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# A Remark on The Global Dynamics of Competitive Systems on Ordered Banach Spaces

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# A REMARK ON THE GLOBAL DYNAMICS OF COMPETITIVE SYSTEMS ON ORDERED BANACH SPACES

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Abstract. A well-known result in [Hsu-Smith-Waltman, Trans. AMS (1996)] states that in a competitive semiflow defined on $X^+ = X_1^+ \times X_2^+$, the product of two cones in respective Banach spaces, if $(u^*, 0)$ and $(0, v^*)$ are the global attractors in $X_1^+ \times \{0\}$ and $\{0\} \times X_2^+$ respectively, then one of the following three outcomes is possible for the two competitors: either there is at least one coexistence steady state, or one of $(u^*, 0), (0, v^*)$ attracts all trajectories initiating in the order interval $I = [0, u^*] \times [0, v^*]$. However, it was demonstrated by an example that in some cases neither $(u^*, 0)$ nor $(0, v^*)$ is globally asymptotically stable if we broaden our scope to all of $X^+$. In this paper, we give two sufficient conditions that guarantee, in the absence of coexistence steady states, the global asymptotic stability of one of $(u^*, 0)$ or $(0, v^*)$ among all trajectories in $X^+$. Namely, one of $(u^*, 0)$ or $(0, v^*)$ is (i) linearly unstable, or (ii) is linearly neutrally stable but zero is a simple eigenvalue. Our results complement the counter example mentioned in the above paper as well as applications that frequently arise in practice.

## Introduction

It is well-known [6] that if there are two steady states $w_i$ $(i = 1, 2)$ of a monotone semiflow in a Banach space with an ordered cone, so that $w_1 < w_2$, and if there are no other steady states lying within the order interval

$$[w_1, w_2] := \{w : w_1 \leq w \leq w_2\},$$

then one of the steady states $w_i$ is unstable, and the remaining one attracts all trajectories initiating in $[w_1, w_2] \setminus \{w_1, w_2\}$. The theorem is extended to competitive systems in [8], where they generalized the result to allow the existence of a repelling (trivial) equilibrium on the boundary of the order interval. We state their results more precisely in the following setting. Let $X = X_1 \times X_2$, $X^+ = X_1^+ \times X_2^+$, and $K = X_1^+ \times (-X_2^+)$. $X^+$ is a cone in $X$ with nonempty interior given by $\operatorname{Int} X^+ = \operatorname{Int} X_1^+ \times \operatorname{Int} X_2^+$. It generates the order relations $\leq, <, \ll$ in the usual way. In particular, if $w = (u, v)$ and $\bar{w} = (\bar{u}, \bar{v})$, then $w \leq \bar{w}$ if and only if $u \leq \bar{u}$ and $v \leq \bar{v}$. For the study of competitive systems, the more important cone is $K$ which also has nonempty interior given by $\operatorname{Int} K = \operatorname{Int} X_1^+ \times (-\operatorname{Int} X_2^+)$. The cone

$K$ generates the partial order relations $\leq_K, <_K, \ll_K$. In this case

$$w \leq_K \bar{w} \quad \Leftrightarrow \quad u \leq \bar{u} \quad \text{and} \quad \bar{v} \leq v.$$

A similar statement holds with $\ll_K$ replacing $\leq_K$ and $\ll$ replacing $\leq$. Consider

$$\begin{cases} \frac{du}{dt} = A_1 u + f(u,v) \\ \frac{dv}{dt} = A_2 v + g(u,v) \end{cases} \tag{1.1}$$

where $A_i$ are sectorial operators on $X_i$ ($i = 1, 2$) respectively; $f : X_1 \times X_2 \to X_1$ and $g : X_1 \times X_2 \to X_2$ are differentiable functions. We denote the continuous semiflow generated by the above system by $T_t$. (See, e.g. [5, Ch. 3].) The semiflow properties are (i) $T_0(u_0, v_0) = (u_0, v_0)$ for all $(u_0, v_0) \in X_1^+ \times X_2^+$, and (ii) $T_t \circ T_s = T_{t+s}$ for $t, s \geq 0$. Recall the following characterization of a competition system of two viable species, i.e. ones which persist in the absence of competition, that was given in [8].

- **(H1):** $T$ is strictly order-preserving with respect to $<_K$. That is, $w <_K \bar{w}$ implies $T_t(w) <_K T_t(\bar{w})$. For each $t > 0$, $T_t : X^+ \to X^+$ is order compact.
- **(H2):** $T_t(0) = 0$ for all $t \geq 0$ and $0$ is a repelling equilibrium, i.e. there exists a neighborhood $U$ of $(0, 0)$ in $X^+$ such that for each $(u_0, v_0) \in U$, $(u_0, v_0) \neq 0$, there is a $t_0 > 0$ such that $T_{t_0}(u_0, v_0) \notin U$.
- **(H3):** $T_t(X_1^+ \times \{0\}) \subset X_1^+ \times \{0\}$ for all $t \geq 0$. There exists $u^* \gg 0$ such that $T_t((u^*, 0)) = (u^*, 0)$ for all $t \geq 0$, and $T_t((u_0, 0)) \to (u^*, 0)$ as $t \to \infty$ for all $u_0 \neq 0$. The symmetric conditions hold for $T$ on $\{0\} \times X_2^+$ with equilibrium point $(0, v^*)$.
- **(H4):** If $(u_0, v_0) \in X^+$ satisfies $u_0 \neq 0$ and $v_0 \neq 0$, then $T_t(u_0, v_0) \gg 0$ for $t > 0$. If $w, \bar{w} \in X^+$ satisfy $w <_K \bar{w}$ and either $w$ or $\bar{w}$ belongs to $\operatorname{Int} X^+$, then $T_t(w) \ll_K T_t(\bar{w})$ for $t > 0$.

Here we recall that order compactness means that for all $u_1 \in X_1^+$ and $v_1 \in X_2^+$, $T_t([0, u_1] \times [0, v_1])$ has compact closure. Also, as a consequence of (H3), we have

$$f(0, v) = 0 \quad \text{for all } v \in X_2^+ \quad \text{and} \quad g(u, 0) = 0 \quad \text{for all } u \in X_1^+.$$

**Theorem 1.1** ([8, Theorem B]). *Let* (H1)-(H4) *hold. Then the omega limit set of every orbit is contained in $I$, where*

$$I = [(u^*, 0), (0, v^*)] = \{(u_0, v_0) \in X : (u^*, 0) \leq_K (u_0, v_0) \leq_K (0, v^*)\}.$$

*If $I$ has no other equilibrium than $(u^*, 0)$ and $(0, v^*)$, then exactly one of the of the following holds.*

- **(a):** $T_t(w) \to (u^*, 0)$ *as* $t \to \infty$ *for every* $w = (u_0, v_0) \in I$ *with* $u_0, v_0 \neq 0$.
- **(b):** $T_t(w) \to (0, v^*)$ *as* $t \to \infty$ *for every* $w = (u_0, v_0) \in I$ *with* $u_0, v_0 \neq 0$.

*Finally, for all $w = (u_0, v_0) \in X^+ \setminus I$ and $u_0, v_0 \neq 0$, either $T_t(w) \to (u^*, 0)$ or $T_t(w) \to (0, v^*)$ as $t \to \infty$.*

As remarked in [8], the result may seem a bit unsatisfactory in the sense that one do not conclude in case (a) (resp. (b)) that $T_t(w) \to (u^*, 0)$ (resp. $T_t(w) \to (0, v^*)$) for all $w = (u_0, v_0) \in X^+$ such that $u_0, v_0 \neq 0$. In fact, the following example given in [8] shows that neither $(u^*, 0)$ nor $(0, v^*)$ is globally asymptotically stable in $X^+$, i.e. case (a) of Theorem 1.1 may hold yet some open set of initial data outside $I$ is attracted to $(0, v^*)$, instead of $(u^*, 0)$. The example is the following planar system in $\mathbb{R}^2_+$.

$$\begin{cases} u' = u(1 - u - v) \\ v' = v(1 - v - \mu u)^3 \end{cases}$$

where $\mu > 1$. It is easy to verify that all positive solutions beginning in $I = [0,1] \times [0,1]$ are attracted to $(u^*, 0) = (1, 0)$ but that solutions starting at $(u_0, v_0)$ near $(0, v^*) = (0, 1)$ and satisfying $v_0 > 1$, $0 < u_0 < (v_0 - 1)^2$ are attracted to $(0, v^*) = (0, 1)$.

Actually, one can observe that in the above example, the unstable equilibrium $(0, v^*) = (0, 1)$ is linearly neutrally stable, and has a two-dimensional center manifold. As we shall see, in a quite general setting, these are indeed crucial for the unstable equilibrium to attract any trajectories starting in $X^+$.

In this paper, we give two sufficient criteria for one of the semitrivial steady states $(u^*, 0)$, $(0, v^*)$ to be globally asymptotically stable. We assume in addition the following regularity of $f$ and $g$:

**(H5):** For each $t > 0$, $T_t : X^+ \to X^+$ is $C^1$.

We define the linear operator $L$ whose spectrum defines the linear stability of $(0, v^*)$.

**Definition 1.2.** Define the linear operator $L : D(A) \times D(B) \to X$ by

$$L \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} A\phi + f_u(0, v^*)[\phi] \\ B\psi + g_u(0, v^*)[\phi] + g_v(0, v^*)[\psi] \end{pmatrix} \tag{1.2}$$

We say that the semi-trivial steady state $(0, v^*)$ is linearly unstable if $L$ has a positive eigenvalue $\lambda$ which possesses an eigenvector $(\hat{\phi}, \hat{\psi}) \in \operatorname{Int} K$.

Our first main result states that if one of the semitrivial steady states, $(u^*, 0)$ or $(0, v^*)$, is linearly unstable, then the other one must be globally asymptotically stable in $\{(u, v) \in X^+ : u_0 \neq 0 \text{ and } v_0 \neq 0\}$.

**Theorem 1.3.** *Suppose* (H1) - (H5) *hold, and*

(i) (1.1) *does not have any steady states in* $\operatorname{Int} X^+$, *and*
(ii) (*Linear Instability of* $(0, v^*)$) $L$ *has a positive eigenvalue* $\hat{\lambda}$ *with positive eigenvector* $(\hat{\phi}, \hat{\psi}) \in \operatorname{Int} K$.

*Then for any* $(u_0, v_0) \in X^+$ *so that* $u_0 \neq 0$ *and* $v_0 \neq 0$,

$$T_t(u_0, v_0) \to (u^*, 0) \quad \text{as } t \to \infty.$$

Our second result deals with the case when $(0, v^*)$ is linearly neutrally stable.

**Theorem 1.4.** *Suppose* (H1) - (H5) *hold, and*

(i) (1.1) *has no steady states in* $\operatorname{Int} X^+$,
(ii) (*Linear Neutral Stability of* $(0, v^*)$) *There exists constant* $\beta > 0$ *such that* $\sigma(L) \subseteq \{0\} \cup \{z \in \mathbb{C} : \operatorname{Re} z < -\beta\}$ *and* $0$ *is a simple eigenvalue of* $L$ *with eigenvector* $(\hat{\phi}, \hat{\psi}) \in \operatorname{Int} K$.
(iii) *Some trajectory starting in* $\operatorname{Int} I$ *does not converge to* $(0, v^*)$, *where*

$$I = [0, u^*] \times [0, v^*] = \{(u, v) \in X : 0 \leq u \leq u^* \text{ and } 0 \leq v \leq v^*\}.$$

*Then for any* $(u_0, v_0) \in X^+$ *so that* $u_0 \neq 0$ *and* $v_0 \neq 0$,

$$T_t(u_0, v_0) \to (u^*, 0) \quad \text{as } t \to \infty.$$

*Remark* 1.5. (i) If we define the linear operator $\tilde{L} : D(A) \to X_1$ by

$$\tilde{L}\phi := A\phi + f_u(0, v^*)[\phi], \tag{1.3}$$

and assume in addition that $(0, v^*)$ is linearly stable as a steady state of the restricted flow in $\{0\} \times X_2^+$, then we see that $(0, v^*)$ is linearly unstable if and only if $\tilde{L}$ has a positive eigenvalue $\lambda$ which possess an eigenvector $\hat{\phi} \in \operatorname{Int} X_1^+$.

(ii) If $(0, v^*)$ is linearly unstsable with positive eigenvalue $\lambda$ and eigenvector $(\hat{\phi}, \hat{\psi})$, then for each $t > 0$, $D_{(u,v)}(T_t)(0, v^*)[\hat{\phi}, \hat{\psi}] = e^{\lambda t}(\hat{\phi}, \hat{\psi})$.

In the case that $T_t$ is strongly monotone, then the principal eigenvalue $\lambda$ of $L$ is always simple, with $\sup\{\operatorname{Re}\lambda' : \lambda' \in \sigma(L) \setminus \{\lambda\}\} < \lambda$ and the principal eigenvector $(\phi, \psi)$ may be chosen so that $(\phi, \psi) \in \operatorname{Int} K$ [11], i.e. Theorem 1.3 or Theorem 1.4 is applicable. See, for instance, [2] for patch models, [3] for nonlocal operators, and [1, 4, 9, 10] for reaction-diffusion models. In particular, for the system considered in [10], both of the semi-trivial steady states, $(u^*, 0)$ and $(0, v^*)$, are linearly neutrally stable, but the principal eigenvalue zero is simple in both cases.

This article presents global dynamical results in the absence of coexistence steady states, i.e. when there are exactly three steady states. In general, the set of quasiconvergent points (i.e. points whose omega limiting sets belong to the set of steady states) is dense in strongly monotone dynamical systems. See [6] and also [7] for some recent results concerning existence of asymptotically stable steady states in analytic semiflows. In contrast to their results, which proves local stability of steady states, our main effort in this paper is to characterize the local stable manifold of the unstable steady state in $C^1$ semiflows.

## Proofs of Theorems 1.3 and 1.4

**Proof of Theorem 1.3.** By the assumptions and Remark 1.5(ii), there exist $(\hat{\phi}, \hat{\psi}) \in \operatorname{Int} K$ and $\hat{\lambda} > 0$ such that

$$DT(0, v^*)[\hat{\phi}, \hat{\psi}] = e^{\hat{\lambda}}(\hat{\phi}, \hat{\psi}),$$

where $T = T_1$ is the time-one map. Fix positive constants $1 < c_1 < c_2 < e^{\hat{\lambda}}$. By (H5), there exists $\epsilon_0 > 0$ such that for all $(\tilde{u}, \tilde{v})$ in a neighborhood $\mathcal{N}$ of $(0, v^*)$,

$$DT(\tilde{u}, \tilde{v})[\hat{\phi}, \hat{\psi}] \geq_K c_2(\hat{\phi}, \hat{\psi}). \tag{2.1}$$

By Theorem 1.1 and the fact that (1.1) has no steady states in $\operatorname{Int} X^+$, any trajectory starting in $\operatorname{Int} X^+$ converges either to $(u^*, 0)$ or $(0, v^*)$. Therefore, it suffices to show that no trajectory starting in $\operatorname{Int} X^+$ converges to $(0, v^*)$. Suppose to the contrary that for some $(u_0, v_0) \in \operatorname{Int} X^+$, $T_t(u_0, v_0) \to (0, v^*)$ as $t \to \infty$. Denote

$$(u_n, v_n) = T^n(u_0, v_0),$$

and define, for each $n \geq 0$,

$$\epsilon_n = \sup\{\epsilon > 0 : \epsilon\hat{\phi} \leq u_n\},$$

then $\epsilon_n > 0$ as $u_n \in \operatorname{Int} X_1^+$ for all $n$. Moreover, $\epsilon_n \to 0$ since $(u_n, v_n) \to (0, v^*)$.

We claim that

$$\epsilon_{n+1} \geq c_1\epsilon_n \quad \text{for all } n \text{ sufficiently large.} \tag{2.2}$$

Since $c_1 > 1$, this contradicts the fact that $u_n \to 0$. To prove (2.2), denote $P_1 : X \to X_1$ to be the projector operator, i.e. $P_1(u, v) = u$ for all $(u, v) \in X$. Then

$$\begin{aligned}
u_{n+1} &= P_1(T(u_n, v_n)) \geq P_1(T(\epsilon_n\hat{\phi}, v_n)) \\
&= P_1(T(\epsilon_n\hat{\phi}, v_n) - T(0, v_n - \epsilon_n\hat{\psi})) \quad \text{by (H3)}, \\
&= P_1(DT(0, v_n - \epsilon_n\hat{\psi})[\epsilon_n\hat{\phi}, \epsilon_n\hat{\psi}] + o(\epsilon_n))
\end{aligned}$$

Since $(0, v_n - \epsilon_n \hat{\psi}) \to (0, v^*)$ as $n \to \infty$, we conclude from (2.1) that for all $n$ sufficiently large,

$$u_{n+1} \geq c_2 \epsilon_n \hat{\phi} + o(\epsilon_n) \geq c_1 \epsilon_n \hat{\phi}.$$

But this proves (2.2), from which we obtain $\epsilon_n \to \infty$, which is a contradiction. □

**Proof of Theorem 1.4.** Again, by Theorem 1.1, it suffices to show that no trajectory starting in $\operatorname{Int} X^+$ converges to $(0, v^*)$. By (ii), there exists $\beta > 0$ such that $\sigma(L) \subseteq \{0\} \cup \{z \in \mathbb{C} : \operatorname{Re} z < -\beta\}$.

**Claim 2.1.** *There exists $\epsilon_3 > 0$ such that if $\|(u_0, v_0) - (0, v^*)\| < \epsilon_3$, then*

$$\|T_t(u_0, v_0) - (0, v^*)\| \leq O(e^{-\beta t/3}) \tag{2.3}$$

*implies $u_0 = 0$. In which case $T_t(u_0, v_0) \in \{0\} \times X_2^+$ for all $t \geq 0$.*

To see the claim, consider for $\delta \in [0, \beta/3)$ the semigroup $T_t^\delta$ generated by the following slightly more general problem

$$\begin{cases} \frac{du}{dt} = Au + f(u, v) + \delta u, \\ \frac{dv}{dt} = Bv + g(u, v). \end{cases} \tag{2.4}$$

We claim that the linear operator $L^\delta : D(A) \times D(B) \to X$ defined by

$$L^\delta \begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} A\phi + f_u(0, v^*)[\phi] + \delta\phi \\ B\psi + g_u(0, v^*)[\phi] + g_v(0, v^*)[\psi] \end{pmatrix}$$

whose spectrum (which determines the linear stability of $(0, v^*)$) satisfies

$$\sigma(\tilde{L}^\delta) \subseteq \{\delta\} \cup \{z \in \mathbb{C} : \operatorname{Re} z < -2\beta/3\}, \tag{2.5}$$

with $\delta$ being a simple eigenvalue of $L^\delta$ with positive eigenvector $(\hat{\phi}, \hat{\hat{\psi}}) \in \operatorname{Int} K$, where $\hat{\hat{\psi}}$ is the unique solution of $B\psi + g_v(0, v^*)[\psi] - \delta\psi = g_u(0, v^*)[\hat{\phi}]$. Note that $\hat{\hat{\psi}}$ is well-defined since $(0, v^*)$ is stable with respect to the restricted flow in $\{0\} \times X_2^+$ by (H3), i.e. $\sigma(B + g_v(0, v^*)) \subset \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$. In other words, the simple eigenvalue 0 of $L$ becomes a simple eigenvalue $\delta$ of $L^\delta$ while the rest of the spectrum is shifted in $\mathbb{C}$ at most by $\delta \in (0, \beta/3)$. Define the $(\beta/3)-$stable manifold of $(0, v^*)$ by

$$\mathcal{S}^\delta_{\beta/3} := \left\{ (u_0, v_0) \in X^+ : \lim_{t \to \infty} e^{\beta t/3} \|T_t^\delta(u_0, v_0) - (0, v^*)\| = 0 \right\}.$$

Then by the proof of Theorem 1.3, for any $\delta \in (0, \beta/3)$, $\mathcal{S}^\delta_{\beta/3} \subset \{(0, v_0) : v_0 \in X_2^+ \setminus \{0\}.\}$. Moreover, $\mathcal{S}^\delta_{\beta/3}$ depends continuously in $\delta \in [0, \beta/3)$ by the arguments in the proof of [5, Theorem 5.2.1]. Precisely, write $(u, v) = (0, v^*) + z$, and (2.4) as

$$z_t = L^\delta z + F(z) \tag{2.6}$$

where $F'(0) = 0$. By the arguments in the proof of [5, Theorem 5.2.1], for each $z_0 \in S^\delta_{\beta/3}$, the solution $z(t)$ with initial condition $z_0$ satisfies

$$z(t) = e^{L_2^\delta t} a + \int_0^t e^{L_2^\delta (t-s)} E_2 F(z(s))\, ds - \int_t^\infty e^{L_1^\delta (t-s)} E_1 F(z(s))\, ds, \tag{2.7}$$

where $a = E_2 z_0$, $E_1, E_2$ are the projections associated with the bounded spectral set $\{\delta\}$ and $\sigma(L^\delta) \setminus \{\delta\}$ respectively, and $L_i^\delta = E_i L^\delta$ $(i = 1, 2)$ are the restriction of $L^\delta$ to the invariant subspaces $E_i X$ $(i = 1, 2)$. In fact, for each $a \in E_2 X$, $\|a\| \ll 1$, the right-hand side of (2.7) defines a contraction map of the space of continuous $z : [0, \infty) \to X$ with $\sup_{t>0} e^{\beta t/3} \|z(t)\| \leq \rho$, for some small $\rho$. Therefore, locally near $(0, v^*)$, $S^{\delta}_{\beta/3}$ is in one-one correspondence with an open set in $E_2X$ containing the origin i.e. for each such $a \in E_2X$, there exists a unique $z(t; a)$ satisfying (2.7) such that $E_2 z(0; a) = a$. Since all the expressions depend on the parameter $\delta$ continuously, we obtain the continuous dependence of $S^{\delta}_{\beta/3}$ on $\delta \in [0, \beta/3)$. Finally, Claim 2.1 follows by letting $\delta \to 0$.

It is easy to see that $\mathcal{S}^0_{\beta/3}$ is a subset of the stable manifold of $\{(0, v^*)\}$, which we shall denote by $\mathcal{S}$. We shall show now that actually $\mathcal{S} = \mathcal{S}^0_{\beta/3}$. Suppose to the contrary that $\mathcal{S}^0_{\beta/3}$ is a proper subset of $\mathcal{S}$, then there exists $(u_0, v_0) \in X^+$ such that $u_0 \neq 0$ and $T_t(u_0, v_0) \to (0, v^*)$. We may assume without loss of generality that $\|(u_0, v_0) - (0, v^*)\| < \epsilon_3$. Then by Claim 2.1, (2.3) must not hold. That is, there exists $t_i \to \infty$ such that

$$\|T_{t_i}(u_0, v_0) - (0, v^*)\| \geq e^{-\beta t_i/3}. \tag{2.8}$$

On the other hand, by [5, Theorem 1.5.2], we may decompose

$$T_t(u_0, v_0) - (0, v^*) = \xi_1(t)(\hat{\phi}, \hat{\psi}) + \xi_2(t),$$

where $\xi_1(t) \in \mathbb{R}$ and $\xi_2(t) \in E_2(X)$ ($E_2$ being the spectral projection onto the part of $\sigma(L) \cap \{z \in \mathbb{C} : \operatorname{Re} z < -\beta\}$). By the proof of [5, Theorem 5.1.1], there exists $\delta_1 > 0$ such that if $|\xi_1(t)| < \delta_1$ for all $t > 0$, (which we may assume as $T_t(u_0, v_0) \to (0, v^*)$) then

$$\|\xi_2(t)\| \leq C e^{-2\beta t/3} \quad \text{for all } t.$$

Then (2.8) implies that for all large $t_i$,

$$|\xi_1(t_i)| \geq \frac{1}{2} e^{-\beta t_i/3}.$$

Hence along $t_i \to \infty$, we have

$$\frac{\|\xi_2(t_i)\|}{|\xi_1(t_i)|} \to 0,$$

whence

$$\begin{aligned} T_{t_i}(u_0, v_0) - (0, v^*) &= \xi_1(t_i) \left[ (\hat{\phi}, \hat{\psi}) + \frac{\xi_2(t_i)}{\xi_1(t_i)} \right] \\ &= \xi_1(t_i) \left[ (\hat{\phi}, \hat{\psi}) + o(1) \right] \in \operatorname{Int} K \end{aligned}$$

for all large $i$. i.e. $T_{t_i}(u_0, v_0) \in \operatorname{Int} I$ for all large $i$. Now, the assumption (iii) and Theorem 1.1 imply that $(u^*, 0)$ attracts all trajectories in $\operatorname{Int} X^+$. Hence $T_t(u_0, v_0) \to (u^*, 0)$ as $t \to \infty$, a contradiction to the assumption that $T_t(u_0, v_0) \to (0, v^*)$ as $t \to \infty$. $\square$


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## References


1. I. Averill and Y. Lou, D. Munther, *On several conjectures from evolution of dispersal*, J. Biol. Dyn. **6** (2012), 117-130.
2. R.S. Cantrell, C. Cosner and Y. Lou, *Evolutionary stability of ideal free dispersal strategies in patchy environments*, J. Math. Biol. **65** (2012), 943-965.
3. R.S. Cantrell, C. Cosner, Y. Lou and D. Ryan, *Evolutionary stability of ideal free dispersal strategies: a nonlocal dispersal model*, Can. Appl. Math. Q. **20** (2012), 15-38.
4. X. He and W.-M. Ni, *The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system I: Heterogeneity vs. homogeneity*, J. Differential Equations **254** (2013), 528-546.
5. D. Henry, Geometric theory of semilinear parabolic equations. Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981. iv+348 pp. ISBN: 3-540-10557-3
6. M. Hirsch, *Stability and convergence in strongly monotone dynamical systems*, J. Reine Angew. Math. **383** (1988), 1-53.
7. M. Hirsch and H. L. Smith, *Asymptotically stable equilibria for monotone semiflows*, Discrete Contin. Dyn. Syst. **14** (2006), 385-398.
8. S.B. Hsu, H.L. Smith and P. Waltman, *Competitive exclusion and coexistence for competitive systems on ordered banach spaces*, Trans. Amer. Math. Soc. **348** (1996), 4083-4094.
9. K.-Y. Lam and W.-M. Ni, *Uniqueness and complete dynamics in heterogeneous competition-diffusion systems*, SIAM J. Appl. Math. **72** (2012), 1695-1712.
10. K.-Y. Lam and D. Munther, *Invading the ideal free distribution*, Discrete Contin. Dyn. Syst. Ser. B. **19** (2014), 3219-3244.
11. H. L. Smith, Monotone dynamical systems, An introduction to the theory of competitive and cooperative systems, Mathematical Surveys and Monographs, **41** American Mathematical Society, Providence, RI, 1995.