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John R. Akeroyd
University of Arkansas

Pratibha G. Ghatage
Cleveland State University, p.ghatage@csuohio.edu

Maria Tjani
University of Arkansas

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Closed-Range Composition Operators on \mathbb{A}^2 and the Bloch Space

John R. Akeroyd, Pratibha G. Ghatage and Maria Tjani

Abstract. For any analytic self-map φ of $\{z : |z| < 1\}$ we give four separate conditions, each of which is necessary and sufficient for the composition operator C_φ to be closed-range on the Bloch space \mathcal{B} . Among these conditions are some that appear in the literature, where we provide new proofs. We further show that if C_φ is closed-range on the Bergman space \mathbb{A}^2 , then it is closed-range on \mathcal{B} , but that the converse of this fails with a vengeance. Our analysis involves an extension of the Julia-Carathéodory Theorem.

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Preliminaries

Let \mathbb{D} denote the unit disk $\{z : |z| < 1\}$ and let \mathbb{T} denote the unit circle $\{z : |z| = 1\}$. We let A denote two-dimensional Lebesgue measure on \mathbb{D} . The Bergman space \mathbb{A}^2 is the collection of functions f that are analytic in \mathbb{D} such that

$$\|f\|_{\mathbb{A}^2}^2 := \int_{\mathbb{D}} |f|^2 dA < \infty.$$

As a closed subspace of $L^2(A)$, \mathbb{A}^2 forms a Hilbert space with respect to the inner product $\langle f, g \rangle := \int_{\mathbb{D}} f \bar{g} dA$. The Bloch space \mathcal{B} is the collection of functions f that are analytic in \mathbb{D} such that

$$\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

Now $\|\cdot\|_{\mathcal{B}}$ defines a norm on \mathcal{B} , and under this norm \mathcal{B} forms a Banach space. Moreover, $\|f\|_{\mathbb{A}^2} \leq 3\|f\|_{\mathcal{B}}$ for any function f that is analytic in \mathbb{D} , and hence $\mathcal{B} \subseteq \mathbb{A}^2$. A function φ that is analytic in \mathbb{D} and that satisfies $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ is

called an *analytic self-map* of \mathbb{D} . Analytic automorphisms of \mathbb{D} are Möbius transformations of the form $z \mapsto c \frac{\alpha-z}{1-\bar{\alpha}z}$, where c is some unimodular constant and α is some point in \mathbb{D} ; we let $\varphi_\alpha(z) = \frac{\alpha-z}{1-\bar{\alpha}z}$. The so-called *pseudohyperbolic metric* on \mathbb{D} is given by $\rho(z, w) = |\varphi_w(z)|$; and is indeed a metric. For any z in \mathbb{D} and any r , $0 < r < 1$, we let $D(z, r)$ denote the pseudohyperbolic disk of radius r about z , namely, $\{w \in \mathbb{D} : \rho(z, w) < r\}$. Now if φ is an analytic self-map of \mathbb{D} , then the composition operator C_φ , given by $C_\varphi(f) := f \circ \varphi$, is a bounded operator on both \mathbb{A}^2 and \mathcal{B} . This result for the Bloch space is a simple consequence of the Schwarz-Pick Lemma (cf., [7, page 2]), and for the Bergman space case one may consult [13, page 17]. Moreover, if φ is not constant, then C_φ is one-to-one on these spaces and hence, by the Open Mapping Theorem, is closed-range if and only if it is bounded below. For any analytic self-map φ of \mathbb{D} , define τ_φ on \mathbb{D} by

$$\tau_\varphi(z) := \frac{(1 - |z|^2)\varphi'(z)}{1 - |\varphi(z)|^2}.$$

For $\varepsilon > 0$, let $\Lambda_\varepsilon = \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}$ and let $F_\varepsilon = \varphi(\Lambda_\varepsilon)$. We say that F_ε satisfies the *reverse Carleson condition* if there exist s and c , $0 < s, c < 1$, such that

$$A(F_\varepsilon \cap D(z, s)) \geq cA(D(z, s)),$$

for all z in \mathbb{D} ; cf., [10] for seminal work regarding this condition. It has been shown that C_φ is closed-range on \mathcal{B} if and only if there exists $\varepsilon > 0$ such that F_ε satisfies the reverse Carleson condition; cf. [9] and [3]. In fact, in [3] it is shown that, what appears to be a weaker condition than the one stated above, is indeed equivalent. To be specific, if there exist $\varepsilon > 0$ and s , $0 < s < 1$, such that $F_\varepsilon \cap D(z, s) \neq \emptyset$ for all z in \mathbb{D} , then C_φ is closed-range on \mathcal{B} . One of the first results of this paper adds one more equivalent condition to this list, and we give a brief and rather novel proof that each of the three conditions are equivalent to C_φ being closed-range on \mathcal{B} ; see Theorem 2.2. We then turn to connections between the Bloch and Bergman space settings. In the Bergman space setting there is an analogue of Λ_ε that takes center stage. Indeed, if φ is an analytic self-map of \mathbb{D} and $\varepsilon > 0$, then we let $\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$ and let $G_\varepsilon = \varphi(\Omega_\varepsilon)$. In [1] it is shown that C_φ is closed-range on \mathbb{A}^2 if and only if there exists $\varepsilon > 0$ such that G_ε satisfies the reverse Carleson condition; that is, there exist s and c , $0 < s, c < 1$, such that

$$A(G_\varepsilon \cap D(z, s)) \geq cA(D(z, s)),$$

for all z in \mathbb{D} . Here, we establish an extension of the Julia-Carathéodory Theorem (see Theorem 3.4) and use it to show that if C_φ is closed-range on \mathbb{A}^2 , then there exist ε and s , $0 < \varepsilon, s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq F_\varepsilon$; see Theorem 3.5. From this we easily have the implication that if C_φ is closed-range on \mathbb{A}^2 , then it is also closed-range on \mathcal{B} ; see Corollary 3.6. We also (by examples) show that the converse of Corollary 3.6 fails, without remedy. Indeed, we construct a *thin* Blaschke product that fixes zero and that has no angular derivative anywhere on \mathbb{T} , whence C_B is norm preserving on \mathcal{B} and

yet is compact on \mathbb{A}^2 ; see Example 3.8. And we also construct a univalent analytic self-map h of \mathbb{D} that has no unimodular nontangential boundary values on \mathbb{T} , and thus has no angular derivative anywhere on \mathbb{T} (whence, C_h is compact on \mathbb{A}^2), such that C_h is closed-range on \mathcal{B} ; see Example 3.10. We close the paper with a result that follows easily from work done in [1] and a remark concerning Fredholm operators; see Sect. 4.

Regarding the Bloch Space

Recall that, for any analytic self-map φ of \mathbb{D} and any $\varepsilon > 0$,

$$\tau_\varphi(z) := \frac{(1 - |z|^2)\varphi'(z)}{1 - |\varphi(z)|^2}, \quad \text{and} \quad \Lambda_\varepsilon := \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}.$$

Lemma 2.1. *For any $\varepsilon > 0$ there exist r and s , $0 < r, s < 1$, such that if $z \in \Lambda_\varepsilon$, then*

- i) $D(z, r) \subseteq \Lambda_{\frac{\varepsilon}{2}}$,
- ii) φ is univalent in $D(z, r)$ and
- iii) $D(\varphi(z), s) \subseteq \varphi(D(z, r))$.

Proof. (i) By [8], τ_φ is Lipschitz with respect to the pseudohyperbolic metric. Indeed, there is a positive constant c , independent of φ and of z and w in \mathbb{D} , such that

$$|\tau_\varphi(z) - \tau_\varphi(w)| \leq c\rho(z, w).$$

Let $r = \frac{\varepsilon}{2c}$ and suppose that $|\tau_\varphi(w)| \leq \frac{\varepsilon}{2}$. Then, for z in Λ_ε ,

$$\frac{\varepsilon}{2} < ||\tau_\varphi(z)| - |\tau_\varphi(w)|| \leq |\tau_\varphi(z) - \tau_\varphi(w)| \leq c\rho(z, w).$$

Therefore, if $z \in \Lambda_\varepsilon$ and $\rho(z, w) < \frac{\varepsilon}{2c}$, then $w \in \Lambda_{\frac{\varepsilon}{2}}$.

(ii) Suppose that $a \in \Lambda_\varepsilon$ and $\alpha := \varphi(a)$. Notice that $\varphi_\alpha \circ \varphi \circ \varphi_a$ is an analytic self-map of the unit disk that maps 0 to 0 and that

$$|(\varphi_\alpha \circ \varphi \circ \varphi_a)'(0)| = |\tau_{\varphi_\alpha \circ \varphi \circ \varphi_a}(0)| = |\tau_{\varphi \circ \varphi_a}(0)| = |\tau_\varphi(a)| > \varepsilon.$$

We argue that $\varphi_\alpha \circ \varphi \circ \varphi_a$ is univalent in $\{z : |z| < r\}$; where, as in (i), $r := \frac{\varepsilon}{2c}$. Multiplying $\varphi_\alpha \circ \varphi \circ \varphi_a$ by an appropriate unimodular constant we may assume that $(\varphi_\alpha \circ \varphi \circ \varphi_a)'(0)$ is a positive real number (greater than ε). And using the facts that $\tau_{\varphi_\alpha \circ \varphi \circ \varphi_a}$ is Lipschitz with respect to the pseudohyperbolic metric, with the same Lipschitz constant c , and that $\varphi_\alpha \circ \varphi \circ \varphi_a$ maps 0 to 0, we find that

$$\operatorname{Re}((\varphi_\alpha \circ \varphi \circ \varphi_a)'(z)) > \frac{\varepsilon}{2}, \tag{2.1.1}$$

whenever $|z| < r$. Now let z and w be distinct points both of which have modulus less than r , and define γ on $[0, 1]$ by $\gamma(t) = (1 - t)z + tw$. Then, by (2.1.1),

$$0 \neq (w - z) \cdot \int_0^1 (\varphi_\alpha \circ \varphi \circ \varphi_a)'(\gamma(t)) dt = (\varphi_\alpha \circ \varphi \circ \varphi_a)(w) - (\varphi_\alpha \circ \varphi \circ \varphi_a)(z),$$

and hence $\varphi_\alpha \circ \varphi \circ \varphi_\alpha$ is univalent in $\{z : |z| < r\}$. It now follows that φ is univalent in $D(a, r)$.

(iii) Given the terminology of part (ii), $h(z) := \frac{1}{r\varepsilon}(\varphi_\alpha \circ \varphi \circ \varphi_\alpha)(rz)$ is analytic and univalent in \mathbb{D} , $h(0) = 0$ and $|h'(0)| > 1$. Therefore, by the Koebe One-Quarter Theorem (cf., [13, page 154]),

$$\left\{z : |z| < \frac{1}{4}\right\} \subseteq h(\mathbb{D}).$$

From this it follows that

$$\left\{z : |z| < \frac{r\varepsilon}{4}\right\} \subseteq (\varphi_\alpha \circ \varphi \circ \varphi_\alpha)(\{z : |z| < r\}).$$

With $s := \frac{r\varepsilon}{4}$ we then find that $D(\varphi(a), s) \subseteq \varphi(D(a, r))$. \square

As before, let φ be an analytic self-map of \mathbb{D} , let $\tau_\varphi(z) = \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2}$ and for $\varepsilon > 0$, let $\Lambda_\varepsilon = \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}$ and let $F_\varepsilon = \varphi(\Lambda_\varepsilon)$. We now give two conditions, each of which is equivalent to C_φ being closed-range on \mathcal{B} ; cf., [9] and [3], or Theorem 2.2 below.

- (*) There exist $\varepsilon > 0$ and constants c and s , $0 < c, s < 1$, such that $A(F_\varepsilon \cap D(z, s)) \geq cA(D(z, s))$ for all z in \mathbb{D} .
- (#) There exist $\varepsilon > 0$ and s , $0 < s < 1$, such that $F_\varepsilon \cap D(z, s) \neq \emptyset$ for all z in \mathbb{D} .

Theorem 2.2. *Let φ be an analytic self-map of \mathbb{D} . Then the following are equivalent.*

- i) C_φ is closed-range on \mathcal{B} .
- ii) Condition (*) holds.
- iii) Condition (#) holds.
- iv) There are constants r, s and c , $0 < r, s, c < 1$, such that, for any w in \mathbb{D} , there exists z_w in \mathbb{D} with the property: φ is univalent on $D(z_w, s)$, $\varphi(D(z_w, s)) \subseteq D(w, r)$ and $A(\varphi(D(z_w, s))) \geq c(1 - |w|^2)^2$.

Proof. (i) \implies (iii). Since any Frostman shift of φ (i.e., $\varphi_\alpha \circ \varphi$, where $\alpha \in \mathbb{D}$) gives rise to a closed-range composition operator on \mathcal{B} if and only if φ does, we may assume that $\varphi(0) = 0$. Now suppose that (iii) does not hold. Then we can find sequences $\{r_n\}_{n=1}^\infty$, where $0 < r_n < 1$ and $\lim_{r \rightarrow \infty} r_n = 1$, and $\{w_n\}_{n=1}^\infty$ in \mathbb{D} , where $\lim_{n \rightarrow \infty} |w_n| = 1$, such that

$$\sup\{|\tau_\varphi(z)| : z \in \varphi^{-1}(D(w_n, r_n))\} \longrightarrow 0,$$

as $n \rightarrow \infty$. Let $\Delta_n = \varphi^{-1}(D(w_n, r_n))$ and let $D_n = \mathbb{D} \setminus \Delta_n$; for $n = 1, 2, 3, \dots$. Now

$$\begin{aligned} \|\varphi_{w_n} \circ \varphi\|_{\mathcal{B}/C} &:= \sup\{(1 - |z|^2)|(\varphi_{w_n} \circ \varphi)'(z)| : z \in \mathbb{D}\} \\ &= \sup\{[1 - \rho^2(w_n, \varphi(z))]| \tau_\varphi(z) | : z \in \mathbb{D}\} \\ &\leq \sup\{[1 - \rho^2(w_n, \varphi(z))]| \tau_\varphi(z) | : z \in \Delta_n\} \\ &\quad + \sup\{[1 - \rho^2(w_n, \varphi(z))]| \tau_\varphi(z) | : z \in D_n\} \longrightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Yet $\|\varphi_{w_n}\|_{\mathcal{B}/C} = 1$, for all n . By Theorem 0 of [9] it now follows that C_φ is not closed-range on \mathcal{B} .

(iii) \implies (ii). We assume (iii), that (#) holds. Then, by Lemma 2.1, (*) holds for $\frac{\varepsilon}{2}$.

(ii) \implies (i). This follows immediately from Proposition 1 and Theorem 1 of [9].

At this point we have established the equivalence of (i), (ii) and (iii).

(iii) \implies (iv). This follows immediately from Lemma 2.1.

(iv) \implies (iii). Assuming (iv),

$$\int_{D(z_w, s)} |\varphi'(z)|^2 dA(z) \geq c(1 - |w|^2)^2,$$

and hence

$$\int_{D(z_w, s)} \frac{|\varphi'(z)|^2}{(1 - |w|^2)^2} dA(z) \geq c.$$

Thus we can find a positive constant ε , dependent only on r and s , such that

$$\int_{D(z_w, s)} |\tau_\varphi(z)|^2 dA(z) \geq \varepsilon^2 A(D(z_w, s)).$$

Therefore, $|\tau_\varphi(z)| \geq \varepsilon$ for some z in $D(z_w, s)$, and hence $F_\varepsilon \cap D(w, r) \neq \emptyset$ for each w in \mathbb{D} ; which gives us (iii). The proof is now complete. \square

A special case of our next result is given by Theorem 2 of [9]; namely, the case that φ is a univalent, analytic self-map of \mathbb{D} . As is indicated in the proof of Theorem 2.2, if $f \in \mathcal{B}$, then $\|f\|_{\mathcal{B}/C} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)|$.

Corollary 2.3. *Let φ be an analytic self-map of \mathbb{D} . Then C_φ is closed-range on \mathcal{B} if and only if there exists $\delta > 0$ such that, for all α in \mathbb{D} , $\|\varphi_\alpha \circ \varphi\|_{\mathcal{B}/C} \geq \delta$.*

Proof. We may assume that $\varphi(0) = 0$ here since any Frostman shift of φ gives rise to a closed-range composition operator on \mathcal{B} if and only if φ does, and since the collection of analytic automorphisms of \mathbb{D} forms a group under the operation of composition. Moreover, notice that $\|\varphi_\alpha\|_{\mathcal{B}/C} = 1$ for all α in \mathbb{D} . So, if C_φ is closed-range on \mathcal{B} , then, by Theorem 0 of [9], there exists $\delta > 0$ such that $\|\varphi_\alpha \circ \varphi\|_{\mathcal{B}/C} \geq \delta$ for all α in \mathbb{D} . Conversely, suppose that there exists $\delta > 0$ such that $\|\varphi_\alpha \circ \varphi\|_{\mathcal{B}/C} \geq \delta$ for all α in \mathbb{D} . Then, by Proposition 2 of [9], (iii) of Theorem 2.2 holds and hence C_φ is closed-range on \mathcal{B} . \square

The Context of \mathbb{A}^2 Versus that of \mathcal{B}

Let φ be an analytic self-map of \mathbb{D} and, for $\varepsilon > 0$, let $\Omega_\varepsilon := \{z \in \mathbb{D} : \frac{1 - |z|^2}{1 - |\varphi(z)|^2} > \varepsilon\}$, let $G_\varepsilon = \varphi(\Omega_\varepsilon)$ and let $K = \mathbb{T} \cap \overline{\Omega}_\varepsilon$. By the Julia-Carathéodory Theorem (cf., [13, page 57]), φ has an angular derivative at each point ξ in K , which we denote by $\varphi'(\xi)$. Indeed, $\varphi'(\xi) = \zeta \bar{\xi} d$, where $\zeta := \varphi(\xi) := \angle \lim_{z \rightarrow \xi} \varphi(z)$ and d is given by

$$d := \liminf_{z \rightarrow \xi} \frac{1 - |\varphi(z)|}{1 - |z|} \left(= \liminf_{z \rightarrow \xi} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right).$$

The Julia-Carathéodory Theorem tells us that $d > 0$. And since $\xi \in K$, $d \leq \frac{1}{\varepsilon}$.

Proposition 3.1. *Given the terminology of the above discussion, φ is continuous on $\bar{\Omega}_\varepsilon$ and φ' is continuous on K .*

Proof. The continuity of φ on $\bar{\Omega}_\varepsilon$ was established in [1]; see Remark 2.6 in this reference. Now let $\{\xi_n\}_{n=1}^\infty$ be a sequence in K that converges to ξ_0 in K , and let $d_n = |\varphi'(\xi_n)|$, for $n = 0, 1, 2, \dots$. Since φ is continuous on K , the continuity of φ' on K will follow if we show that $d_n \rightarrow d_0$, as $n \rightarrow \infty$. Now by the discussion just prior to this proposition, $\{d_n\}_{n=1}^\infty$ is bounded. And so, passing to a subsequence if necessary, we may assume that $d_n \rightarrow d$, as $n \rightarrow \infty$. Thus our goal here is to show that $d = d_0$. To this end, by the Julia-Carathéodory Theorem we can find a sequence $\{r_n\}_{n=1}^\infty$ in $(0, 1)$, such that $\lim_{n \rightarrow \infty} r_n = 1$ and $|d_n - \frac{1-|\varphi(r_n \xi_n)|}{1-r_n}| < \frac{1}{n}$, for $n = 1, 2, 3, \dots$. Hence, $\{r_n \xi_n\}_{n=1}^\infty$ is a sequence in \mathbb{D} that converges to ξ_0 and $\{\frac{1-|\varphi(r_n \xi_n)|}{1-r_n}\}_{n=1}^\infty$ converges to d . Julia's Theorem (cf., [13, page 63]) now tells us that $d = d_0$. \square

We now set the stage for two subsequent results.

Discussion 3.2. For any point ξ in \mathbb{T} and any θ , $0 < \theta < \pi$, we let $S(\xi, \theta)$ denote the interior of closed convex hull of $\{\xi\} \cup \{z : |z| \leq \sin(\frac{\theta}{2})\}$. We call $S(\xi, \theta)$ the *Stolz region* based at ξ with vertex angle θ . For our purposes here it is sufficient that we keep the vertex angles of our Stolz regions fixed at $\frac{\pi}{2}$, though our arguments carry through for any fixed θ in the aforementioned range. Let φ be an analytic self-map of \mathbb{D} and, for $\varepsilon > 0$, let $\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{1-|z|^2}{1-|\varphi(z)|^2} > \varepsilon\}$ and let $K = \mathbb{T} \cap \bar{\Omega}_\varepsilon$. Define W_ε by

$$W_\varepsilon = \bigcup_{\xi \in K} S(\xi, \frac{\pi}{2}).$$

Suppose that $\{z_n\}_{n=1}^\infty$ is a sequence in W_ε that converges to a point ξ_0 in K . So, we can find a sequence $\{\xi_n\}_{n=1}^\infty$ in K such that $z_n \in S(\xi_n, \frac{\pi}{2})$ (for $n = 1, 2, 3, \dots$) and $\lim_{n \rightarrow \infty} \xi_n = \xi_0$. Now

- $\zeta_n := \varphi(\xi_n) := \angle \lim_{z \rightarrow \xi_n} \varphi(z)$, and
- $\angle \lim_{z \rightarrow \xi_n} \varphi'(z) := \varphi'(\xi_n) = \zeta_n \bar{\xi}_n d_n$ - the angular derivative of φ at ξ_n , where $d_n := |\varphi'(\xi_n)|$.

By Proposition 3.1, $\varphi'(\xi_n) \rightarrow \varphi'(\xi_0) = \zeta_0 \bar{\xi}_0 d_0$, as $n \rightarrow \infty$, where $\zeta_0 := \varphi(\xi_0) := \angle \lim_{z \rightarrow \xi_0} \varphi(z)$ and $d_0 := |\varphi'(\xi_0)|$. Since $0 < d_0 < \infty$, we can find $M > 1$ such that $\frac{1}{M} \leq d_n \leq M$ for all n .

Lemma 3.3. *Assuming the terminology of Discussion 3.2, for any $\varepsilon > 0$, there exist s , $0 < s < 1$, and N (in \mathbb{N}) such that*

$$\left| d_n - \frac{1-|\varphi(z)|}{1-|z|} \right| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

Proof. If not, then we can find $d \neq d_0$, a subsequence $\{\xi_{n_k}\}_{k=1}^\infty$ of $\{\xi_n\}_{n=1}^\infty$ and a sequence $\{z'_k\}_{k=1}^\infty$ such that

- $z'_k \in S(\xi_{n_k}, \frac{\pi}{2})$ for all k ,
- $|z'_k - \xi_{n_k}| \rightarrow 0$ and hence $|z'_k - \xi_0| \rightarrow 0$ (as $k \rightarrow \infty$), and
- $(1 - |\varphi(z'_k)|)/(1 - |z'_k|) \rightarrow d$, as $k \rightarrow \infty$.

By Julia's Theorem this would then tell us that

$$d = |\varphi'(\xi_0)| = d_0;$$

a contradiction. □

Theorem 3.4. *Assuming the terminology of Discussion 3.2, φ' is continuous on \overline{W}_ε .*

Proof. Our proof here is based on Lemma 3.3 and some observations concerning the proof of the Julia-Carathéodory Theorem in [13]. By Proposition 3.1, all we need to show is that, given the hypothesis of Discussion 3.2, $\varphi'(z_n) \rightarrow \varphi'(\xi_0)$, as $n \rightarrow \infty$.

Claim A. *For any $\varepsilon > 0$ there exist $s, 0 < s < 1$, and N (in \mathbb{N}) such that*

$$\left| \zeta_n \bar{\xi}_n d_n - \frac{\zeta_n - \varphi(z)}{\xi_n - z} \right| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

To justify this claim we first observe that, by Lemma 3.3, for any $\eta > 0$, there exist $\sigma, 0 < \sigma < 1$, and ν (in \mathbb{N}) such that

$$\left| d_n - \frac{1 - |\varphi(r\xi_n)|}{1 - r} \right| < \eta \quad \text{and} \quad \left| d_n - \frac{1 - |\varphi(r\xi_n)|^2}{1 - r^2} \right| < \eta \quad (3.4.1)$$

provided $\sigma \leq r < 1$ and $n \geq \nu$. Mimicking the proof of JC (1) \implies JC (2) (in Sect. 4.5 of [13]), for $n \geq \nu$ we carry the discussion to the right half-plane $\{w : \operatorname{Re}(w) > 0\}$. Let φ_n and ψ_n be the Möbius transformations given by $\varphi_n(z) := \frac{\xi_n + z}{\xi_n - z}$ and $\psi_n(z) := \frac{\zeta_n + z}{\zeta_n - z}$. Define Φ_n and γ_n on $\{w : \operatorname{Re}(w) > 0\}$ by $\Phi_n(w) := (\psi_n \circ \varphi \circ \varphi_n^{-1})(w)$ and $\gamma_n(w) := \Phi_n(w) - c_n w$, where $c_n := \frac{1}{d_n}$. Now by (3.4.1), if $n \geq \nu$ and $\sigma \leq r < 1$, then

$$d_n - \eta < \frac{1 - |\varphi(r\xi_n)|}{1 - r}, \frac{1 - |\varphi(r\xi_n)|^2}{1 - r^2} < d_n + \eta$$

and hence, by Julia's Theorem and with $w_{n,r} := \varphi_n(r\xi_n) (= \frac{1+r}{1-r})$,

$$\frac{1}{d_n} \leq \frac{\operatorname{Re}(\Phi_n(w_{n,r}))}{\operatorname{Re}(w_{n,r})} = \left(\frac{1 - |\varphi(r\xi_n)|^2}{1 - r^2} \right) \frac{(1 - r)^2}{|\zeta_n - \varphi(r\xi_n)|^2} < \frac{d_n + \eta}{(d_n - \eta)^2}.$$

Therefore, if n is sufficiently large (allowing η to be sufficiently small), one can force

$$\frac{\operatorname{Re}(\gamma_n(w_{n,\sigma}))}{\operatorname{Re}(w_{n,\sigma})}$$

to be less than any prescribed positive real number; and $w_{n,\sigma} = \frac{1+\sigma}{1-\sigma}$, which clearly does not vary with n . We let $w_{n,\sigma}$ play the role of w_0 in the proof of JC (1) \implies JC (2) (in Sect. 4.5 of [13]). And since the image under φ of any compact subset of \mathbb{D} is a compact subset of \mathbb{D} , $\{|\gamma_n(w_{n,\sigma})|\}_{n=1}^\infty$ is bounded. Thus, following through with the argument in [13], we find that, for any

$\tau > 0$, there is a positive real number R such that if $w \in \varphi_n(S(\xi_n, \frac{\pi}{2}))$ and $|w| > R$, then

$$\left| \frac{\gamma_n(w)}{w} \right| < \tau, \quad (3.4.2)$$

provided n is sufficiently large. Now, via the correspondence $w = \varphi_n(z)$, routine calculations give that

$$\frac{w+1}{\Phi_n(w)+1} = \xi_n \bar{\zeta}_n \left(\frac{\zeta_n - \varphi(z)}{\xi_n - z} \right),$$

and hence,

$$\left| \frac{\gamma_n(w)+1}{w+1} \right| = \left| \frac{\xi_n - z}{\zeta_n - \varphi(z)} - \frac{\xi_n \bar{\zeta}_n c_n w}{w+1} \right|.$$

We now find that Claim (A) follows from (3.4.2).

Claim B. For any $\varepsilon > 0$ there exist $s, 0 < s < 1$, and N (in \mathbb{N}) such that

$$|\zeta_n \bar{\xi}_n d_n - \varphi'(z)| < \varepsilon,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$.

Now Claim (B) follows directly from Claim (A) and the proof of JC (2) \implies JC (3) (in Sect. 4.6 of [13]). And by Claim (B) and the fact that $\varphi'(\xi_n) \longrightarrow \varphi'(\xi_0)$, as $n \rightarrow \infty$, we find that

$$\varphi'(z_n) \longrightarrow \varphi'(\xi_0),$$

as $n \rightarrow \infty$; which completes our proof. \square

Theorem 3.5. Let φ be an analytic self-map of \mathbb{D} . If C_φ is closed-range on \mathbb{A}^2 , then there exist ε and $s, 0 < \varepsilon, s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq F_\varepsilon$.

Proof. Suppose that C_φ is closed-range on \mathbb{A}^2 . Then there exists $\varepsilon > 0$ such that $G_\varepsilon := \varphi(\Omega_\varepsilon)$ satisfies the reverse Carleson condition; cf., [1]. In particular, $\mathbb{T} \subseteq \overline{G_\varepsilon}$. So, for each point v_0 in \mathbb{T} , we can find a sequence $\{w_n\}_{n=1}^\infty$ in Ω_ε such that $\{\varphi(w_n)\}_{n=1}^\infty$ converges to v_0 . Passing to a subsequence if necessary, we may assume that $\{w_n\}_{n=1}^\infty$ converges to some point ω_0 in $K := \mathbb{T} \cap \overline{\Omega_\varepsilon}$. Therefore, by Julia's Theorem, $v_0 = \varphi(\omega_0) := \angle \lim_{w \rightarrow \omega_0} \varphi(w)$. Thus, $\varphi(K) = \mathbb{T}$. We proceed indirectly and suppose that the conclusion of this theorem fails. Then we can find a sequence $\{z_n\}_{n=1}^\infty$ in $\mathbb{D} \setminus \{0\}$, such that $\{|z_n|\}_{n=1}^\infty$ converges to 1 and

$$\sup\{|\tau_\varphi(w)| : \varphi(w) = z_n\} \longrightarrow 0, \quad (3.5.1)$$

as $n \rightarrow \infty$. Since $\varphi(K) = \mathbb{T}$, there exists $\{\xi_n\}_{n=1}^\infty$ in K such that $\varphi(\xi_n) = \zeta_n := \frac{z_n}{|z_n|}$, for $n = 1, 2, 3, \dots$. Passing to a subsequence if need be, we may assume that $\{\xi_n\}_{n=1}^\infty$ converges to some point ξ_0 in K . Since, by Proposition 3.1, φ is continuous on K , indeed, continuous on $\overline{\Omega_\varepsilon}$, we find that $\{\zeta_n\}_{n=1}^\infty$ converges to $\zeta_0 := \varphi(\xi_0)$. Now, by Theorem 3.4 and its proof, there exist δ and $s, 0 < \delta, s < 1$, and N in \mathbb{N} such that

$$|\tau_\varphi(z)| \geq \delta,$$

whenever $z \in S(\xi_n, \frac{\pi}{2})$, $|z| > s$ and $n \geq N$. Moreover, by Claim (A) in the proof of Theorem 3.4 (that speaks to the conformality of φ at ξ_n), we can find σ , $0 < \sigma < 1$, and ν in \mathbb{N} such that

$$\{r\zeta_n : \sigma \leq r < 1\} \subseteq \varphi(\{z \in S(\xi_n, \frac{\pi}{2}) : |z| > s\}),$$

whenever $n \geq \nu$. Since $z_n \in \{r\zeta_n : \sigma \leq r < 1\}$, if n is sufficiently large, we find that (3.5.1) above cannot occur; and our proof is complete. \square

Our next result is an immediate consequence of Theorem 3.5 and Theorem 2.2; and so we state it without proof.

Corollary 3.6. *Let φ be an analytic self-map of \mathbb{D} . If C_φ is closed-range on \mathbb{A}^2 , then it is also closed-range on \mathcal{B} .*

A slight modification of the proof of Theorem 3.5 gives us the following rather surprising result. It also can be viewed as a byproduct of the nice behavior of φ on \overline{W}_ε , as indicated by Theorem 3.4.

Theorem 3.7. *Let φ be an analytic self-map of \mathbb{D} . Then the following are equivalent.*

- i) C_φ is closed-range on \mathbb{A}^2 .
- ii) There exist ε , s and c , $0 < \varepsilon, s, c < 1$, such that

$$A(G_\varepsilon \cap D(z, s)) \geq cA(D(z, s)),$$

for all z in \mathbb{D} .

- iii) There exist ε and s , $0 < \varepsilon, s < 1$, such that $\{z : s \leq |z| < 1\} \subseteq G_\varepsilon$.

Proof. The equivalence between (i) and (ii) was established in [1]. And clearly (iii) implies (ii). So we need only establish that (i) implies (iii). To this end, assume that C_φ is closed-range on \mathbb{A}^2 and mimic the proof of Theorem 3.5, replacing $|\tau_\varphi(z)|$ by $\frac{1-|z|^2}{1-|\varphi(z)|^2}$, throughout. The argument carries over with this modification to give us (iii). \square

By Theorem 2.5 of [1], the only univalent analytic self-maps of \mathbb{D} that give rise to closed-range composition operators on \mathbb{A}^2 are the analytic automorphisms of \mathbb{D} . This is in contrast with the Bloch space setting. Indeed, if ψ is any conformal mapping from \mathbb{D} one-to-one and onto $\mathbb{D} \setminus [0, 1)$, then C_ψ is closed-range on \mathcal{B} ; cf., Example 2 of [9]. So, the converse of Corollary 3.6 fails. Our next two examples show that the converse fails with a vengeance. Our first is an example of a *thin* Blaschke product B that fixes zero and has no angular derivative at any point of the unit circle \mathbb{T} ; and by thin we mean that $(1 - |a_n|^2)|B'(a_n)| \rightarrow 1$, as $n \rightarrow \infty$, where $\{a_n\}_{n=1}^\infty$ are the zeros of B . Therefore, C_B is norm preserving on \mathcal{B} (cf., [5], or [11]) and yet is compact on \mathbb{A}^2 (cf., [13, pages 52 and 195]). And since C_B is compact and not of finite rank on \mathbb{A}^2 , it is not closed-range on \mathbb{A}^2 . This first example is a factor of the one produced by J. Shapiro on page 185 of [13].

Example 3.8. Let B^* be the Blaschke product constructed by J. Shapiro on page 185 of [13] and let $\{a_n\}_{n=1}^\infty$ be the zeros of B^* . Associated with each a_n is an arc I_n of length $\frac{1}{n}$ of the form $I_n = \{e^{i\theta} : \theta_n \leq \theta \leq \theta_{n+1}\}$. The zeros a_n

are given by: $a_n := r_n e^{i\omega_n}$, where $r_n := 1 - \frac{1}{n^2}$ and $\omega_n := \frac{1}{2}(\theta_n + \theta_{n+1})$. A theorem of O. Frostman (cf., [13, page 183]) is then used to show that B^* has no angular derivative anywhere on \mathbb{T} . For each positive integer ν , we define the ν^{th} “layer” of zeros of B^* as $[a_\nu] := \{a_\nu, a_{\nu+1}, \dots, a_{N_\nu}\}$, where N_ν is the unique positive integer that satisfies:

$$\mathbb{T} \subseteq \bigcup_{n=\nu}^{N_\nu} I_n, \text{ yet } \mathbb{T} \not\subseteq \bigcup_{n=\nu}^{N_\nu-1} I_n.$$

Since $\sum_{n=\nu}^{N_\nu-1} \frac{1}{n} < 2\pi$, it follows that $N_\nu < 540\nu$. For any positive integer ν , let B_ν be the Blaschke product with (simple) zeros $[a_\nu]$. For any a_k in $[a_\nu]$, let B_ν^k denote B_ν with the Blaschke factor involving a_k deleted. And choose a_{k^*} in $[a_\nu] \setminus \{a_k\}$ such that $\rho(a_k, a_{k^*}) \leq \rho(a_k, a_l)$, whenever $a_l \in [a_\nu] \setminus \{a_k\}$. Then, for such l ,

$$\left| \frac{a_k - a_l}{1 - \bar{a}_l a_k} \right|^2 - 1 \geq - \frac{(1 - r_k^2)(1 - r_{k^*}^2)}{1 - 2r_k r_{k^*} \cos(\theta_k - \theta_{k^*}) + r_k^2 r_{k^*}^2}.$$

Now, $|\theta_k - \theta_{k^*}| \geq \frac{1}{4k}$ and so, for ν sufficiently large,

$$1 - 2r_k r_{k^*} \cos(\theta_k - \theta_{k^*}) + r_k^2 r_{k^*}^2 \geq \frac{1}{20k^2}.$$

Hence,

$$\left| \frac{a_k - a_l}{1 - \bar{a}_l a_k} \right|^2 - 1 \geq - \frac{80}{(k^*)^2} \geq - \frac{80}{\nu^2},$$

independent of k and l in our range here. Therefore,

$$\begin{aligned} 0 &> \sum_{k \neq l = \nu}^{N_\nu} \left(\left| \frac{a_k - a_l}{1 - \bar{a}_l a_k} \right|^2 - 1 \right) \\ &\geq (540\nu) \left(- \frac{80}{\nu^2} \right) = - \frac{43,200}{\nu} \longrightarrow 0, \end{aligned}$$

as $\nu \rightarrow \infty$; uniformly in k , $\nu \leq k \leq N_\nu$. From this it follows that

$$|B_\nu^k(a_k)| \longrightarrow 1, \tag{3.7.1}$$

as $\nu \rightarrow \infty$; uniformly in k , $\nu \leq k \leq N_\nu$. Now since B^* is a Blaschke product,

$$|B_\nu| \longrightarrow 1 \tag{3.7.2}$$

uniformly on compact subsets of \mathbb{D} , as $\nu \rightarrow \infty$. And since, for any fixed ν , B_ν is a finite Blaschke product,

$$|B_\nu(z)| \longrightarrow 1 \tag{3.7.3}$$

uniformly in z , as $|z| \rightarrow 1^-$. Using (3.7.1)–(3.7.3), one can find a (rapidly) increasing sequence $\{\nu_j\}_{j=1}^\infty$ of positive integers such that $[a_{\nu_k}] \cap [a_{\nu_l}] = \emptyset$ if $k \neq l$, and such that

$$B := \prod_{j=1}^{\infty} B_{\nu_j},$$

whose (simple) zeros we enumerate as $\{\alpha_n\}_{n=1}^\infty$, satisfies

$$|B^{\hat{n}}(\alpha_n)| \longrightarrow 1,$$

as $n \rightarrow \infty$; where $B^{\hat{n}}$ denotes B with the Blaschke factor involving α_n deleted. And we may assume that $\nu_1 = 1$. Hence, B is a thin Blaschke product that fixes zero. Since the zeros of B consist of infinitely many disjoint layers of the zeros of B^* , one can argue as in [13, page 185], and find that

$$\sum_{n=1}^{\infty} \frac{1 - |\alpha_n|}{|\zeta - \alpha_n|^2} = \infty,$$

for each ζ in \mathbb{T} . Thus, by a theorem of O. Frostman (cf., [13, page 183]), we conclude that B has no angular derivative at any point in \mathbb{T} .

Remark 3.9. The converse of Theorem 3.5 does not hold. Indeed, by Theorem 2.7 of [4], if B is the Blaschke product that we produced in Example 3.8, then

$$\mathbb{D} \subseteq F_{\frac{1}{2}};$$

and yet C_B is far from closed-range on \mathbb{A}^2 .

We now produce a univalent analytic self-map h of \mathbb{D} that has no angular derivative at any point of \mathbb{T} (whence, C_h is compact on \mathbb{A}^2) such that C_h is closed-range on \mathcal{B} . This dramatically improves upon our understanding of what is possible in the univalent case; cf., Example 2 of [9]. And since $h(\mathbb{D})$ contains no annulus with outer boundary equal to \mathbb{T} (and similarly for Example 2 of [9]), there is no analogue of Theorem 3.7 in the context of the Bloch space.

Example 3.10. Here we construct a conformal mapping h from \mathbb{D} one-to-one and onto an *infinite ribbon* G that spirals out to \mathbb{T} such that C_h is closed-range on \mathcal{B} . So h will have no unimodular nontangential boundary values on \mathbb{T} , and thus no angular derivative anywhere on \mathbb{T} . We write h as the composition of three conformal mappings:

- $\zeta = i \left(\frac{1+z}{1-z} + e \right)$, which maps \mathbb{D} univalently onto $G_1 := \{\zeta : \text{Im}(\zeta) > e\}$,
- $\xi = \log(\zeta)$, which maps G_1 univalently onto a smoothly bounded subregion G_2 of the swath $\{\xi : \text{Re}(\xi) > 1 \text{ and } 0 < \text{Im}(\xi) < \pi\}$ that asymptotically approximates this swath, and
- $w = \xi^i$, which maps G_2 univalently onto an infinite ribbon G that spirals out to \mathbb{T} .

Thus, $h(z) = \left[\log \left(i \left(\frac{1+z}{1-z} + e \right) \right) \right]^i$. Clearly h has no unimodular nontangential boundary values on \mathbb{T} and thus has no angular derivative anywhere on \mathbb{T} . As we noted just prior to Example 3.8, this tells us that C_h is compact and hence not closed-range on \mathbb{A}^2 . One may also refer to Theorem 2.5 of [1] to obtain that C_h is not closed-range on \mathbb{A}^2 . Now let $\Gamma = h([0, 1))$, which is an arc of infinite length that spirals out to \mathbb{T} . Our strategy in showing that C_h is closed-range on \mathcal{B} is to first establish that there exists $\varepsilon > 0$ such that $\Gamma \subseteq F_\varepsilon$ and then establish that there exists s , $0 < s < 1$, such that

$\Gamma \cap D(z, s) \neq \emptyset$ for all z in \mathbb{D} . Theorem 2.2 then gives us the conclusion. In what follows we use the symbol \sim between real-valued functions f and g defined on $[0, 1)$ (viz., $f \sim g$) to indicate that there is a constant $M > 1$ such that $\frac{1}{M}f(x) \leq g(x) \leq Mf(x)$ for all x in $[0, 1)$. Now, for x in $[0, 1)$,

$$\begin{aligned} h(x) &= \left[\log \left(i \left(\frac{1+x}{1-x} + e \right) \right) \right]^i \\ &= \left[\log \left(\frac{1+x}{1-x} + e \right) + \frac{i\pi}{2} \right]^i. \end{aligned}$$

Denoting $\log \left(\frac{1+x}{1-x} + e \right) + \frac{i\pi}{2}$ by ξ_x , we have:

$$h(x) = e^{i \log(\xi_x)} = e^{-\arg(\xi_x)} \cdot e^{i \log|\xi_x|}.$$

Hence,

$$1 - |h(x)| \sim \arg(\xi_x) \sim \frac{1}{\log \left(\frac{1+x}{1-x} + e \right)}.$$

Thus, for x in $[0, 1)$,

$$\frac{1-x}{1-|h(x)|} \sim (1-x) \log \left(\frac{1+x}{1-x} + e \right).$$

And, for such x , $h'(x) = \frac{e^{i \log(\xi_x)}}{\log \left(\frac{1+x}{1-x} + e \right) + \frac{i\pi}{2}} \cdot \frac{2}{(1-x^2) + e(1-x)^2}$; whence

$$|h'(x)| \sim \frac{1}{(1-x) \log \left(\frac{1+x}{1-x} + e \right)}.$$

Evidently, $|\tau_h(x)| \sim 1$, and so there exists $\varepsilon > 0$ such that $\Gamma \subseteq F_\varepsilon$. Now, as x increases to 1 in $[0, 1)$, $h(x)$ traverses Γ through infinitely many counterclockwise rotations about 0 as it works its way toward \mathbb{T} . To complete our argument here it is important that we obtain a good estimate on the ratio between $1 - |h(x')|$ and $1 - |h(x)|$, if $[x, x']$ is a subinterval of $[0, 1)$ over which h makes precisely one rotation about 0. Recalling that $h(x) = e^{-\arg(\xi_x)} \cdot e^{i \log|\xi_x|}$, we find that this reduces to an examination of

$$h^*(y) := e^{-\frac{1}{y}} \cdot e^{i \log(y)},$$

as y in $[1, \infty)$ increases to ∞ . Notice that h^* winds through 2π radians on any subinterval of $[1, \infty)$ of the form $[y, e^{2\pi}y]$. And, independent of y , $\frac{1-|h^*(e^{2\pi}y)|}{1-|h^*(y)|}$ is boundedly equivalent to $\frac{1}{e^{2\pi}}$. This then tells us that $\mathbb{D} \setminus \Gamma$ does not contain pseudohyperbolic disks of radius arbitrarily near 1. Hence, there exists s , $0 < s < 1$, such that $\Gamma \cap D(z, s) \neq \emptyset$, for all z in \mathbb{D} . Since, as we have shown, $\Gamma \subset F_\varepsilon$, for some $\varepsilon > 0$, we can now refer to Theorem 2.2 and conclude that C_h is closed-range on \mathcal{B} .

Closing Remarks

In this final section we give a result in the context of \mathbb{A}^2 for singular inner functions and we point out some implications of our work here to the theory of Fredholm operators. In our discussion we let m denote normalized Lebesgue measure on \mathbb{T} . Recall that a compact subset E of \mathbb{T} is said to be *porous* if there exists ε , $0 < \varepsilon < 1$, such that whenever I is a arc of \mathbb{T} with $I \cap E \neq \emptyset$, then there is a subarc J of I where $m(J) > \varepsilon m(I)$ and $J \cap E = \emptyset$. In [12] it is shown that E is a porous subset of \mathbb{T} if and only if E has the property: For any singular measure μ supported on E , every nontrivial Frostman shift of the singular inner function S_μ is a Carleson–Newman Blaschke product; that is, a finite product of interpolating Blaschke products. The proof of Corollary 3.11 in [1] also establishes our next result.

Proposition 4.1. *Let E be a porous subset of \mathbb{T} . If μ is any singular measure with support in E , then C_{S_μ} is closed-range on \mathbb{A}^2 .*

Remark 4.2. We close the paper with some thoughts concerning Fredholm operators. We first recall that the little Bloch space \mathcal{B}_0 is the collection of functions f in \mathcal{B} for which

$$\lim_{r \rightarrow 1} \sup_{r < |z| < 1} (1 - |z|^2) |f'(z)| = 0.$$

And the Dirichlet space \mathcal{D} is the collection of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in \mathbb{D} , such that

$$\|f\|_{\mathcal{D}}^2 := \sum_{n=0}^{\infty} (n+1) |a_n|^2 < \infty.$$

An operator between two Banach spaces is called a *Fredholm operator* if its range is closed and both the operator and its adjoint have finite dimensional kernel. If φ is an analytic self-map of \mathbb{D} and C_φ is a Fredholm operator on a Hilbert space of analytic functions that contains \mathcal{D} , then φ is a disk automorphism; cf., [6, page 153]. Now $\mathcal{D} \subseteq \mathcal{B}_0$, but we will show that the situation is different for \mathcal{B}_0 . Indeed, there exists Fredholm composition operators on \mathcal{B}_0 whose symbols are not disk automorphisms. The *minimal Besov space* B_1 is the collection of all functions f that are analytic in \mathbb{D} of the form

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n \varphi_{w_n}(z), \tag{4.2.1}$$

where $\{w_n\}_{n=1}^{\infty} \subseteq \overline{\mathbb{D}}$, and $\{a_n\}_{n=1}^{\infty} \in l^1$. The norm on B_1 is given by

$$\|f\|_{B_1} := \inf \left\{ \sum_{n=0}^{\infty} |a_n| : (4.2.1) \text{ holds} \right\}.$$

Now B_1 is a Banach space with respect to this norm and is invariant under disk automorphisms. Under the pairing $(f, g) = \int_{\mathbb{D}} f'(z) \overline{g'(z)} dA(z)$, the dual

of \mathcal{B}_0 is B_1 and the dual of B_1 is \mathcal{B} ; cf., [2]. Notice that, for g in \mathcal{B}_0 and w in \mathbb{D} ,

$$(g, \varphi_w) = - \int_{\mathbb{D}} g'(z) \frac{1 - |w|^2}{(1 - w\bar{z})^2} dA(z) = -(1 - |w|^2)g'(w),$$

and therefore,

$$(g, C_\varphi^*(\varphi_w)) = \langle C_\varphi(g), \varphi_w \rangle = -(1 - |w|^2) (g \circ \varphi)'(w) = -\tau_\varphi(w)(g, \varphi_{\varphi(w)}).$$

If $w \in \mathbb{D}$, then

$$C_\varphi^*(\varphi_w) = -\tau_\varphi(w)\varphi_{\varphi(w)}, \tag{4.2.2}$$

and if $|w| = 1$, then $\varphi_w = w$ and

$$C_\varphi^* \varphi_w = 0. \tag{4.2.3}$$

By (4.2.2) and (4.2.3) it is easy to see that the kernel of $C_\varphi^* : B_1 \rightarrow B_1$ consists of the constant functions. Also, a non-constant composition operator is always one-to-one, and therefore $C_\varphi : \mathcal{B}_0 \rightarrow \mathcal{B}_0$ will be a Fredholm operator if it is closed-range. It is shown in [9] that if ψ is a conformal mapping from \mathbb{D} onto $\mathbb{D} \setminus [0, 1)$, then C_ψ is bounded below on \mathcal{B} . Any univalent self-map of \mathbb{D} is in \mathcal{B}_0 , and thus $\psi \in \mathcal{B}_0$ and C_ψ is a Fredholm operator on \mathcal{B}_0 .

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