volume of the elements. As mentioned earlier the closed form expressions neglects the contribution from the shear and for the region between the inner and outer support span the principal stress will be larger than the longitudinal stress thus explaining the difference in the effective area values between the two approaches. For the three-point flexure effective area values the difference between the longitudinal and the principal stresses when integrated over the gage area will result in the maximum difference in comparison to the four-point flexure specimen.

Three-point flexure - Effective Volume

The effective volume values obtained for the circular three-point flexure specimen are plotted as a function of Weibull modulus (m) in Figure 4.34. These results are shown as a function of the mesh density (listed in Table 3.3) for both the numerical and the ASTM closed form solution. With the increasing value of Weibull modulus the effective volume curves exhibit a knee in the curve for both the methods.

For the three-point flexure case similar to the earlier discussed four-point flexure case the fine mesh (Fine 'F') model in comparison to all other mesh density models represent the best numerical approximation of the effective volume values obtained

utilizing the numerical approach. From Figure 4.34 the difference between the estimated numerical effective volume values and the closed form effective volume values can be observed. The fine mesh three-point flexure specimen shows the maximum difference in the effective volume values in comparison to the other coarser mesh models.

Figure 4.34 Effective volume (V_{eff}) as a function of Weibull modulus (m) for circular flexure specimen ($L_i / L_o = 0.0$)

Figure 4.35 shows a plot of the percentage difference between the numerical and the closed form effective volume values for the circular three-point flexure specimen. For all the different mesh densities considered for the specimen, the numerical effective volume curves plot above the closed form curve. The percentage difference between the numerical and the closed form value increases with the increase in the Weibull modulus.

Consider the case of fine mesh density case with a span ratio of 0.5 as an example, with the Weibull moduli value of 5 the difference between the numerical and the closed form value is 8.1% and this difference increases to about 40.9% for the Weibull moduli value of 25. Considering the contribution of effective volume value on the reliability calculations the 40.9% difference can be significant.

Figure 4.35 Percentage difference in V_{eff} (numerical over ASTM) against Weibull modulus for circular flexure specimen ($L_i / L_o = 0.0$)

Three-point flexure - Effective Area

The effective area values for the circular three-point flexure specimen are plotted in Figure 4.36 as a function of the Weibull modulus (m). The curves plotted as a function of the mesh density (as listed in Table 3.3) are obtained utilizing both the numerical and

the ASTM closed form expressions. With the increasing value of Weibull modulus all the effective area curves exhibit a knee similar to that observed for the effective volume curves. From the Figure 4.36 the difference between the closed form values and estimated numerical values is evident. For the reasons explained earlier for the four-point flexure case all the numerical effective area curves consistently plot above the closed form effective area curve maintaining nearly constant difference between the two methods regardless of the mesh density of the specimen.

Figure 4.36 Effective area (A_{eff}) as a function of Weibull modulus (m) for circular flexure specimen ($L_i / L_o = 0.0$)

Figure 4.37 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for circular flexure specimen ($L_i/L_o = 0.0$)

Figure 4.37 shows the percentage difference between the numerical effective area and the ASTM closed form effective area values. Contrary to the four-point flexure case the curves for the calculated numerical values plot above the closed form curve yielding constant difference between the two approaches regardless of the mesh density of the specimen and this difference increases with the increasing value of the Weibull modulus. Consider the fine mesh density case as an example, the difference between the numerical and the closed form value is about 9.2% for the Weibull moduli value of 5 and this difference increases to about 48.3% for the Weibull moduli value of 25. This difference is important from the viewpoint of overall design of components failing due to surface flaws as highlighted by the reliability discussions detailed in the later chapters. Again

observing that all the curves are coincident and thus pointing to the insensitivity of effective area values to the variation in the mesh density of the specimen for the reasons explained earlier for the case of four-point flexure case.

C – RING TEST SPECIMEN (ASTM C1323-2006^[5])

Effective volume

The work published by Duffy et al.^[9] on C-ring specimen is reviewed here. Their work as well as Jadaan's^[16] effort prompted this work. Their efforts suggests that the two-dimensional ASTM closed form expressions for effective volume and area are valid for only a limited range of the C-ring geometry parameters. The effective volume curves for the C-ring obtained utilizing both the numerical and the ASTM closed form approach exhibit a knee in the curve with the increasing value of Weibull modulus similar to that has been observed for the flexure bars. Their work is presented for completeness.

Figure 4.38 is from Duffy et al.^[9] and shows the percentage difference between the numerical and the closed form effective volume values plotted as a function of Weibull modulus. From the figure it can be observed that the curves for the calculated numerical effective volume values plot above the closed form effective volume curve for all the mesh densities considered. On the basis of the arguments presented earlier for the bend bars, considering all the models with different mesh densities the fine mesh model represents the best numerical approximation of the effective volume values obtained through the numerical approach. Note that the fine mesh has the largest percentage difference for the C-ring geometry considered in the study. The numerical solution for the fine mesh model deviates from the closed form solution in the range of about 12% for the Weibull moduli value of 5 to about 45% for the Weibull moduli value of 30. The mesh sensitivity study performed for the numerical calculations of effective volume shows an asymptotic convergence with values other than the closed form values.

Figure 4.38 Percentage difference in effective volume ‘kV’ (numerical over ASTM) as a function of Weibull modulus for C-ring specimen (see Duffy et al.^[9])

Duffy et al.^[9] also discussed the effect of changing geometry over the effective volume values. The closed form solutions present in the ASTM C1323 (derived from Jadaan’s^[16] work) are derived from the classical strength of materials theories which are based on assumptions of the two-dimensional stress behavior in the C-ring specimen. These assumptions are fairly accurate in the limits for small length (b in Figure 3.19) to thickness ratios (outer radius - inner radius) i.e. less than about 1.0. As the ratio of length to the thickness of the C-ring increases beyond 1.0 the percentage difference between the numerical effective volume values and closed form values increases as can be seen from Figure 4.39. Duffy et al. reported that the numerical effective volume calculations are

quite sensitive to Weibull modulus and length of the C-ring if other geometry parameters are kept constant.

Figure 4.39 Percentage difference in effective volume ‘kV’ (numerical over ASTM) as a function of specimen geometry for $m = 10$ for C-ring specimen (see Duffy et al.^[9])

The reader is referred to the work of Duffy et al.^[9] for a complete discussion on the topic of effective areas for the C-ring specimens. Results similar to those reported here were presented in the earlier work.

O-RING TEST SPECIMENS (ASTM WK88) (Duffy, Segall communications)

Effective volume

The results presented by Kranendonk and Sinnema^[19] suggest that the two-dimensional plane strain and plane stress results in the O-ring specimens are limited by geometry. As the ratio of length (width) to thickness of the O-ring increases the classical solutions (elasticity, strength of materials) deviates from the experimental (and numerical) results. As suggested in the paper^[19] for the O-ring (as well as C-ring) with a length to thickness ratio of 6 the plane stress assumption fails and as length increases above the threshold of two-dimensional regime longitudinal stresses become non-neglecting.

Work performed by Duffy et al.^[12] further suggests that the closed form solutions for V_{eff} and A_{eff} are not valid for the O-ring specimen geometry investigated in their work. Absence of the plane strain and/or plane stress conditions in O-ring specimens as suggested by Kranendonk and Sinnema^[19] makes the estimation of longitudinal stress from the closed form circumferential stress equation unachievable (see Duffy et al.^[12]). Thus the effective volume and closed form equations can not be improved to include the effects of influential longitudinal stresses. The neglect of the contributions from the longitudinal stresses to the effective volume and area calculations for such cases will yield error in the results.

Figure 4.40 from Duffy et al.^[12] shows the percentage difference between the numerical and the closed form effective volume values plotted as a function of Weibull modulus. From the figure it can be observed that the curves for the calculated numerical effective volume values plot above the closed form effective volume curve for all the mesh densities considered. On the basis of the arguments presented earlier for the bend bars, considering all the models with different mesh densities the fine mesh model represents the best numerical approximation of the effective volume values obtained through the numerical approach. The effective volume curves for the O-ring obtained utilizing both the numerical and the ASTM closed form approach exhibit a knee in the curve with the increasing value of Weibull modulus similar to that has been observed for the flexure bars as well as for the C-ring.

Figure 4.40 Percentage difference in effective volume 'kV' (numerical over ASTM) as a function of Weibull modulus for O-ring specimen (see Duffy et al.^[12])

It can be observed from the Figure 4.40 that the numerical solution for the fine mesh model deviates from the closed form solution in the range of about 1% for the Weibull moduli value of 5 to about 55% for the Weibull moduli value of 30. The mesh sensitivity study performed for the numerical calculations of effective volume shows an asymptotic convergence with values other than the closed form values.

Effective area

Asymptotic convergence similar to the numerical effective volume values were observed in the numerical effective area values for the O-ring specimens (see Duffy et al.^[12]). Figure 4.41 shows the percentage difference between the numerical and the closed form effective area as a function of Weibull modulus. It can be observed from the figure that the numerical effective area solution for the fine mesh model deviates from the closed form solution in the range of about 12% for the Weibull moduli value of 5 to about 64% for the Weibull moduli value of 30. The mesh sensitivity study performed for the numerical calculations of effective area shows an asymptotic convergence with values other than the closed form values.

Figure 4.41 Percentage difference in effective area 'kA' (numerical over ASTM) as a function of Weibull modulus for O-ring specimen (see Duffy et al.^[12])

The reader is referred to the work of Duffy et al.^[12] for a complete discussion on the topic of effective areas for the O-ring specimens. Results similar to those reported here were presented in the earlier work.

RING-ON-RING TEST SPECIMEN (ASTM C1499-01^[6])

Effective volume

For the ring-on-ring biaxial disc specimen the effective volume values obtained with the numerical approach utilizing the PIA reliability model along with the ASTM closed form expressions are plotted as a function of Weibull modulus (m), and as a function of the mesh density (listed in Table 3.5) in Figure 4.42. The effective volume curves obtained utilizing both the methods exhibit a knee in the curve with increasing value of Weibull modulus.

Figure 4.42 Effective volume (V_{eff}) as a function of Weibull modulus (m) for ring-on-ring disc specimen using PIA reliability model

In Figure 4.42 the effective volume solution calculated by the closed form equations plots above all the curves plotted for the numerical solution calculated using the PIA reliability model for all the six mesh types listed in Table 3.5. Figure 4.43 shows the effective volume values plotted as a function of number of elements and as a function of Weibull modulus. The number of elements represents the range of mesh densities utilized in the work. As the degree of fineness (from coarse to fine) of the mesh increases the numerical effective volume curves show an asymptotic convergence to a value different than the closed form value. Amongst all the mesh densities the fine mesh model represents the best numerical approximation of the effective volume values obtained through the numerical approach. The fine mesh model shows the maximum difference in the effective volume values as compared to the closed form solution. Figure 4.44 bolsters the above arguments. Figure 4.44 shows a plot of effective volume (V_{eff}) plotted as a function of 1/thickness of the bottom layer elements for the ring-on-ring biaxial disc specimen. The curves exhibit the asymptotic convergence of the numerical effective volume values similar to that discussed above.

Figure 4.43 Effective volume (V_{eff}) as a function of number of elements for ring-on-ring biaxial disc specimen

Figure 4.44 Effective volume (V_{eff}) as a function of 1/thickness for bottom layer elements for ring-on-ring biaxial disc specimen

The percentage difference between the numerical and the ASTM closed form effective volume values for the PIA reliability model are plotted in Figure 4.45. For all the mesh models, the curves for the numerically generated effective volume values plot below the closed form effective volume curve. It is interesting to note that with the increasing value of Weibull modulus (for finer meshes) the percentage difference between the numerical and the closed form value also increases. As an example, for the fine mesh density model and the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 9% and the difference increases to about 21% for the Weibull moduli value of 25.

Figure 4.45 Percentage difference in V_{eff} (numerical over ASTM) as a function of Weibull modulus for ring-on-ring specimen (PIA reliability model)

For the ring-on-ring biaxial disc specimen the effective volume values obtained with the numerical approach utilizing the Batdorf reliability model along with the ASTM closed form expressions are plotted as a function of Weibull modulus in Figure 4.46. The effective volume curves obtained utilizing both the approaches exhibit a knee in the curve with the increasing value of Weibull modulus. From the figure it can be noted that the numerical effective volume solutions calculated utilizing the Batdorf reliability model plots above the closed form value curve for all the mesh types (listed in Table 3.5). The difference between the estimated numerical effective volume values and the ASTM closed form effective volume values is evident from the figure.

Figure 4.46 Effective volume (V_{eff}) as a function of Weibull modulus (m) for ring-on-ring specimen using Batdorf reliability model

The percentage difference between the numerical and the ASTM closed form effective volume values for the Batdorf reliability model are plotted in Figure 4.47. For all the mesh models, the curves for the calculated numerical effective volume values plots above the closed form effective volume curve. It is interesting to note that with the increasing value of Weibull modulus the percentage difference between the numerical and the closed form value also increases. As an example, for the fine mesh density model and the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 29% and the difference increases to about 147% for the Weibull moduli value of 25.

Figure 4.47 Percentage difference in V_{eff} (numerical over ASTM) as a function of Weibull modulus for ring-on-ring specimen (Batdorf reliability model)

Effective area

For the ring-on-ring biaxial disc specimen the effective area values obtained with the numerical approach utilizing the PIA reliability model along with the ASTM closed form expressions are plotted as a function of Weibull modulus in Figure 4.48. The effective area curves obtained utilizing both the methods exhibit a knee in the curve with increasing value of Weibull modulus.

Figure 4.48 Effective area (A_{eff}) as a function of Weibull modulus (m) for ring-on-ring specimen using PIA reliability model

In Figure 4.48 the effective area solution calculated by the closed form equations plots above all the curves plotted for the numerical solution calculated using the PIA reliability model for all the six mesh types listed in Table 3.5. The difference between

the estimated numerical effective area values and the ASTM closed form effective area values is evident from the Figure 4.48. Figure 4.49 shows the effective area values plotted as a function of number of elements and as a function of Weibull modulus. The number of elements represents the range of mesh densities utilized in the work. As the degree of fineness (from coarse to fine) of the mesh increases the numerical effective area curves show an asymptotic convergence to a value different than the closed form value. Amongst all the mesh densities the fine mesh model represents the best numerical approximation of the effective area values obtained through the numerical approach. The fine mesh model shows the maximum difference in the effective area values as compared to the closed form solution. Figure 4.50 bolsters the above arguments. Figure 4.50 shows a plot of effective area (A_{eff}) plotted as a function of 1/thickness of the bottom layer elements for the ring-on-ring biaxial disc specimen. The curves exhibit the asymptotic convergence of the numerical effective area values similar to that discussed above.

Figure 4.49 Effective area (A_{eff}) as a function of number of elements for ring-on-ring biaxial disc specimen

Figure 4.50 Effective area (A_{eff}) as a function of 1/thickness for bottom layer elements for ring-on-ring biaxial disc specimen

The percentage difference between the numerical and the ASTM closed form effective area values for the PIA reliability model are plotted in Figure 4.51. For all the mesh models, the curves for the calculated numerical effective area values plots below the closed form effective area curve. It is interesting to note that with the increasing value of Weibull modulus (for finer meshes) the percentage difference between the numerical and the closed form value also increases. As an example, for the fine mesh density model and the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 9% and the difference increases to about 20% for the Weibull moduli value of 25.

Figure 4.51 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for ring-on-ring specimen (PIA reliability model)

For the ring-on-ring biaxial disc specimen the effective area values obtained with the numerical approach utilizing the Batdorf reliability model along with the ASTM closed form expressions are plotted as a function of Weibull modulus in Figure 4.52. The effective area curves obtained utilizing both the approaches exhibit a knee in the curve with the increasing value of Weibull modulus. From the figure it can be noted that the numerical effective area solutions calculated utilizing the Batdorf reliability model plots above the closed form value curve for all the mesh types (listed in Table 3.5). The difference between the estimated numerical effective area values and the ASTM closed form effective area values is evident from the figure.

Figure 4.52 Effective area (A_{eff}) as a function of Weibull modulus (m) for ring-on-ring specimen using Batdorf reliability model

The percentage difference between the numerical and the ASTM closed form effective area values for the Batdorf reliability model are plotted in Figure 4.53. For all the mesh models, the curves for the calculated numerical effective area values plots above the closed form effective area curve. It is interesting to note that with the increasing value of Weibull modulus the percentage difference between the numerical and the closed form value also increases. As an example, for the fine mesh density model and the Weibull moduli value of 5 the difference between the numerical and the closed form value is about 29% and the difference increases to about 149% for the Weibull moduli value of 25.

Figure 4.53 Percentage difference in A_{eff} (numerical over ASTM) as a function of Weibull modulus for ring-on-ring specimen (Batdorf reliability model)

PRESSURIZED (BURST) TUBE SPECIMEN

Effective volume

Due to the absence of closed form solutions for this type of specimen the results of Duffy, Lara-cuzaro and Carter^[11] are presented here for completeness. One could simply compute the effective volume and effective area assuming the test specimen was a simple pressure vessel, but this would ignore the complexities of the boundary conditions. In their original work Duffy, Lara-cuzaro and Carter^[11] treated the problem using contact stress elements. Thus the effective volume and area values obtained by utilizing their numerical approach are presented. The asymptotic convergence in the numerical effective volume values was observed by utilizing the mesh sensitivity approach where several models were considered with increasing levels of mesh refinements. Figure 4.54 shows the effective volume values plotted as a function of Weibull modulus and as a function of mesh density for the burst tube specimen. The typical knee in the effective volume curves is evident from the figure.

Figure 4.54 Effective volume as a function of Weibull modulus and as a function of mesh density for burst tube specimen^[11]

The authors of the original study also discussed the effects of the end conditions on the stress state in tube ends from a recessed elastomer plug which was used to accommodate the load platens (see Figure 3.32). A sensitivity study was performed by including various recess lengths for the elastomer plug and it was concluded that the recess dimensions may affect the results considerably. The recess length that is too long may be inefficient while too short may interrogate the effects of surface flaws at the top and bottom surface of the tube interfering with the investigation of volume flaws.

On another note the effects of the elastic material properties of the ceramic tube and elastomer plug were also studied by the reference. It was reported that the change in the elastic material properties such as Young's modulus and Poisson's ratio for the

different candidate ceramic materials considered for the work showed no major effects on the calculated effective volume and area values. Even changing Young's modulus of elastomer plug does not show any change in the effective volume and area values.

Effective area

Figure 4.55 shows the effective volume area plotted as a function of Weibull modulus and as a function of mesh density for the burst tube specimen. The typical knee as observed in the effective volume values is evident in the effective area curves from the figure.

Figure 4.55 Effective area as a function of Weibull modulus and as a function of mesh density for burst tube specimen^[11]

CHAPTER V

SUMMARY

The difference in the closed form effective volume and area values (mostly published in various ASTM standards) as compared to the best numerical values has been discussed throughout the earlier chapters for various specimen types. In this concluding chapter the preceding work is summarized. In addition the effect of potential error associated with the closed form effective volume and area values on the component probability of failure calculations are discussed.

ERROR IN COMPUTING COMPONENT RELIABILITY

By utilizing the estimated material parameters $\tilde{m}$ and $\tilde{\sigma}_\theta$, the Weibull material scale parameter σ_o can be computed using the following equation

$$(\sigma_o)_V = (V_{eff})^{1/m_V} (\sigma_\theta)_V \quad (2.7)$$

Using the effective volume or area from either the numerical procedure or the available closed form values with the calculated value of σ_o from the above expression, the probability of failure of the component is calculated by using the following equation

$$P_f = 1 - \exp \left[-V_{eff} \left(\frac{\sigma_{max}}{\sigma_o} \right)^m \right] \quad (2.6)$$

Both expressions were given earlier. The asymptotic convergence of V_{eff} and A_{eff} for a range of Weibull modulus values suggest a convergence to more theoretically rigorous values using the numerical approach since the complete stress state and the entire stressed region of the component is accounted for. The closed form solutions for V_{eff} and A_{eff} are approximations for all specimen geometries considered. The assumptions that underpin the closed form solutions have been outlined earlier.

From the arguments presented by Duffy et al.^[9], consider the error introduced in component reliability calculations if the closed form solution is used for V_{eff} . Assume that when compared to “theoretically correct values” the closed form V_{eff} is in error with an associated error coefficient “*Error*,” that is

$$(V_{eff})_{closed\ form} = Error \cdot (V_{eff})_{true} \quad (5.1)$$

where the subscript “*true*” refers to theoretically correct value, and *Error* is a coefficient that can be greater than, or less than one. Thus

$$(\sigma_{oV})_{closed\ form} = (Error)^{1/m_V} (V_{eff})_{true}^{1/m} (\sigma_{\theta})_V \quad (5.2)$$

and this leads to

$$(P_f)_{closed\ form} = 1 - \exp \left[- \left(\frac{1}{Error} \right) \int_V \left(\frac{\sigma}{(\sigma_{oV})_{true}} \right)^{m_V} dV \right] \quad (5.3)$$

or

$$(P_f)_{closed\ form} = 1 - \left(1 - (P_f)_{true} \right)^{1/Error} \quad (5.4)$$

where in Equation (5.3)

$$(\sigma_{oV})_{true} = (V_{eff})_{true}^{1/m_V} \sigma_\theta \quad (5.5)$$

is the theoretically correct Weibull material scale parameter. In terms of reliability Equation (5.4) can be expressed as

$$R_{closed\ form} = (R_{true})^{1/Error} \quad (5.6)$$

A non-conservative trend from the above equation is indicated by the fact that the closed form reliability will approach one as the error coefficient becomes large (i.e., greater than one). The associated closed form probability of failure tends to zero. On the other hand as the error coefficient approaches zero in the limit (from one) then the reliability and the probability of failure computations obtained from the closed form approach become overly optimistic. No error corresponds to $Error = 1.0$. At this point of the discussion, the numerically generated values of V_{eff} and A_{eff} are treated as the more rigorous values and the closed form values are deemed to have an “*Error*” associated with them. Table 5.1 lists the *Error* coefficients associated with effective volume values. These values were obtained by assuming that the V_{eff} generated by the numerical approach are close in

value to the “theoretical correct” values alluded to above. Similarly the *Error* coefficients associated with effective area values are listed in Table 5.2.

Table 5.1 *Error* coefficient for effective volume associated with the closed form solutions relative to the numerical (fine mesh) solutions

Specimen	<i>Error</i> Coefficient (Effective Volume)				
	$m = 5$	$m = 10$	$m = 15$	$m = 20$	$m = 25$
Rectangular bar					
Four-point flexure					
$L_i/L_o = 0.5$	1.0507	1.1029	1.1561	1.2112	1.2682
$L_i/L_o = 0.4$	1.0411	1.0826	1.1244	1.1673	1.2114
$L_i/L_o = 0.3$	1.0341	1.0673	1.1004	1.1341	1.1685
$L_i/L_o = 0.2$	1.0285	1.0544	1.0794	1.1047	1.1301
$L_i/L_o = 0.1$	1.0310	1.0564	1.0786	1.1001	1.1212
Three-point flexure	0.9320	0.9033	0.8022	0.7485	0.7028
Circular rod					
Four-point flexure					
$L_i/L_o = 0.5$	1.0226	1.0435	1.0638	1.0877	1.1080
$L_i/L_o = 0.4$	1.0192	1.0366	1.0535	1.0739	1.0907
$L_i/L_o = 0.3$	1.0166	1.0316	1.0460	1.0638	1.0780
$L_i/L_o = 0.2$	1.0148	1.0282	1.0411	1.0573	1.0698
$L_i/L_o = 0.1$	1.0135	1.0264	1.0387	1.0543	1.0662
Three-point flexure	0.9251	0.8585	0.8004	0.7529	0.7096
Ring-on-ring biaxial disc					
PIA reliability model	1.0998	1.1171	1.1573	1.2063	1.2599
Batdorf reliability model	0.7748	0.5671	0.4809	0.4342	0.4055

Table 5.2 *Error* coefficient for effective area associated with the closed form
solutions relative to the numerical (fine mesh) solutions

Specimen	<i>Error</i> Coefficient (Effective Area)				
	$m = 5$	$M = 10$	$m = 15$	$m = 20$	$m = 25$
Rectangular					
Four-point flexure					
$L_i/L_o = 0.5$	1.0458	1.0949	1.1467	1.2007	1.2567
$L_i/L_o = 0.4$	1.0366	1.0751	1.1159	1.1582	1.2018
$L_i/L_o = 0.3$	1.0294	1.0593	1.0910	1.1241	1.1579
$L_i/L_o = 0.2$	1.0233	1.0450	1.0682	1.0925	1.1174
$L_i/L_o = 0.1$	1.0249	1.0439	1.0629	1.0827	1.1026
Three-point flexure	0.9252	0.8488	0.7818	0.7255	0.6785
Circular					
Four-point flexure					
$L_i/L_o = 0.5$	1.0185	1.0376	1.0567	1.0796	1.0991
$L_i/L_o = 0.4$	1.0152	1.0308	1.0466	1.0660	1.0821
$L_i/L_o = 0.3$	1.0127	1.0257	1.0388	1.0557	1.0692
$L_i/L_o = 0.2$	1.0111	1.0222	1.0337	1.0489	1.0608
$L_i/L_o = 0.1$	1.0098	1.0196	1.0299	1.0442	1.0551
Three-point flexure	0.9154	0.8400	0.7750	0.7220	0.6744
Ring-on-ring biaxial disc					
PIA reliability model	1.0985	1.1118	1.1492	1.1960	1.2477
Batdorf reliability model	0.7728	0.5637	0.4771	0.4302	0.4013

Consider the case of the rectangular four-point flexure specimen with the inner load span to the outer support span ratio of 0.5 and the Weibull moduli value of 25. From Table 5.1 the closed form effective volume value for this specimen has an associated error coefficient of, $Error = 1.2682$. Now consider a theoretically correct value of Weibull material scale parameter results in say 1 failure in 100,000 components, which gives reliability of 0.99999 for the component. The reliability of the component as calculated from Equation (5.6) will be 0.9999921 or in terms of component failure 1 in 126,816. Based on these calculations the error in the component probability of failure will be of 0.0000079 instead of the theoretically correct value of 0.00001 that is about 21.15% lower (see Table 5.3). Consequently the probability of failure is underestimated (and reliability overestimated) and the error proves to be a non-conservative error.

Now consider the case of the rectangular three-point flexure specimen with the Weibull moduli value of 25. From Table 5.1 the closed form effective volume value for this specimen has an associated error coefficient of, $Error = 0.7028$. Again consider a theoretically correct value of Weibull material scale parameter results in say 1 failure in 100,000 components, which gives reliability of 0.99999 for the component. The reliability of the component as calculated from Equation (5.6) will be 0.99998577 or in terms of component failure 1 in 70,274. Based on these calculations the error in the component probability of failure will be of 0.0000142 instead of the theoretically correct value of 0.00001 that is about 42.3% higher (see Table 5.2). Consequently the probability of failure is overestimated (and reliability underestimated) and the error proves to be a conservative error.

Similar observations can be made for component probability of failure calculations based on the error associated with the effective area calculations in three-point and four-point rectangular flexure specimens. The above arguments show that the error in effective volume and area calculations can affect the overall probability of failure of the component to a significant extent.

Table 5.3 delineates the extent of this impact on the probability of failure values associated with the use of the closed form effective volume solutions. From the table it can be observed that the difference in the probability of failure value increases with the increasing value of Weibull modulus for all the investigated specimens. The effect of neglecting the shear stresses in three-point flexure specimen for both the rectangular and the circular cross-sections on the probability of failure calculations are indicated. Also it is evident that the closed form expressions for V_{eff} and A_{eff} were not determined utilizing the Batdorf reliability model. Table 5.4 shows trends similar to that present in Table 5.3 for the probability of failure values associated with the use of the closed form effective area solutions

Table 5.3 Difference in the probability of failure using the closed form effective
volume solutions relative to the numerical solutions (fine mesh)

Specimen	P_f Difference (Effective Volume)				
	$m = 5$	$m = 10$	$m = 15$	$m = 20$	$m = 25$
Rectangular bar					
Four-point flexure					
$L_i / L_o = 0.5$	-4.82%	-9.33%	-13.50%	-17.44%	-21.15%
$L_i / L_o = 0.4$	-3.95%	-7.63%	-11.06%	-14.34%	-17.45%
$L_i / L_o = 0.3$	-3.30%	-6.31%	-9.12%	-11.83%	-14.42%
$L_i / L_o = 0.2$	-2.77%	-5.16%	-7.36%	-9.47%	-11.51%
$L_i / L_o = 0.1$	-3.01%	-5.34%	-7.29%	-9.10%	-10.81%
Three-point flexure	+7.29%	+10.70%	+24.66%	+33.60%	+42.30%
Circular rod					
Four-point flexure					
$L_i / L_o = 0.5$	-2.21%	-4.17%	-5.99%	-8.06%	-9.75%
$L_i / L_o = 0.4$	-1.88%	-3.54%	-5.08%	-6.88%	-8.32%
$L_i / L_o = 0.3$	-1.63%	-3.06%	-4.40%	-6.00%	-7.24%
$L_i / L_o = 0.2$	-1.45%	-2.75%	-3.94%	-5.42%	-6.53%
$L_i / L_o = 0.1$	-1.33%	-2.58%	-3.73%	-5.15%	-6.21%
Three-point flexure	+8.10%	+16.48%	+24.94%	+32.83%	+40.92%
Ring-on-ring biaxial disc					
PIA reliability model	-9.07%	-10.48%	-13.59%	-17.10%	-20.63%
Batdorf reliability model	+29.07%	+76.34%	+107.96%	+130.31%	+146.58%

Table 5.4 Difference in the probability of failure using the closed form effective area solutions relative to the numerical solutions (fine mesh)

Specimen	P_f Difference (Effective Area)				
	$m = 5$	$M = 10$	$m = 15$	$m = 20$	$m = 25$
Rectangular bar					
Four-point flexure					
$L_i / L_o = 0.5$	-4.38%	-8.67%	-12.79%	-16.71%	-20.42%
$L_i / L_o = 0.4$	-3.53%	-6.99%	-10.38%	-13.66%	-16.79%
$L_i / L_o = 0.3$	-2.85%	-5.59%	-8.34%	-11.04%	-13.64%
$L_i / L_o = 0.2$	-2.27%	-4.30%	-6.38%	-8.46%	-10.50%
$L_i / L_o = 0.1$	-2.43%	-4.20%	-5.92%	-7.63%	-9.31%
Three-point flexure	+8.09%	+17.81%	+27.91%	+37.84%	+47.39%
Circular rod					
Four-point flexure					
$L_i / L_o = 0.5$	-1.82%	-3.62%	-5.37%	-7.38%	-9.02%
$L_i / L_o = 0.4$	-1.50%	-2.99%	-4.45%	-6.19%	-7.58%
$L_i / L_o = 0.3$	-1.26%	-2.51%	-3.74%	-5.28%	-6.47%
$L_i / L_o = 0.2$	-1.09%	-2.17%	-3.26%	-4.66%	-5.73%
$L_i / L_o = 0.1$	-0.97%	-1.92%	-2.90%	-4.23%	-5.23%
Three-point flexure	+9.24%	+19.05%	+29.03%	+38.50%	+48.27%
Ring-on-ring biaxial disc					
PIA reliability model	-8.97%	-10.05%	-12.99%	-16.39%	-19.85%
Batdorf reliability model	+29.40	+77.40	+109.61	+132.47	+149.18

Table 5.5 shows the Hazard rate associated with the use of closed form effective volume solutions for 1 failure in 100,000 components as discussed earlier. Table 5.5 shows discernible trends similar to earlier discussed trends. Similarly Table 5.6 shows

the Hazard rate associated with the use of closed form effective area solutions for 1 failure in 100,000 components.

Table 5.5 Hazard rate for effective volume values calculated for closed form solutions against numerical approach with 1 in 100,000 failure

Specimen	Hazard rate for 1 in 100,000 failure (Effective volume)				
	$m = 5$	$M = 10$	$m = 15$	$m = 20$	$m = 25$
Rectangular bar					
Four-point flexure					
$L_i/L_o = 0.5$	1/105,068	1/110,290	1/115,613	1/121,118	1/126,816
$L_i/L_o = 0.4$	1/104,113	1/108,259	1/112,441	1/116,735	1/121,140
$L_i/L_o = 0.3$	1/103,410	1/106,734	1/110,040	1/113,412	1/116,849
$L_i/L_o = 0.2$	1/102,846	1/105,444	1/107,942	1/110,465	1/113,013
$L_i/L_o = 0.1$	1/103,100	1/105,636	1/107,863	1/110,010	1/112,118
Three-point flexure	1/93,202	1/90,334	1/80,217	1/74,848	1/70,275
Circular rod					
Four-point flexure					
$L_i/L_o = 0.5$	1/102,261	1/104,347	1/106,376	1/108,767	1/110,801
$L_i/L_o = 0.4$	1/101,919	1/103,665	1/105,352	1/107,390	1/109,071
$L_i/L_o = 0.3$	1/101,660	1/103,160	1/104,597	1/106,379	1/107,803
$L_i/L_o = 0.2$	1/101,476	1/102,824	1/104,105	1/105,726	1/106,983
$L_i/L_o = 0.1$	1/101,351	1/102,644	1/103,869	1/105,431	1/106,620
Three-point flexure	1/92,510	1/85,854	1/80,038	1/75,285	1/70,963
Ring-on-ring biaxial disc					
PIA reliability model	1/109,978	1/111,710	1/115,729	1/120,628	1/125,992
Batdorf reliability model	1/77,479	1/56,710	1/48,086	1/43,420	1/40,555

Table 5.6 Hazard rate for effective area values calculated for closed form solutions
against numerical approach with 1 in 100,000 failure

Specimen	Hazard rate for 1 in 100,000 failure (Effective area)				
	$m = 5$	$m = 10$	$m = 15$	$m = 20$	$m = 25$
Rectangular					
Four-point flexure					
$L_i/L_o = 0.5$	1/104,578	1/109,491	1/114,669	1/120,068	1/125,665
$L_i/L_o = 0.4$	1/103,659	1/107,511	1/111,585	1/115,817	1/120,175
$L_i/L_o = 0.3$	1/102,938	1/105,926	1/109,103	1/112,406	1/115,795
$L_i/L_o = 0.2$	1/102,326	1/104,495	1/106,817	1/109,246	1/111,736
$L_i/L_o = 0.1$	1/102,492	1/104,386	1/106,293	1/108,265	1/110,264
Three-point flexure	1/92,518	1/84,882	1/78,180	1/72,549	1/67,846
Circular					
Four-point flexure					
$L_i/L_o = 0.5$	1/101,849	1/103,756	1/105,671	1/107,963	1/109,912
$L_i/L_o = 0.4$	1/101,524	1/103,084	1/104,657	1/106,603	1/108,205
$L_i/L_o = 0.3$	1/101,274	1/102,570	1/103,885	1/105,572	1/106,918
$L_i/L_o = 0.2$	1/101,105	1/102,221	1/103,367	1/104,892	1/106,080
$L_i/L_o = 0.1$	1/100,983	1/101,961	1/102,991	1/104,417	1/105,515
Three-point flexure	1/91,540	1/83,999	1/77,502	1/72,202	1/67,445
Ring-on-ring biaxial disc					
PIA reliability model	1/109,854	1/111,176	1/114,924	1/119,605	1/124,769
Batdorf reliability model	1/77,280	1/56,368	1/47,708	1/43,015	1/40,131

CONCLUSION

As discussed earlier the study done by Duffy et al.^{[9][12]} for the C-ring and the O-ring specimens highlight the need for revisiting the assumptions made in deriving closed form expressions for effective volume and area that are available in the literature. The results presented in this chapter and Chapter 4 underscores the fact that the closed form equations for the effective volume and effective area for a broad range of test specimens are inaccurate. This was demonstrated through the use of the numerical approach outlined earlier which accommodates multi-axial states of the stress present in most test specimens via finite element analysis. This issue of utilizing the correct effective volume (or area) has a decided impact on component reliability calculations. In this chapter it was shown that the most widely accepted test specimen in use today, i.e., the four-point bend specimen, produced decidedly non-conservative component probability of failure predictions when the closed form expressions for effective volume (or area) were employed.

As Duffy et al.^[11] points out the discussion above must be tempered by the fact that loss in fidelity in calculating component reliability can be caused by several factors. They include sampling error associated with estimating Weibull parameter from failure data, experimental error in obtaining failure data, as well as using finite element analysis to obtain the stress field throughout a component. The discussion here focused on effective volumes and effective areas and how they affect the component probability of failure.

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