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Criteria for components of a function space to be homotopy equivalent

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Abstract

We give a general method that may be effectively applied to the question of whether two components of a function space $\text{map}(X, Y)$ have the same homotopy type. We describe certain group-like actions on $\text{map}(X, Y)$. Our basic results assert that if maps $f, g: X \rightarrow Y$ are in the same orbit under such an action, then the components of $\text{map}(X, Y)$ that contain f and g have the same homotopy type.

Introduction

Let X and Y be connected, countable CW complexes and let $\text{map}(X, Y)$ denote the space of all continuous (not necessarily based) maps between X and Y with the compact-open topology. The space $\text{map}(X, Y)$ is generally disconnected with path components in one-to-one correspondence with the set $\langle X, Y \rangle$ of (free) homotopy classes of maps. Furthermore, different components may – and frequently do – have distinct homotopy types. A basic problem in homotopy theory is to determine whether two components are homotopy equivalent or, more generally, to classify the path components of $\text{map}(X, Y)$ up to homotopy equivalence. For $x_0 \in X$ a choice of basepoint, we have the evaluation map $\omega: \text{map}(X, Y) \rightarrow Y$, defined by $\omega(g) = g(x_0)$, which is a fibration. Let $\text{map}(X, Y; f)$ denote the path component of $\text{map}(X, Y)$ that contains a given map $f: X \rightarrow Y$. We may also ask for a finer classification, up to fibre-homotopy equivalence, of the evaluation fibrations $\omega_f: \text{map}(X, Y; f) \rightarrow Y$, obtained by restricting ω to the component of f .

Work on these classification problems dates back to the 1940s. Whitehead considered the case $X = S^n$ and $Y = S^m$, in which a component corresponds to $\alpha \in \pi_n(S^m)$, and proved that $\text{map}(S^n, S^m; \alpha)$ is homotopy equivalent to $\text{map}(S^n, S^m; 0)$ if and only if the evaluation fibration ω_α admits a section [28, Theorem 2.8]. Hansen, and later McLendon, extended this analysis ([10, 11, 18]). In [12], Hansen obtained a classification of components of $\text{map}(M^n, S^n)$, where M^n is a suitably restricted n -manifold. Sutherland extended this result in [24]. Møller [20] gave a classification of components of $\text{map}(\mathbb{C}P^m, \mathbb{C}P^n)$ for $1 \leq m \leq n$. The case in which X is a manifold and $Y = BG$, the classifying space of a Lie group, has been the subject of extensive recent research by Crabb, Kono, Sutherland, Tsukuda and others (see e.g. [2, 15, 16, 17, 25, 26]). Our purpose in this paper is to give a general method

that may be applied to show that (evaluation fibrations of) components of $\text{map}(X, Y)$ are (fibre-) homotopy equivalent. In addition to yielding many new results, our method allows some of the particular cases just mentioned to be viewed as special cases within a general framework.

Our basic results are presented in Section 2. We consider the orbit of a point in $\text{map}(X, Y)$ under a group-like action on $\text{map}(X, Y)$ and observe in Theorem 2.2 that two distinct components of $\text{map}(X, Y)$ are homotopy equivalent whenever each overlaps with any one orbit – not in the same point, obviously. Now, in the situations that we have in mind, the action on $\text{map}(X, Y)$ arises from a group-like action on Y . In this case, we have a corresponding group action on the set of homotopy classes of maps $\langle X, Y \rangle$. Write \mathcal{O} for the orbit set of this group action. Then we obtain a surjection

$$\mathcal{O} \twoheadrightarrow \frac{\{\text{components of } \text{map}(X, Y)\}}{\simeq} \quad (1)$$

of sets, where \simeq denotes homotopy equivalence (Theorem 2.4). This may be applied “locally,” to analyze whether two particular components are homotopy equivalent. It may also be applied “globally,” to deduce a finite – or even a concrete upper bound on the – number of distinct homotopy types amongst the (usually infinitely many) components of $\text{map}(X, Y)$. We illustrate both approaches in Section 3. For based spaces X and Y , we may also consider $[X, Y]$, the set of based-homotopy equivalence classes of based maps. Ignoring basepoints gives a surjection $[X, Y] \twoheadrightarrow \langle X, Y \rangle$ of sets of homotopy classes. Once more, in the situations that we have in mind, the group action on $\langle X, Y \rangle$ that we referred to above actually restricts to one on $[X, Y]$. Writing \mathcal{O}_* for the corresponding orbit set, we may compose the surjection (1) with the surjection $\mathcal{O}_* \twoheadrightarrow \mathcal{O}$ of orbit sets. Although \mathcal{O}_* is *a priori* larger than \mathcal{O} , it is more familiar in homotopy theory and in many cases may be analyzed effectively. With further restrictions on X and Y , we may sharpen these results, replacing the right-hand set in (1) by fibre-homotopy equivalence classes of evaluation fibrations $\omega_f: \text{map}(X, Y; f) \rightarrow Y$. Also, we may readily adapt the methods used here to study homotopy types of components of $\text{map}_*(X, Y)$, the function space of basepoint-preserving maps – see the comment at the end of Section 2 and the discussion that ends the paper.

In Section 3 we focus our general method on actions on $\text{map}(X, Y)$ that arise from certain specific actions on Y . We first consider the holonomy action of ΩB on the fibre Y of a fibration $Y \rightarrow E \rightarrow B$. In Theorem 3.2, we show that if two based maps $f, g: X \rightarrow Y$ satisfy $j \circ f \sim_* j \circ g: X \rightarrow E$, where $j: Y \rightarrow E$ denotes the fibre inclusion, then the components $\text{map}(X, Y; f)$ and $\text{map}(X, Y; g)$ have the same homotopy type. With some restrictions on X and Y , we are able to conclude more strongly that the evaluation fibrations $\omega_f: \text{map}(X, Y; f) \rightarrow Y$ and $\omega_g: \text{map}(X, Y; g) \rightarrow Y$ are fibre-homotopy equivalent. We illustrate these ideas in Example 3.3 and Example 3.5, which give simple cohomological conditions under which two components of $\text{map}(X, G/H)$ are homotopy equivalent, or there are finitely many homotopy types amongst the components of $\text{map}(X, G/H)$, where H is a closed subgroup of a Lie group G . Next we focus on the universal fibration with fibre Y and obtain a link between the classification problem for components of a function space and the class of *cyclic maps* (see [27]). In this context, we extend the result of Whitehead mentioned above to prove that the evaluation fibrations $\omega_f: \text{map}(X, Y; f) \rightarrow Y$ and $\omega_0: \text{map}(X, Y; 0) \rightarrow Y$ are fibre-homotopy equivalent if and only if ω_f admits a section (Theorem 3.7). We obtain further results in the case in which X is a co-H-space, including a connection between computations of the Gottlieb groups of spheres and Hansen’s results on

the classification of the components of $\text{map}(S^n, S^m)$ (cf. Example 3.11). We end the paper with a brief discussion of comparable results about components of the based mapping space $\text{map}_*(X, Y)$, but with the action arising from cogroup-like actions on X .

Group-Like Actions on a Function Space

We begin by setting conventions and notation. First, we make clear that *homotopy* (homotopic maps, homotopy equivalence, etc.) generally refers to *free homotopy*: we use “ \sim ” and “ \simeq ” to denote (free) homotopy and (free) homotopy equivalence, respectively. If *based homotopy* is intended, we will be specific and use “ \sim_* ” and “ \simeq_* ” in that case.

A *fibration* $p: E \rightarrow B$ means a Hurewicz fibration [29, p.29]. Recall that, for $p_1: E_1 \rightarrow B$ and $p_2: E_2 \rightarrow B$ fibrations over a space B , a based map $f: E_1 \rightarrow E_2$ is a *fibre homotopy equivalence* if there exists $g: E_2 \rightarrow E_1$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identities by based homotopies F and G satisfying $p_1 \circ F(x, t) = p_1(x)$ and $p_2 \circ G(y, t) = p_2(y)$ for $x \in E_1, y \in E_2$ and $t \in I$.

An *H-space* is a based space G together with a based multiplication $m: G \times G \rightarrow G$ that satisfies $m \circ J \sim_* \nabla: G \vee G \rightarrow G$ where $J: G \vee G \rightarrow G \times G$ is the inclusion and $\nabla: G \vee G \rightarrow G$ is the folding map. We note that the homotopy can be replaced by strict equality provided the basepoint of G is non-degenerate [29, Theorem III-4.7]. The H-space is *homotopy-associative* if $m \circ (m \times 1) \sim_* m \circ (1 \times m): G \times G \times G \rightarrow G$. By a *group-like space*, we mean a homotopy-associative H-space G together with a based inverse map $\iota: G \rightarrow G$ that satisfies $m \circ (\iota \times 1) \circ \Delta \sim_* 0$ and $m \circ (1 \times \iota) \circ \Delta \sim_* 0$, where $\Delta: G \rightarrow G \times G$ is the diagonal map.

By a *homotopy-associative action* of a homotopy-associative H-space G on a based space Y , we mean a based map $A: G \times Y \rightarrow Y$ that satisfies $A \circ i_2 \sim_* 1: Y \rightarrow Y$ and $A \circ (1 \times A) \sim_* A \circ (m \times 1): G \times G \times Y \rightarrow Y$, where $i_2: G \rightarrow Y \times G$ is the inclusion. We say the action is *strictly unital* if we have $A \circ i_2 = 1$. The argument in [29, Theorem III-4.7] mentioned above easily extends to show an action may be taken to be strictly unital when the basepoint of G is non-degenerate. Given $g \in G$ and $x \in Y$, we will usually write $g \cdot x$ for $A(g, x)$.

For the rest of the paper, we assume (at least) that X and Y are based, connected, countable CW complexes with fixed choices of non-degenerate basepoints. While these hypotheses are not strictly necessary for all that we do, they seem to provide a reasonable level of generality. Despite these restrictions on X and Y – indeed, despite further restrictions (e.g. X is frequently assumed to be a finite complex) – we must allow for much greater generality when considering the function space $\text{map}(X, Y)$. Lemma 2.1 and Lemma 2.3 below deal with technical points that become issues when we consider the function space.

LEMMA 2.1. *Suppose U and V are path-connected spaces with non-degenerate basepoints. Then we have:*

- (1) *Given $f: U \rightarrow V$, there exists a based map $f': U \rightarrow V$ with $f \sim f'$;*
- (2) *If U and V are homotopy equivalent, then they are based homotopy equivalent with respect to the non-degenerate basepoints.*

Proof. Part (1) is [29, III-1.4]. For (2), suppose that $f: U \rightarrow V$ is a (free) homotopy equivalence. Let $u_0 \in U$ and $v_0 \in V$ be non-degenerate basepoints. Since u_0 is non-degenerate, f is homotopic to a based map $f': U \rightarrow V$ by (1). Since f is a homotopy equivalence, so too is f' . But since $f'(u_0) = v_0$, and both u_0 and v_0 are non-degenerate, it follows that f' is a based homotopy equivalence (see, e.g., [13, Proposition 6.18]).

It is well-known that all components of a group-like space have the same homotopy type (see [13, Proposition 5.28]). We generalize this fact in the following result:

THEOREM 2.2. *Let $A: G \times Y \rightarrow Y$ be a homotopy-associative action of a group-like space G on a space Y . For each $x \in Y$, let $Y_x \subseteq Y$ denote the path component of Y that contains x . Then for each $g \in G$, the components Y_x and $Y_{g \cdot x}$ have the same homotopy type. If Y_x and $Y_{g \cdot x}$ both have non-degenerate basepoints, then Y_x and $Y_{g \cdot x}$ have the same based homotopy type.*

Proof. Let $m: G \times G \rightarrow G$ be the multiplication and $\iota: G \rightarrow G$ the inverse map. Let $e \in G$ denote the basepoint. For each $g \in G$, we may define “translation by g ” to be the map $\tau_g: Y \rightarrow Y$, where $\tau_g(x) = g \cdot x$ for each $x \in Y$. Then τ_g restricts to a map $\tau_g: Y_x \rightarrow Y_{g \cdot x}$. On the other hand, we have the translation $\tau_{\iota(g)}: Y \rightarrow Y$. Let $i_g: Y \rightarrow G \times Y$ be the inclusion defined by $i_g(x) = (g, x)$ for each $x \in Y$. Then we have

$$\begin{aligned} \tau_{\iota(g)} \circ \tau_g &= A \circ (1 \times A) \circ ((\iota, 1) \times 1) \circ i_g \sim A \circ (m \times 1) \circ ((\iota, 1) \times 1) \circ i_g \\ &\sim A \circ (0 \times 1) \circ i_g \sim \tau_e \sim 1_Y. \end{aligned}$$

Let $H: Y \times I \rightarrow Y$ be a homotopy from $\tau_{\iota(g)} \circ \tau_g$ to 1. $H(x, t)$ gives a path from $\iota(g) \cdot (g \cdot x)$ to x and it follows that $\tau_{\iota(g)}$ restricts to a map $\tau_{\iota(g)}: Y_{g \cdot x} \rightarrow Y_x$. Furthermore, the homotopy H restricts to a homotopy $H: Y_x \times I \rightarrow Y_x$ between the composition of the restrictions $\tau_{\iota(g)} \circ \tau_g$ and the restriction of the identity to Y_x . That is, the restriction of $\tau_{\iota(g)}$ to $Y_{g \cdot x}$ is a left-homotopy inverse for the restriction of τ_g to Y_x . A similar argument shows that $\tau_{\iota(g)}$ is a two-sided inverse, and thus $\tau_g: Y_x \rightarrow Y_{g \cdot x}$ is a homotopy equivalence. The last assertion follows from Lemma 2.1 (2).

Our interest in Theorem 2.2 lies in its implications for function spaces. By Lemma 2.1 (1), any map $f: X \rightarrow Y$ is homotopic to a based map. Therefore, when identifying a component of $\text{map}(X, Y)$ as $\text{map}(X, Y; f)$ for some map $f: X \rightarrow Y$, we may assume that f is a based map. Also, these hypotheses ensure that the evaluation map $\omega_f: \text{map}(X, Y; f) \rightarrow Y$ is a Hurewicz fibration by [29, Theorem I.7.8]. We write $\text{map}_*^*(X, Y; f) = \omega_f^{-1}(*)$ for the fibre over the basepoint of Y . Note that the space $\text{map}_*^*(X, Y; f)$ consists of based maps $g: X \rightarrow Y$ which are (freely) homotopic to f . Thus $\text{map}_*(X, Y; f) \subseteq \text{map}_*^*(X, Y; f)$ and the inclusion can be strict.

Lemma 2.1 indicates that we will want $\text{map}(X, Y)$ to have non-degenerate basepoints. Since we have not been able to find an explicit reference for what we want in the literature, we provide the following result that is suited to our purposes.

LEMMA 2.3. *Let X be a compact metric space and Y a countable CW complex. Then every point of $\text{map}(X, Y)$ is non-degenerate.*

Proof. Suppose given $a_0 \in \text{map}(X, Y)$. We want to show that $\{a_0\} \hookrightarrow \text{map}(X, Y)$ is a cofibration. By [5, Theorem XV.7.4], it is sufficient to show that there exists a neighbourhood of a_0 in $\text{map}(X, Y)$, of which $\{a_0\}$ is a strong deformation retract. By [19, Lemma 3], $\text{map}(X, Y)$ is “ELCX.” In particular, we may choose an open set V in $\text{map}(X, Y)$ that contains a_0 , and for which there exists a homotopy $\lambda: V \times V \times I \rightarrow \text{map}(X, Y)$ that satisfies $\lambda(a, b, 0) = a$, $\lambda(a, b, 1) = b$, and $\lambda(a, a, t) = a$ for all $a, b \in V$ (cf. the discussion above [19, Lemma 3]). We define $H: V \times I \rightarrow \text{map}(X, Y)$ by $H(b, t) = \lambda(a_0, b, t)$ and check that this displays a_0 as a strong deformation retract of V , as required.

Suppose $A: G \times Y \rightarrow Y$ is a homotopy-associative action of a group-like space G on Y . The based function space $\text{map}_*(X, G)$ is then a group-like space as well, with multiplication defined to be pointwise multiplication of functions [29, Theorem III-5.18]. We have an induced action

$$\mathcal{A}: \text{map}_*(X, G) \times \text{map}(X, Y) \rightarrow \text{map}(X, Y)$$

of $\text{map}_*(X, G)$ on $\text{map}(X, Y)$ defined by

$$\mathcal{A}(\gamma, g)(x) = A(\gamma(x), g(x))$$

for $\gamma \in \text{map}_*(X, G)$, $g \in \text{map}(X, Y)$. As above, we write $\gamma \cdot g$ for $\mathcal{A}(\gamma, g)$. We note that the following result holds in considerable generality.

THEOREM 2.4. *Let $f: X \rightarrow Y$ be a map between based, connected, countable CW complexes. Let $A: G \times Y \rightarrow Y$ be a homotopy-associative action of a group-like space G on Y . Let $\gamma: X \rightarrow G$ be any based map. Then we have:*

- (A) *the path components $\text{map}(X, Y; f)$ and $\text{map}(X, Y; \gamma \cdot f)$ have the same homotopy type;*
- (B) *if X is a finite complex, then $\text{map}(X, Y; f)$ and $\text{map}(X, Y; \gamma \cdot f)$ have the same based homotopy type;*
- (C) *if X is finite and the action is strictly unital, then the evaluation fibrations $\omega_f: \text{map}(X, Y; f) \rightarrow Y$ and $\omega_{\gamma \cdot f}: \text{map}_{\gamma \cdot f}(X, Y; \gamma \cdot f) \rightarrow Y$ are fibre-homotopy equivalent.*

Proof. Write $\tau_\gamma: \text{map}(X, Y; f) \rightarrow \text{map}(X, Y; \gamma \cdot f)$ for translation by γ . By Theorem 2.2, τ_γ is a homotopy equivalence and (A) follows. Part (B) follows from Theorem 2.2 and Lemma 2.3. For (C), we use results of Dold in [3]. The evaluation fibration $\omega_f: \text{map}(X, Y; f) \rightarrow Y$ has the Weak Covering Homotopy Property [3, Definition 5.1] since it is a Hurewicz fibration. Since the action of G is strictly unital, the diagram

$$\begin{array}{ccc} \text{map}(X, Y; f) & \xrightarrow{\tau_\gamma} & \text{map}(X, Y; \gamma \cdot f) \\ & \searrow \omega_f & \swarrow \omega_{\gamma \cdot f} \\ & Y & \end{array}$$

commutes. The map τ_γ is a based homotopy equivalence by (B). By [3, Theorem 6.1], τ_γ is thus a fibre-homotopy equivalence.

We can recast Theorem 2.4 as follows: Write

$$\mathcal{A}_\#: [X, G] \times \langle X, Y \rangle \rightarrow \langle X, Y \rangle$$

for the induced action of the group $[X, G]$ induced on the set $\langle X, Y \rangle$ of homotopy classes of maps. We write \mathcal{O} for the set of orbits of $\langle X, Y \rangle$ under this action.

COROLLARY 2.5. *Let $f: X \rightarrow Y$ be a map between CW complexes. Let $A: G \times Y \rightarrow Y$ be a homotopy-associative action of a group-like space G on Y . Let \mathcal{O} be the set of orbits of the induced action of the group $[X, G]$ on $\langle X, Y \rangle$. Then:*

- (A) *we have a surjection of sets*

$$\mathcal{O} \twoheadrightarrow \frac{\{\text{components of } \text{map}(X, Y)\}}{\simeq};$$

(B) if X is a finite complex then

$$\mathcal{O} \twoheadrightarrow \frac{\{\text{components of } \text{map}(X, Y)\}}{\simeq_*};$$

(C) if X is a finite complex and the group-like action on Y is strictly unital then

$$\mathcal{O} \twoheadrightarrow \frac{\{\text{evaluation fibrations } \omega_f: \text{map}(X, Y; f) \rightarrow Y\}}{\text{fibre-homotopy equivalence}};$$

In particular, if \mathcal{O} is a finite set, then there are finitely many distinct homotopy types amongst the components of $\text{map}(X, Y)$.

We observe that the discussion of this section can be given with $\text{map}_*(X, Y)$ replacing $\text{map}(X, Y)$. We will see in the next section that there is a further situation that gives rise to an action on $\text{map}_*(X, Y)$, to which we may apply our methods.

Holonomy Actions and Universal Actions

A standard source for an action on a space Y is fibration sequence

$$Y \xrightarrow{j} E \xrightarrow{p} B \quad (2)$$

in which Y occurs as the fibre. For then we have the holonomy action $A: \Omega B \times Y \rightarrow Y$ of the group-like space ΩB on Y . As above, this yields an induced action

$$A: \text{map}_*(X, \Omega B) \times \text{map}(X, Y) \longrightarrow \text{map}(X, Y) \quad (3)$$

of $\text{map}_*(X, \Omega B)$ on $\text{map}(X, Y)$. In this situation, we may be quite precise about the orbits.

LEMMA 3·1. *Let $f, g: X \rightarrow Y$ be based maps. With reference to the action (3) induced from the fibration (2), the following are equivalent:*

- (A) $g \sim_* \gamma \cdot f$ for some $\gamma \in \text{map}_*(X, \Omega B)$, that is, f and g are in the same orbit;
- (B) $g \sim_* A \circ (\gamma \times f) \circ \Delta$;
- (C) $j \circ f \sim_* j \circ g: X \rightarrow E$.

Proof. (A) and (B) are equivalent from the definitions. To see (C) is equivalent, consider the Puppe sequence

$$\cdots \longrightarrow [X, \Omega B] \xrightarrow{\partial_*} [X, Y] \xrightarrow{j_*} [X, E] \xrightarrow{p_*} [X, B] \quad (4)$$

corresponding to the fibration (2). As is well-known, $[X, \Omega B]$ acts on $[X, Y]$ as described in (B) (see e.g. [29, p.140]). Furthermore, an orbit of $[f] \in [X, Y]$ under this action is precisely the pre-image of $j_*([f])$ ([29, III·6·20]). The equivalence of (B) and (C) follows.

THEOREM 3·2. *Let X and Y be connected, countable CW complexes with non-degenerate basepoints. Let $j: Y \rightarrow E$ be the fibre inclusion of a fibration in which Y occurs as the fibre. Suppose $j \circ f \sim_* j \circ g: X \rightarrow E$ for maps $f, g: X \rightarrow Y$. Then $\text{map}(X, Y; f)$ and $\text{map}(X, Y; g)$ are homotopy equivalent. If X is a finite complex then the evaluation fibrations $\omega_f: \text{map}(X, Y; f) \rightarrow Y$ and $\omega_g: \text{map}(X, Y; g) \rightarrow Y$ are fibre-homotopy equivalent.*

Proof. The result follows directly from Theorem 2·4 and Lemma 3·1.

Example 3.3. Consider a compact, connected Lie group G and a toral subgroup $T \subseteq G$. Then we have a fibre sequence $G/T \rightarrow BT \rightarrow BG$ with fibre inclusion $j: G/T \rightarrow BT = \coprod K(\mathbb{Z}, 2)$. Given a CW complex X and based maps $f, g: X \rightarrow G/T$ we see that $H^2(f) = H^2(g): H^*(G/T; \mathbb{Z}) \rightarrow H^*(X; \mathbb{Z})$ implies $j \circ f \sim_* j \circ g$. We conclude from Theorem 3.2 that $H^2(f) = H^2(g)$ implies the components $\text{map}(X, G/T; f)$ and $\text{map}(X, G/T; g)$ are homotopy equivalent.

We may develop Theorem 3.2 as follows.

COROLLARY 3.4. *Let X and Y be connected, countable CW complexes with non-degenerate basepoints. Let $j: Y \rightarrow E$ be the fibre inclusion of a fibration in which Y occurs as the fibre. If the image of $j_*: [X, Y] \rightarrow [X, E]$ is a finite set in $[X, E]$, then there are finitely many distinct homotopy types amongst the components of $\text{map}(X, Y)$. If, further, X is finite, there are finitely many fibre-homotopy types amongst the evaluation fibrations $\omega_f: \text{map}(X, Y; f) \rightarrow Y$ for $f: X \rightarrow Y$.*

So, for instance, returning to the situation of Example 3.3, we may say that if $H^2(X; \mathbb{Z})$ is finite, then there are finitely many distinct homotopy types amongst the components of $\text{map}(X, G/T)$. We offer a further example along these lines.

Example 3.5. Let G be a connected Lie group and H a closed subgroup. Suppose that $\text{Hom}(H^*(G/H; \mathbb{Q}), H^*(X; \mathbb{Q})) = 0$, for a finite complex X . (These hypotheses hold, for instance, whenever H is a subgroup of maximal rank and X is any finite complex with $H^{\text{even}}(X; \mathbb{Q}) = 0$.) Then there are finitely many fibre-homotopy types amongst the evaluation fibrations $\omega_f: \text{map}(X, G/H; f) \rightarrow Y$, for maps $f: X \rightarrow G/H$. To see why, observe that BH is rationally a product of Eilenberg–Mac Lane spaces, and hence the hypotheses imply that each $j \circ f: X \rightarrow BH$ is null-homotopic after rationalization, where $j: G/H \rightarrow BH$ is the fibre inclusion of the fibre sequence $G/H \rightarrow BH \rightarrow BG$ and $f: X \rightarrow G/H$ is any map. Since rationalization of homotopy sets is a finite-to-one map [21, Corollary II.5.4], it follows that $j_*: [X, G/H] \rightarrow [X, BH]$ has finite image. Now we may apply Corollary 3.4.

We next observe that the universal action on a space Y is that induced by the evaluation map of the identity component. Precisely, observe that the space $\text{map}(X, X; 1)$ is a strictly associative H -space with multiplication given by composition of functions. Define the action

$$A_\infty: \text{map}(Y, Y; 1) \times Y \rightarrow Y$$

by $A_\infty(g, y) = g(y)$ for $g \in \text{map}(Y, Y; 1)$ and $y \in Y$. Given any H -action $A: G \times Y \rightarrow Y$ we obtain, by adjointness, an H -map $\hat{A}: G \rightarrow \text{map}(Y, Y; 1)$ which commutes with the actions in the sense that $A_\infty(\hat{A}(g), y) = A(g, y)$ for all $g \in G$ and $y \in Y$. Conversely, any H -map $\hat{A}: G \rightarrow \text{map}(Y, Y; 1)$ induces an action $A: G \times Y \rightarrow Y$. We remark that, according to Gottlieb [8], this universal action corresponds to the holonomy action in the universal fibration with fibre Y

$$Y \xrightarrow{j_\infty} E_\infty \xrightarrow{p_\infty} B_\infty, \quad (5)$$

(cf. [1, 4, 23]). We will need the following consequence of the classifying fibration:

THEOREM 3.6. *Let Y be a CW complex. Then $\text{map}(Y, Y; 1)$ is a group-like space.*

Proof. Since Y is a CW complex, combining [1] with [6, Satz.7.3] gives an H -equivalence $\text{map}(Y, Y; 1) \simeq \Omega_0 B_\infty$, where $\Omega_0 B_\infty$ denotes the component of the constant loop. Thus $\text{map}(Y, Y; 1)$ is group-like by [29, Corollary III.5.17].

Now write

$$A_\infty : \text{map}_*(X, \text{map}(Y, Y; 1)) \times \text{map}(X, Y) \rightarrow \text{map}(X, Y)$$

for the action induced by A_∞ on $\text{map}(X, Y)$, and

$$(\mathcal{A}_\infty)_\# : [X, \text{map}(Y, Y; 1)] \times [X, Y] \rightarrow [X, Y]$$

for the corresponding group action on the set $[X, Y]$. Given a homotopy class $[f] \in [X, Y]$ we write $O_\infty([f])$ for the orbit of $[f]$ under this action.

We recall that a based map $f : X \rightarrow Y$ is called *cyclic* if the map $(f \mid 1) : X \vee Y \rightarrow Y$ admits some extension $\Gamma : X \times Y \rightarrow Y$ [27]. We write $G(X, Y) \subseteq [X, Y]$ for the set of based homotopy classes of cyclic maps. In the special case in which $X = S^n$, $G(S^n, Y)$ is just $G_n(Y) \subseteq \pi_n(Y)$, the n th *Gottlieb group* of Y [9].

It is a direct consequence of adjointness that $f : X \rightarrow Y$ is cyclic if and only if the evaluation fibration $\omega_f : \text{map}(X, Y; f) \rightarrow Y$ admits a section. We also have

$$O_\infty([0]) = G(X, Y). \quad (6)$$

For suppose $\gamma : X \rightarrow \text{map}(Y, Y; 1)$ is a based map. We define a section $s : Y \rightarrow \text{map}(X, Y; \gamma \cdot 0)$ by the rule $s(y) = \gamma(x)(y)$. Conversely, if $s : Y \rightarrow \text{map}(X, Y; f)$ is a section for some based map $f : X \rightarrow Y$ then $f \sim_* \gamma \cdot 0$ where $\gamma : X \rightarrow \text{map}(Y, Y; 1)$ is given by $\gamma(x)(y) = s(y)(x)$. As a consequence, we obtain the following result which extends [28, Theorem 2.8] and its generalization by Yoon [30, Theorem 4.5].

THEOREM 3.7. *Let X and Y be CW complexes with non-degenerate basepoints. Let $f : X \rightarrow Y$ be a map. If $\omega_f : \text{map}(X, Y; f) \rightarrow Y$ has a section then $\text{map}(X, Y; f)$ is homotopy equivalent to $\text{map}(X, Y; 0)$. If X is a finite complex then the following are equivalent:*

- (A) *the map $f : X \rightarrow Y$ is cyclic;*
- (B) *the evaluation fibration $\omega_f : \text{map}(X, Y; f) \rightarrow Y$ has a section;*
- (C) *the evaluation fibration $\omega_f : \text{map}(X, Y; f) \rightarrow Y$ is fibre-homotopy equivalent to $\omega_0 : \text{map}(X, Y; 0) \rightarrow Y$.*

Proof. The first statement follows from (6) and Theorem 2.4 (A). The equivalence of (A) and (B) is a consequence of adjointness, as mentioned above. We obtain (A) implies (C) by observing that the universal action A_∞ is strictly unital and applying Theorem 2.4 (C) and (6). Finally, note that (C) implies (B) since the evaluation fibration $\omega_0 : \text{map}(X, Y; 0) \rightarrow Y$ admits the section $s(y)(x) = y$.

COROLLARY 3.8. *Let Y be a finite CW complex. Then Y is an H -space if and only if for every finite CW complex X the evaluation fibrations $\omega_f : \text{map}(X, Y; f) \rightarrow Y$ are fibre-homotopy equivalent for all maps $f : X \rightarrow Y$.*

Proof. The result follows from Theorem 3.7 and the equivalences:

$$Y \text{ is an } H\text{-space} \iff 1 : Y \rightarrow Y \text{ is cyclic} \iff \text{every map } f : X \rightarrow Y \text{ is cyclic}$$

which are direct from definitions.

We now consider the above action \mathcal{A}_∞ in the special case in which X is a co-H-space. Suppose the coproduct is $\sigma: X \rightarrow X \vee X$. The map σ induces a pairing which we denote ‘+’ in the set $[X, Y]$. By [27, Theorem 1.5] the set of cyclic maps $G(X, Y)$ is a subgroup of $[X, Y]$ when X is a co-group-like space. When X is merely a co-H-space, Varadarajan’s proof gives that the set $G(X, Y)$ is closed under addition. We show that, when X is a co-H-space, the orbit of a class $[f] \in [X, Y]$ under the action of $(\mathcal{A}_\infty)_\#$ is just the set of translates of $[f]$ by $G(X, Y)$, that is, we have

$$\mathcal{O}_\infty([f]) = \{d + f \mid d \in G(X, Y)\} \quad (7)$$

This result is a direct consequence of the following:

LEMMA 3.9. *Let X be a co-H-space. Let $\gamma: X \rightarrow \text{map}(Y, Y; 1)$ and $f: X \rightarrow Y$ be based maps. Let $d: X \rightarrow Y$ be defined by $d(x) = \gamma(x)(y)$. Then*

$$\gamma \cdot f \sim_* d + f.$$

Proof. Let $\Gamma: X \times Y \rightarrow Y$ denote the adjoint of γ . By the definition of d we then have the following homotopy-commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta} & X \times X & \xrightarrow{1 \times f} & X \times Y & \xrightarrow{\Gamma} & Y \\ & \searrow \sigma & \uparrow J & & \uparrow J & \nearrow (d|1) & \\ & & X \vee X & \xrightarrow{1 \vee f} & X \vee Y & & \end{array} .$$

The following consequence was proved by Yoon ([30, Theorem 4.9]) for X a suspension.

THEOREM 3.10. *Suppose X is a CW co-H-space and Y is any CW complex. Let $d \in G(X, Y)$ be any cyclic map. Then for each map $f: X \rightarrow Y$, we have $\text{map}(X, Y; f) \simeq \text{map}(X, Y; f + d)$. If X is a finite co-H-space then the corresponding evaluation fibrations ω_f and ω_{f+d} are fibre-homotopy equivalent.*

Proof. Since $d: X \rightarrow Y$ is cyclic there exists a based map $\Gamma: X \times Y \rightarrow Y$ extending $(d \mid 1): X \vee Y \rightarrow Y$. Let $\gamma: X \rightarrow \text{map}(Y, Y; 1)$ denote the adjoint to Γ . Then, under the universal action we have $\gamma \circ f \sim_* d + f$ by Lemma 3.9 and the result follows from Theorem 2.4.

Note that if X is a suspension, or more generally a cogroup-like space, then (7) gives a bijection

$$\mathcal{O}_\infty \cong [X, Y]/G(X, Y), \quad (8)$$

where \mathcal{O}_∞ denotes the orbits of the action $(\mathcal{A}_\infty)_\#$ on $[X, Y]$ and the right-hand side is simply the quotient group. If, for instance, $X = S^n$, then from Corollary 2.5 (C) we obtain a surjection

$$\pi_n(Y)/G_n(Y) \twoheadrightarrow \frac{\{\text{evaluation fibrations } \omega_f: \text{map}(S^n, Y; f) \rightarrow Y\}}{\text{fibre-homotopy equivalence}}. \quad (9)$$

When Y is simple, the connecting homomorphism in the long exact homotopy sequence of the evaluation fibration $\omega_f: \text{map}(S^n, Y; f) \rightarrow Y$, when viewed as a map

$$\partial: \pi_k(Y) \longrightarrow \pi_{k-1}(\text{map}(S^n, Y; f)) \cong \pi_{k+n-1}(Y),$$

may be described in terms of Whitehead products with the class represented by $f: S^n \rightarrow Y$ (see [29, Section 3]). This fact can be used in special cases to distinguish non-equivalent components of $\text{map}(S^n, Y)$ by calculating homotopy groups. For a recent application of this method see [25]. Hansen uses this method in [11]; his result [11, Theorem 2.3] implies the surjection (9) is actually a bijection when $Y = S^m$ is also a sphere. Thus the homotopy classification of components of $\text{map}(S^n, S^m)$ reduces to the problem of computing the Gottlieb groups $G_n(S^m)$. See [7] for recent results in this direction. Using other recent calculations of Gottlieb groups we obtain the following.

Example 3.11. Let $V_{n+k,k} = O(n+k)/O(n)$ denote the real Stiefel manifold. Suppose that we have $m \leq n$ and $m \equiv 3, 5, 6, 7 \pmod{8}$. Then the evaluation fibrations

$$\omega_f: \text{map}(S^m, V_{n+k,k}; f) \rightarrow V_{n+k,k}$$

are fibre-homotopy equivalent for all $f: S^m \rightarrow V_{n+k,k}$. For by [14, Theorem 3.1], we have $G_m(V_{n+k,k}) = \pi_m(V_{n+k,k})$ in these cases. Similar examples can be formulated for the complex and quaternionic Stiefel manifolds using [14, Theorem 3.2] and [14, Theorem 3.3], respectively.

Our last remark on these topics concerns the case in which Y is a so-called G -space, that is, a space that satisfies $G_n(Y) = \pi_n(Y)$ for each n . Such spaces have been studied by Siegel, Gottlieb and others, and are considered as being “close” to H -spaces from certain points of view. There are examples of G -spaces that are not H -spaces, however [22]. If Y is a G -space, then the surjection (9) yields that all evaluation fibrations $\text{map}(S^n, Y; f) \rightarrow Y$ are fibre-homotopy equivalent to each other (and each has a section). This is a further property that G -spaces share with H -spaces.

In case the function space of based maps is of interest, there is a separate source of actions in addition to those obtained by restricting actions on the unbased function space, as we have done above. Namely, those group-like actions on the based mapping space $\text{map}_*(X, Y)$ that arise from cogroup-like actions on X . We finish the paper with a brief discussion of this topic. The approach here is essentially that observed by Sutherland in [25, Section 4].

Suppose that C is a *co- H -space* with comultiplication $\sigma: C \rightarrow C \vee C$. Then, for any space Y , the based function space $\text{map}_*(C, Y)$ is an H -space with product $m(f, g) = (f | g) \circ \sigma$, where $f, g: C \rightarrow Y$ are based maps. By [29, Theorem III.5.16], $\text{map}_*(C, Y)$ is homotopy-associative, respectively group-like, if C is homotopy-coassociative, respectively cogroup-like. By a *homotopy-coassociative coaction* of a homotopy-coassociative co- H -space C on a based space X , we mean a based map $B: X \rightarrow C \vee X$ that satisfies $p_2 \circ B \sim_* 1: X \rightarrow X$ and $(\sigma | 1) \circ B \sim_* (1 | B) \circ B: X \rightarrow C \vee C \vee X$ where $p_2: C \vee B \rightarrow B$ is the obvious projection.

Suppose C is a homotopy-coassociative co- H -space and $B: X \rightarrow C \vee X$ is a homotopy-coassociative coaction. Define

$$B: \text{map}_*(C, Y) \times \text{map}_*(X, Y) \longrightarrow \text{map}_*(X, Y). \quad (10)$$

by setting $B(\gamma, f) = (\gamma | f) \circ B$. It is direct to check that B defines a homotopy-associative action on $\text{map}_*(X, Y)$. If C is cogroup-like, this is a group-like action. Thus we may apply Theorem 2.2 to this situation, giving:

THEOREM 3.12. *Let $B: X \rightarrow C \vee X$ be a homotopy-coassociative coaction of a cogroup-like space C on X . Let $\gamma: C \rightarrow Y$ be a based map.*

- (A) For any map $f: X \rightarrow Y$, under the resulting action (10) on $\text{map}_*(X, Y)$, the path components $\text{map}_*(X, Y; f)$ and $\text{map}_*(X, Y; \gamma \cdot f)$ of $\text{map}_*(X, Y)$ have the same homotopy type.
- (B) Write

$$\mathcal{B}_\# : [C, Y] \times [X, Y] \rightarrow [X, Y]$$

for the action induced on homotopy sets by the action (10). Let \mathcal{O}' denote the set of orbits of $[X, Y]$ under this action of the group $[C, X]$. There is a surjection of sets

$$\mathcal{O}' \twoheadrightarrow \frac{\{\text{components of } \text{map}_*(X, Y)\}}{\simeq}.$$

In particular, if \mathcal{O}' is a finite set, then there are finitely many distinct homotopy types amongst the components of $\text{map}_*(X, Y)$.

Proof. The proof is a direct consequence of the preceding discussion and Theorem 2.2.

A standard source for a coaction on a space X is a cofibration sequence

$$Z \xrightarrow{i} A \xrightarrow{q} X$$

in which X occurs as the cofibre. For then we have $B: X \rightarrow \Sigma Z \vee X$, the usual coaction of the cogroup-like space ΣZ on X . This then leads to an action as above

$$\mathcal{B}: \text{map}_*(\Sigma Z, Y) \times \text{map}_*(X, Y) \longrightarrow \text{map}_*(X, Y). \quad (11)$$

Consider the Puppe sequence

$$\cdots \longrightarrow [\Sigma Z, Y] \xrightarrow{\partial^*} [X, Y] \xrightarrow{q^*} [A, Y] \xrightarrow{i^*} [Z, Y]. \quad (12)$$

As is well known, $[\Sigma Z, Y]$ acts on $[X, Y]$ and an orbit of $f \in [X, Y]$ under this action is precisely the pre-image of $q^*(f)$ (see [29, III-6.20]). It is easy to see that this action in the Puppe sequence is identical with the action $\mathcal{B}_\#$ induced on $[X, Y]$ by the action (11). Thus the following result is a direct consequence of Theorem 3.12.

THEOREM 3.13. *Let $q: A \rightarrow X$ be the cofibre projection of a cofibration in which X occurs as the cofibre. Suppose $f \circ q \sim_* g \circ q: A \rightarrow Y$ for maps $f, g: X \rightarrow Y$. Then $\text{map}_*(X, Y; f)$ and $\text{map}_*(X, Y; g)$ have the same homotopy type. If the image of $q^*: [X, Y] \rightarrow [A, Y]$ is a finite set in $[A, Y]$, then there are finitely many distinct homotopy types amongst the components of $\text{map}_*(X, Y)$.*

Example 3.14. Suppose X is an n -dimensional manifold. Then X occurs as the cofibre in a cofibration of the form $S^n \rightarrow A \rightarrow X$ where A is an $(n-1)$ -dimensional CW complex. Note that the components of $\text{map}_*(X, S^n)$ are in one-to-one correspondence with $H^n(X)$, by the Hopf–Whitney classification theorem, and so there are generally infinitely many components of $\text{map}_*(X, S^n)$. However, $[A, S^n]$ consists of a single element, namely the homotopy class of the trivial map. By Theorem 3.13, *all components of the based mapping space $\text{map}_*(X, S^n)$ have the same homotopy type.*

Along the same lines, we offer the following:

Example 3.15. Suppose $X = S^n \cup_\alpha e^{r+1}$ is a two-cell complex, with $\alpha \in \pi_r(S^n)$ for some $r \geq n$. Suppose Y is any space with $\pi_n(Y)$ finite. Then there are finitely many distinct

homotopy types amongst the components of the based mapping space $\text{map}_*(X, Y)$. For we have a cofibre sequence $S^r \rightarrow S^n \rightarrow X$, in whose Puppe sequence the map $q^*: [X, Y] \rightarrow [S^n, Y]$ has finite image by hypothesis. The assertion follows from Theorem 3.13.

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