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Evaluation Maps in Rational Homotopy

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1. Introduction

Let X be a based space and let $\mathrm{Map}\,(X, X)$ be the space of unbased, or free, maps from X to itself. In general, $\mathrm{Map}\,(X, X)$ is disconnected; we denote by $\mathrm{Map}\,(X, X; 1)$ its identity component; that is, the path component that consists of self maps that are (freely) homotopic to the identity. Then we have *the evaluation map* $\omega\colon \mathrm{Map}\,(X, X; 1) \to X$ defined by evaluation at the basepoint of X. This map occupies a central place in the homotopy theory of fibrations (cf. [5–8]).

The evaluation map ω and its rationalization will play a distinguished role in this paper. However, our methods and results apply equally well to other "evaluation maps". For example, consider the space $\mathrm{Top}(X, X; 1)$ of self-homeomorphisms of X homotopic to the identity, and the corresponding evaluation map $\mathrm{w}\colon \mathrm{Top}(X, X; 1) \to X$. Likewise, if X is a smooth manifold, then replace $\mathrm{Top}(X, X)$ with $\mathrm{Diff}(X, X)$, and so forth. A further example of an "evaluation map" concerns *configuration spaces*. Let $F(X, k)$ denote the configuration space that consists of ordered k-tuples of distinct points in a space X, and let $(p_1, \ldots, p_k)$ be a choice of basepoint in $F(X, k)$. Then we have an evaluation map $\theta\colon \mathrm{Top}(X, X; 1) \to F(X, k)$ given by $\theta(\alpha) = (\alpha(p_1), \ldots, \alpha(p_k))$.

Motivated by the preceding examples, we now make a formal definition of the evaluation maps that we consider. Recall that a strict H-space is an H-space (E, μ) with a strict unit. By an *action of E on a space X*, we mean a map $A\colon E \times X \to X$ that satisfies $A \circ i_2 = 1\colon X \to X$. We say that *the action is associative* if, in addition, we have $A \circ (\mu \times 1) = A \circ (1 \times A)$.

Definition 1.1. Given a strict H-space E and an associative action $A : E \times X \to X$, define the *generalized evaluation map* associated to A as $\mathrm{w} = A \circ i_1 : E \to X$.

Examples 1.2. (1) The action $A\colon \mathrm{Map}\,(X, X; 1) \times X \to X$ given by $A(f, x) = f(x)$ makes $\omega\colon \mathrm{Map}\,(X, X; 1) \to X$ a generalized evaluation map according to Definition 1.1. Similarly for all the other examples mentioned above.

(2) Suppose G is a connected topological group and $A\colon G \times X \to X$ is a group action. Then the *orbit map* of the action is a generalized evaluation map $G \to X$.

(3) More generally, suppose we are given a fibration $X \to Y \to B$. Then the connecting map $\partial\colon \Omega B \to X$ is a generalized evaluation map. This follows from the usual action of the Moore loops ΩB on the fibre X.

Revert now to the ordinary evaluation map $\omega\colon \mathrm{Map}\,(X, X; 1) \to X$. *For the remainder of the paper, we assume that X is a finite nilpotent complex*, and denote by $X_{\mathbb{Q}}$ its rationalization. Then by [11], the evaluation map for $X_{\mathbb{Q}}$, denoted $\omega_{\mathbb{Q}}$, is the rationalization of ω. We refer to $\omega_{\mathbb{Q}}$ as *the rationalized evaluation map*. Recall that the *nth Gottlieb group of X*, $G_n(X)$, is the subgroup $\mathrm{im}\,\pi_n(\omega) \subset \pi_n(X)$ [7]. An element $[f] \in \pi_n(X)$ belongs to $G_n(X)$ if and only if $f \vee 1 : S^n \vee X \to X$ extends to $S^n \times X$. Recall also that, by a result of Lang [12], we have $G_n(X_{\mathbb{Q}}) \cong G_n(X) \otimes \mathbb{Q}$. These rationalized Gottlieb groups have played an important role in rational homotopy theory (cf. [1,10]). A result of Félix–Halperin [1, Th. III] implies that $G_{2i}(X_{\mathbb{Q}}) = 0$ for all i, and $\dim G_*(X_{\mathbb{Q}}) < \infty$. Suppose $\{[f_1], [f_2], \ldots, [f_r]\}$ is a basis of $G_*(X_{\mathbb{Q}})$ with $f_i : S^{n_i} \to X$. Denote by F_i the extension of f_i to $S^{n_i} \times X$, and by S_X the product of the odd-dimensional rational spheres $S^{n_i}_{\mathbb{Q}}$. Then we form a map $F\colon S_X \times X_{\mathbb{Q}} \to X_{\mathbb{Q}}$ as the composition

$$F = F_1 \circ (1 \times F_2) \circ \cdots \circ (1 \times \cdots \times 1 \times F_r).$$

Now set $\Gamma_X = F \circ i\colon S_X \to X_{\mathbb{Q}}$, where i denotes the inclusion of the product of spheres as the first r factors. We refer to Γ_X as a *total Gottlieb element of $X_{\mathbb{Q}}$*.

The preceding discussion extends naturally to generalized evaluation maps. Suppose we are given $\mathrm{w}\colon E \to X$ any generalized evaluation map. In Section 2, we construct a map $\Gamma_{\mathrm{w}}\colon S_{\mathrm{w}} \to X_{\mathbb{Q}}$ such that $\mathrm{im}\,\pi_*(\Gamma_{\mathrm{w}}) \otimes \mathbb{Q} = \mathrm{im}\,\pi_*(\mathrm{w}) \otimes \mathbb{Q}$. As with S_X above, S_{w} is a finite product of odd-dimensional rational spheres.

Theorem 1.3. *Let $\mathrm{w}\colon E \to X$ be any generalized evaluation map with X a nilpotent, finite complex. Suppose that $\Gamma_{\mathrm{w}}\colon S_{\mathrm{w}} \to X_{\mathbb{Q}}$ is a total Gottlieb element of $X_{\mathbb{Q}}$ with respect to* w. *Then S_{w} is a* *retract of $E_{\mathbb{Q}}$, and $\mathrm{w}_{\mathbb{Q}}$ factors up to homotopy through Γ_{w}. In particular, $G_*(X_{\mathbb{Q}}) = 0$ if and only if $\omega_{\mathbb{Q}} : \mathrm{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \to X_{\mathbb{Q}}$ is null-homotopic.*

We continue with a theorem related to the homotopy behaviour of the maps $\Gamma_{\mathrm{w}} \colon S_{\mathrm{w}} \to X_{\mathbb{Q}}$. Recall that a map $f \colon X \to Y$ is a *homotopy monomorphism in the nilpotent category* if, for any nilpotent space A, the induced map of homotopy sets $f_* \colon [A, X] \to [A, Y]$ is injective [3].

Theorem 1.4. *Let X be a nilpotent, finite complex and $\mathrm{w} \colon E \to X$ be any generalized evaluation map. Then $\Gamma_{\mathrm{w}} \colon S_{\mathrm{w}} \to X_{\mathbb{Q}}$ is a homotopy monomorphism in the nilpotent category. In particular, a rationalized Gottlieb element $f \colon S^n \to X_{\mathbb{Q}}$ is a homotopy monomorphism in the nilpotent category.*

This implies that rationalized Hopf maps are homotopy monomorphisms in the nilpotent category. By contrast, the Hopf map $\eta \colon S^7 \to S^4$ is not a homotopy monomorphism [3].

A further consequence of Theorem 1.4 is the classification, up to rational homotopy, of cyclic maps. A map $f \colon A \to X$ is called *cyclic* if $(f \mid 1) \colon A \vee X \to X$ extends to a map $A \times X \to X$ [17]. Denote by $G(A, X)$ the set of homotopy classes of cyclic maps from A into X. This is a generalization of the nth Gottlieb group of X, which we obtain by taking $A = S^n$. Upon rationalizing a cyclic map, we obtain a map $f_{\mathbb{Q}} \colon A \to X_{\mathbb{Q}}$ in $G(A, X_{\mathbb{Q}})$.

Theorem 1.5. *Let X be a nilpotent, finite complex and let A be any nilpotent space. Then there are bijections of sets*

$$G(A, X_{\mathbb{Q}}) \cong [A, S_X] \cong \oplus_r \mathrm{Hom}(H_r(A; \mathbb{Q}), G_r(X_{\mathbb{Q}})).$$

Proof. The first bijection is given by $(\Gamma_X)_* \colon [A, S_X] \to G(A, X_{\mathbb{Q}})$. This is a bijection by Theorems 1.3 and 1.4. Now remark that S_X has the homotopy type of a product of rational Eilenberg–Mac Lane spaces, $S_X = \prod_{i=1}^r K(\mathbb{Q}, n_i)$. By taking cohomology classes, we thus obtain a bijection $[A, S_X] \xrightarrow{\cong} \oplus_{i=1}^r H^{n_i}(A; \mathbb{Q})$. □

Together with Sam Smith, the second-named author has studied cyclic maps from the rational homotopy point of view in [14]. Many of the results of [14] may be placed in context with the above classification result. For instance, we retrieve [14, Th. 3.2]: If $H^{\mathrm{odd}}(A; \mathbb{Q}) = 0$, then any map $g \colon A \to S_{\mathrm{w}}$ must be null-homotopic.

Our last topic is the (co)homological behavior of generalized evaluation maps which, for the ordinary evaluation map, has been studied by Gottlieb [9], Oprea [15,16] and Halperin [10]. Let $h_X : \pi_*(X) \otimes \mathbb{Q} \to H_*(X; \mathbb{Q})$ denote the rationalized Hurewicz homomorphism. We generalize [15, Th. 1] by proving:

Theorem 1.6. *Let $\mathrm{w} \colon E \to X$ be any generalized evaluation map with X a finite, nilpotent complex. Then, $\dim \mathrm{im}\,(h_X \circ (\mathrm{w}_\# \otimes \mathbb{Q})) = r$ if and only if $\dim \mathrm{im}\, H_*(\mathrm{w}; \mathbb{Q}) = 2^r$. In this case, there is a rational homotopy equivalence $X \simeq_{\mathbb{Q}} S \times Y$, with S a product of r odd-dimensional spheres.*

As $\chi(S \times Y) = 0$, we obtain the following sharpening of [9, Th.3].

Corollary 1.7. *Suppose that $\chi(X) \neq 0$. Then, for every generalized evaluation map $\mathrm{w} \colon E \to X$, we have $\widetilde{H}_*(\mathrm{w}; \mathbb{Q}) = 0 \colon \widetilde{H}_*(E; \mathbb{Q}) \to \widetilde{H}_*(X; \mathbb{Q})$.*

On the other hand, the cohomology algebra structure of a symplectic manifold M, or more generally a c-symplectic space [13], does not allow a decomposition of M as $S^{2n+1} \times Y$. Hence, we get:

Corollary 1.8. *Let M be a simply connected, symplectic manifold. Then every generalized evaluation map* $\mathrm{w}\colon E \to M$ *is trivial on rational homology. Consequently, if G is a connected Lie group and* $a\colon G \to M$ *is the orbit map of any G-action on M, we have* $\widetilde{H}_*(a;\mathbb{Q}) = 0$.

Finally, and directly from Theorems 1.3 and 1.6 we have the following:

Corollary 1.9. *Let* $\mathrm{w}\colon E \to X$ *be an evaluation map with X a nilpotent, finite complex. The following are equivalent:*

(1) *The homomorphism* $H_*(\mathrm{w})\colon H_*(E;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ *is surjective;*
(2) $\Gamma_{\mathrm{w}}\colon S_{\mathrm{w}} \to X$ *is a rational homotopy equivalence.*

The text is divided into four parts. In Section 2, we present the factorization results. Section 3 contains the monomorphism theorem. The homological behaviour of generalized evaluation maps is discussed in Section 4.

We assume the reader's familiarity with rational homotopy theory and use the standard notation and terminology for minimal models as presented in [2]. The basic facts that we use are as follows: each nilpotent space X has a unique Sullivan minimal model $(\mathcal{M}_X, d)$ in the category of commutative DG (differential graded) algebras over $\mathbb{Q}$. This DG algebra $(\mathcal{M}_X, d)$ is of the form $\mathcal{M}_X = \wedge V$, a free graded commutative algebra generated by a positively graded vector space V of finite type. The differential d is decomposable, in that $d(V) \subseteq \wedge^{\geq 2} V$, and V admits a basis $\{v_\alpha\}$ indexed by a well-ordered set such that $d(v_\alpha) \in \wedge(\{v_\beta\}_{\beta<\alpha})$. An H_0-space has a minimal model with zero differential. Each map $f\colon X \to Y$ also has a Sullivan minimal model which is a DG algebra map $\mathcal{M}_f\colon \mathcal{M}_Y \to \mathcal{M}_X$. The Sullivan minimal model is a complete rational homotopy invariant. We have natural isomorphisms $H(\mathcal{M}_X, d) \cong H^*(X;\mathbb{Q})$ and $Q(\mathcal{M}_X) \cong \mathrm{Hom}(\pi_*(X),\mathbb{Q})$ where $Q(\mathcal{M}_X) \cong V$ denotes the vector space of indecomposable elements in $\wedge V$. If $f, g\colon X \to Y$ are maps of *rational spaces*, then f and g are homotopic if and only if $\mathcal{M}_f$ and $\mathcal{M}_g$ are homotopic in an algebraic sense.

2. Factorization of an evaluation fibration

We start this section with a natural generalization of the Félix–Halperin result on Gottlieb groups [1, Theorem 3].

Proposition 2.1. *Let X be a nilpotent space and* $p\colon E \to X$ *be any map with E an H_0-space. If X has finite rational category, that is, if* $\mathrm{cat}_0(X) = r < \infty$, *then* $p_\#\left(\pi_{\mathrm{even}}(E_{\mathbb{Q}})\right) = 0$ *and* $\dim p_\#\left(\pi_{\mathrm{odd}}(E_{\mathbb{Q}})\right) \leq r$.

Proof. For the first assertion, suppose that $\beta \in \pi_{2i}(E_{\mathbb{Q}})$ satisfies $p_\#(\beta) \neq 0$. We identify $K(\mathbb{Q}, 2i) \simeq \Omega\Sigma S^{2i}_{\mathbb{Q}}$. Let $\epsilon\colon X \to \Omega\Sigma X$ denote the adjoint of the (suspension of the) identity. Since $E_{\mathbb{Q}}$ is an H-space, we may choose a retraction $r\colon \Omega\Sigma E_{\mathbb{Q}} \to E_{\mathbb{Q}}$ of $\epsilon\colon E_{\mathbb{Q}} \to \Omega\Sigma E_{\mathbb{Q}}$ so that $r \circ \epsilon = 1$ and the following diagram commutes:

$$\begin{tikzcd}
\Omega\Sigma S^{2i}_{\mathbb{Q}} \arrow[r, "\Omega\Sigma\beta"] & \Omega\Sigma E_{\mathbb{Q}} \arrow[d, bend left, "r"] \\
S^{2i}_{\mathbb{Q}} \arrow[u, "\epsilon"] \arrow[r, "\beta"'] & E_{\mathbb{Q}}. \arrow[u, "\epsilon"]
\end{tikzcd}$$

That is, β extends to a map $\tilde{\beta} = r \circ \Omega\Sigma\beta$ such that $p \circ \tilde{\beta} \colon K(\mathbb{Q}, 2i) \to X_{\mathbb{Q}}$ is injective in homotopy. But then, the mapping theorem of [1] implies that $\infty = \mathrm{cat}_0(K(\mathbb{Q}, 2i)) \leq \mathrm{cat}_0(X) = r$, which is a contradiction. Therefore, we have $p_{\#}\left(\pi_{\mathrm{even}}(E_{\mathbb{Q}})\right) = 0$. For the second assertion, consider any finite, linearly independent subset $\{\alpha_1, \ldots, \alpha_k\}$ of $\pi_{\mathrm{odd}}(X_{\mathbb{Q}})$ in the image of $p_{\#}$. Choose a $\beta_i \in \pi_{n_i}(E_{\mathbb{Q}})$ with $p_{\#}(\beta_i) = \alpha_i$ for each i. Using the multiplication of $E_{\mathbb{Q}}$, we extend the map $\bigvee_i S^{n_i}_{\mathbb{Q}} \to E_{\mathbb{Q}}$, defined as β_i on each summand into a map $\widetilde{\Gamma}_p \colon S_p = \prod_{i=1}^k S^{n_i}_{\mathbb{Q}} \to E_{\mathbb{Q}}$. Since $p \circ \widetilde{\Gamma}_p \colon S_p \to X_{\mathbb{Q}}$ is injective in homotopy groups, by the mapping theorem we have $k = \mathrm{cat}_0(S_p) \leq r$. The second assertion follows. $\square$

In order to study generalized evaluation maps $\mathrm{w} \colon E \to X$, we first present a global structure result concerning maps between H_0-spaces.

Proposition 2.2. *Let $f \colon X \to Y$ be a map between H_0-spaces. Up to homotopy equivalence, $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ decompose as products, $X_{\mathbb{Q}} = A \times B$, $Y_{\mathbb{Q}} = A \times C$ such that*

(a) *$p_A \circ f_{\mathbb{Q}} = p_A : A \times B \to A$, $p_C \circ f_{\mathbb{Q}} : A \times B \to C$ is zero in homotopy groups, and $f_{\mathbb{Q}} \circ i_A = i_A : A \to A \times C$. In particular, if $(f_{\mathbb{Q}})_*$ is surjective, then C is rationally a point.*

(b) *If $f_{\mathbb{Q}}$ is an H-map, then $p_C \circ f_{\mathbb{Q}}$ itself is null-homotopic.*

Proof. To prove the result, we translate the above into minimal models and prove

(a) The map f admits a Sullivan minimal model of the form $\varphi \colon (\wedge(V \oplus R), 0) \to (\wedge(V \oplus S), 0)$ with $\varphi(v) = v$ for $v \in V$, and such that $\varphi(R) \in \wedge^{\geq 2}(V \oplus S) \cap \wedge V \otimes \wedge^+(S)$;

(b) If $f_{\mathbb{Q}}$ is an H-map, then f admits a model of the form $\varphi \colon (\wedge(V \oplus K), 0) \to (\wedge(V \oplus S), 0)$ with $\varphi(v) = v$ for $v \in V$ and $\varphi(K) = 0$.

(a) We denote by V a maximal subspace of T such that $Q(\varphi) \colon V \to W$ is injective. Denote by $R \subseteq T$ a complement of V, and by $S \subseteq W$ a complement of im $Q(\varphi)$ in W. Let $\{v_i\}_{i \in I}$ be a graded basis for V. Then the elements $\varphi(v_i)$ are linearly independent indecomposable elements in $\wedge W$. Denote by $\{r_j\}_{j \in J}$ a graded basis for R, and by $\{s_k\}_{k \in K}$ a graded basis for S. With respect to the generators $\{v_i, r_j\}$ for $\wedge T$ and $\{v'_i = \varphi(v_i), s_k\}$ for $\wedge W$, the map φ satisfies $\varphi(v_i) = v'_i$ and $\varphi(R) \subset \wedge^{\geq 2}(W)$. We can thus suppose that $\varphi(v) = v$, and that $\varphi(R)$ is decomposable. We now change generators in R so that $\varphi(R)$ also belongs to the ideal generated by S. Suppose that this is true for $R^{<n}$, and let r be a generator in R^n. If $\varphi(r) = a + b$ with $a \in \wedge V$ and b in the ideal generated by S, we change the generator to $r' = r - a$. The result follows by induction.

(b) Here, we apply the previous step to write $\varphi \colon \wedge(V \oplus K) \to \wedge(V \oplus S)$ with $\varphi(v) = v$ for $v \in V$ and $\varphi(k)$ both decomposable and in $\wedge V \otimes \wedge^+(S)$ for $k \in K$. We now prove by induction that φ is zero on K.

The existence of multiplications on $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ is reflected in their Sullivan models by morphisms of algebras $\Delta_1 \colon \wedge T \to \wedge T \otimes \wedge T$ and $\Delta_2 \colon \wedge W \to \wedge W \otimes \wedge W$ that satisfy $\Delta_1(v) - (v \otimes 1 + 1 \otimes v) \in \wedge^+ T \otimes \wedge^+ T$, and likewise for Δ_2. Furthermore, since $f_{\mathbb{Q}}$ is an H-map, then $(\varphi \otimes \varphi) \circ \Delta_1 = \Delta_2 \circ \varphi$. Assume inductively that we have $\varphi(K^{\leq n}) = 0$, and suppose that $u \in K^{n+1}$. We write

$$\varphi(u) = \varphi_r(u) + \varphi_{r+1}(u) + \cdots + \varphi_m(u)$$

with $\varphi_r(u) \in \wedge^r(V \oplus S)$. By the definition of K, we have $r \geq 2$. Consider a term in $\varphi_r(u)$ that is of minimal length q in $\wedge S$, for some $1 \leq q \leq r$, $s_{i_1} s_{i_2} \cdots s_{i_q} \nu$ with $\nu \in \wedge^{r-q} V$. Then $\Delta_2\varphi(u)$ contains a contribution $s_{i_1} \otimes s_{i_2} \cdots s_{i_q} \nu$, and this term will appear uniquely as such in $\Delta_2\varphi(u) - (1 \otimes \varphi(u) +$

$\varphi(u) \otimes 1)$. On the other hand, $\Delta_1(u) - (1 \otimes u + u \otimes 1) \in \wedge^+(V \oplus K^{\leq n}) \otimes \wedge^+(V \oplus K^{\leq n})$, and so $(\varphi \otimes \varphi)\Delta_1(u) - (1 \otimes \varphi(u) + \varphi(u) \otimes 1)$ cannot contain any occurrence of a term such as $s_{i_1} \otimes s_{i_2} \cdots s_{i_q} \nu$, by our induction hypothesis. In summary, if $\varphi_r(u)$ contains some non-zero term, then we cannot have $(\varphi \otimes \varphi)\Delta_1(u) = \Delta_2\varphi(u)$, which is a contradiction. It follows by induction that $\varphi(K) = 0$. $\square$

We remark in passing that Proposition 2.2 implies the following results:

Corollary 2.3. *Let $f : X \to Y$ be a map between H_0-spaces that is an H-map after rationalization. If $(f_{\mathbb{Q}})_\#$ is zero, then f is rationally null-homotopic.* $\square$

Corollary 2.4. *Let $g : X \to Y$ and $r : Y \to Z$ be maps between H_0-spaces. If $g_{\mathbb{Q}}$ is an H-map and $(r_{\mathbb{Q}})_\#$ is surjective, then their composition $r \circ g$ admits a Sullivan minimal model of the form $\varphi : (\wedge(V \oplus K), 0) \to (\wedge(V \oplus W), 0)$ with $\varphi(v) = v$ for $v \in V$ and $\varphi(K) = 0$.* $\square$

So now suppose that $p : E \to X$ is any map from an H_0-space E to a nilpotent, finite space X. The image of p in rational homotopy groups is of finite dimension, and we may pick a finite basis $\{[f_1], \ldots, [f_k]\}$ in $\pi_{\mathrm{odd}}(X_{\mathbb{Q}})$ for this image. We denote by $\tilde{f_i}$ a lifting of f_i to $E_{\mathbb{Q}}$, and by $\widetilde{\Gamma}_p : S_p \to E_{\mathbb{Q}}$ the product of the $\tilde{f_i}$. Then set $\Gamma_p = p_{\mathbb{Q}} \circ \widetilde{\Gamma}_p : S_p \to X_{\mathbb{Q}}$. This construction gives a commutative diagram

$$\begin{tikzcd}
 & E_{\mathbb{Q}} \arrow[d, "p_{\mathbb{Q}}"] \\
S_p \arrow[ur, "\widetilde{\Gamma}_p"] \arrow[r, "\Gamma_p"'] & X_{\mathbb{Q}}
\end{tikzcd}$$

in which Γ_p is both injective and onto the image of p in rational homotopy groups.

Definition 2.5. The map $\Gamma_p : S_p \to X_{\mathbb{Q}}$ is called a *total Gottlieb element for $X_{\mathbb{Q}}$ with respect to p.*

In general, there may be many choices of total Gottlieb elements with respect to p, and different lifts of each. We keep the notation $\Gamma_X : S_X \to X_{\mathbb{Q}}$ for a total Gottlieb element with respect to the ordinary evaluation fibration $\omega : \mathrm{Map}\,(X, X; 1) \to X$.

Theorem 2.6. *Let*

$$F \xrightarrow{j} E \xrightarrow{p} X$$

be a fibration sequence of nilpotent spaces in which F and E are H_0-spaces and X is a nilpotent, finite space. Let $\Gamma_p : S_p \to X_{\mathbb{Q}}$ be any total Gottlieb element for $X_{\mathbb{Q}}$ with respect to p, and let $\widetilde{\Gamma}_p$ be any lift of Γ_p through $p_{\mathbb{Q}}$. Assume there is an action $A : F_{\mathbb{Q}} \times E_{\mathbb{Q}} \to E_{\mathbb{Q}}$ of $F_{\mathbb{Q}}$ on $E_{\mathbb{Q}}$ that satisfies $A \circ i_1 = j_{\mathbb{Q}}$ and $p_{\mathbb{Q}} \circ A = p_{\mathbb{Q}} \circ p_2 : F_{\mathbb{Q}} \times E_{\mathbb{Q}} \to X_{\mathbb{Q}}$. Then there is a retraction $r : E_{\mathbb{Q}} \to S_p$ of $\widetilde{\Gamma}_p$ such that $p_{\mathbb{Q}} = \Gamma_p \circ r : E_{\mathbb{Q}} \to X_{\mathbb{Q}}$.

Proof. From Proposition 2.2, we assume an identification $E_{\mathbb{Q}} \simeq Y \times Z$, with Y and Z rational H-spaces, together with maps $i : Y \to E_{\mathbb{Q}}$ and $\phi : Y \to F_{\mathbb{Q}}$, with $i_\#$ an injection onto $\mathrm{im}\,(j_{\mathbb{Q}})_\#$ and $j_{\mathbb{Q}} \circ \phi = i$.

Now consider the following commutative diagram:

$$\begin{tikzcd}
Y \times S_p \arrow[r, "p_2"] \arrow[d, "A\circ(\phi\times\widetilde{\Gamma}_p)"', "\simeq"] & S_p \arrow[d, "\Gamma_p"] \\
E_{\mathbb{Q}} \arrow[r, "p_{\mathbb{Q}}"'] \arrow[u, bend right, dotted, "H"'] & X_{\mathbb{Q}}.
\end{tikzcd}$$

Observe that $A \circ (\phi \times \widetilde{\Gamma}_p) \circ i_1 = i \colon Y \to E_{\mathbb{Q}}$. Furthermore, from the long exact homotopy sequence of the fibration, we find that $A \circ (\phi \times \widetilde{\Gamma}_p) \circ i_2 \colon S_p \to E_{\mathbb{Q}}$ has image in homotopy that is complementary to $\operatorname{im}(j_{\mathbb{Q}})_\#$. Hence, $A \circ (\phi \times \widetilde{\Gamma}_p)$ induces an isomorphism in rational homotopy, and thus is a homotopy equivalence. Consequently, there is an inverse (rational) homotopy equivalence $H \colon E_{\mathbb{Q}} \to Y \times S_p$ as indicated in the diagram. Now set $r = p_2 \circ H \colon E_{\mathbb{Q}} \to S_p$. Then we have $r \circ \widetilde{\Gamma}_p = p_2 \circ H \circ A \circ (\phi \times \widetilde{\Gamma}_p) \circ i_2 = 1 \colon S_p \to S_p$, so that r is a retraction of $\widetilde{\Gamma}_p$. Furthermore, since $p \circ A(\varphi \times \widetilde{\Gamma}_p) = \Gamma_p \circ p_2$, we have $\Gamma_p \circ r = \Gamma_p \circ p_2 \circ H = p_{\mathbb{Q}} \colon E_{\mathbb{Q}} \to X_{\mathbb{Q}}$, which gives the desired factorization. $\square$

Proof of Theorem 1.3. Consider first the case of the standard evaluation map. The action

$$A \colon \mathrm{Map}_*(X, X; 1) \times \mathrm{Map}(X, X; 1) \to \mathrm{Map}(X, X; 1),$$

defined by $A(f, g) = g \circ f$ satisfies the hypothesis of Theorem 2.6. Therefore, we may apply the result to the evaluation fibration sequence

$$\mathrm{Map}_*(X, X; 1) \to \mathrm{Map}(X, X; 1) \xrightarrow{\omega} X.$$

Suppose now that $\mathrm{w} \colon E \to X$ is any evaluation map. Then there is an action $A \colon E \times X \to X$ that restricts to w. The adjoint $g \colon E \to \mathrm{Map}(X, X; 1)$ of this action is a lift of w through $\omega \colon \mathrm{Map}(X, X; 1) \to X$. Since the action is associative, the adjoint g is an H-map. Upon rationalizing, we obtain the commutative diagram

$$\begin{tikzcd}
E_{\mathbb{Q}} \arrow[r, "g_{\mathbb{Q}}"] \arrow[dr, "\mathrm{w}_{\mathbb{Q}}"'] & \mathrm{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \arrow[r, "r"] \arrow[d, "\omega_{\mathbb{Q}}"] & S_X \arrow[dl, "\Gamma_X"] \\
& X_{\mathbb{Q}} &
\end{tikzcd}$$

in which $r \colon \mathrm{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \to S_X$ is a retraction of $\widetilde{\Gamma}_X \colon S_X \to \mathrm{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1)$. Since r is a retraction, $r_\#$ is surjective. So by Corollary 2.4, a model of $r \circ g \colon E \to S_X$ has the form $\varphi \colon (\wedge(V \oplus K), 0) \to (\wedge(V \oplus W), 0)$, with $\varphi(v) = v$ for $v \in V$ and $\varphi(K) = 0$. Thus, φ factors in the form

$$\begin{tikzcd}
\wedge(V \oplus K) \arrow[r, "\mathrm{proj}"'] & \wedge V \arrow[r, "\mathrm{incl}"'] & \wedge(V \oplus W) \arrow[l, bend right, "\mathrm{proj}"']
\end{tikzcd}$$

together with the evident retraction of the inclusion $\wedge V \to \wedge(V \oplus W)$, as indicated. When translated into spaces, this implies that $r \circ g_{\mathbb{Q}}$ factors rationally through a rational H-space Y, $E_{\mathbb{Q}} \xrightarrow{q} Y \xrightarrow{j} S_X$. Notice that $j \colon Y \to S_X$ has as its minimal model the projection $\wedge(V \oplus K) \to \wedge V$. Furthermore, q has a right inverse i. Now consider the commutative diagram

$$\begin{tikzcd} & E_{\mathbb{Q}} \arrow[dl, bend right, "q"'] \arrow[d, "\mathrm{w}_{\mathbb{Q}}"] \\ Y \arrow[ur, "i"] \arrow[r, "\Gamma_X \circ j"'] & X_{\mathbb{Q}}\,. \end{tikzcd}$$

The map $\Gamma_X \circ j\colon Y \to X_{\mathbb{Q}}$ is a total Gottlieb element for $X_{\mathbb{Q}}$ with respect to w. Since we have a retraction q of i, which here serves as our lift of $\Gamma_X \circ j$ through $\mathrm{w}_{\mathbb{Q}}$, this total Gottlieb element satisfies the conclusion of the theorem.

The conclusion now follows for every total Gottlieb element. For, suppose we are given another total Gottlieb element $\Gamma'_p\colon S'_p \to X_{\mathbb{Q}}$ for $X_{\mathbb{Q}}$ with lift $\widetilde{\Gamma}'_p\colon S'_p \to E_{\mathbb{Q}}$. Then the map $h = q \circ \widetilde{\Gamma}'_p\colon S'_p \to Y$ is a homotopy equivalence. Therefore, we may define $r' = h^{-1} \circ q\colon E_{\mathbb{Q}} \to S'_p$, which is easily checked to be a retraction of $\widetilde{\Gamma}'_p$ that satisfies $\Gamma'_p \circ r' = \mathrm{w}_{\mathbb{Q}}$. $\square$

Example 2.7. The following example shows that, for the factorization of Theorem 1.3, it is not sufficient to assume that E and the homotopy fiber F are H_0-spaces. Let $q\colon S^3 \times S^3 \times S^3 \to S^9$ be the map obtained by pinching out all but the top cell of the product. As may be checked by a direct computation, the fiber sequence $F \xrightarrow{j} S^3 \times S^3 \times S^3 \xrightarrow{p} S^3 \times S^9$ with $p = (p_1, q)$ has a fiber that is rationally equivalent to $S^3 \times S^3 \times K(\mathbb{Q}, 8)$. Hence, the fiber inclusion j is a map of H_0-spaces. Now p has an image of dimension 1 on rational homotopy groups. Evidently, however, p does not factor through S^3.

3. Gottlieb groups and homotopy monomorphisms

Let $\mathrm{w}\colon E \to X$ be an evaluation map. By Theorem 1.3, $\mathrm{w}_{\mathbb{Q}}$ factors as $\mathrm{w}_{\mathbb{Q}} = \Gamma_{\mathrm{w}} \circ r$, where $r\colon E_{\mathbb{Q}} \to S_{\mathrm{w}}$ is a left inverse of $\widetilde{\Gamma}_{\mathrm{w}}$. As a retraction, r has $\widetilde{\Gamma}_{\mathrm{w}}$ as a right inverse, and so is a *homotopy epimorphism*. We will show that

$$(\Gamma_{\mathrm{w}})_*\colon [A, S_{\mathrm{w}}] \to [A, X_{\mathbb{Q}}]$$

is injective for any nilpotent space A. In order to show this, we establish some technical points concerning Gottlieb groups and rational homotopy monomorphisms.

Suppose X has minimal model $(\wedge W, d)$. By changing generators if necessary, we assume that any element $w \in W$ that satisfies $d(w + \chi) = 0$ for some decomposable χ is itself a cocycle.

The Gottlieb group $G_*(X_{\mathbb{Q}})$ may be identified with the subspace of $\mathrm{Hom}(W, \mathbb{Q})$ formed by those linear maps that extend to derivations of $\wedge W$ that commute with d (see [2] for a discussion of this). Denote by $\overline{\theta}_i$, $1 \le i \le r$, a linear basis of $G_*(X_{\mathbb{Q}})$, and by v_i elements of W with $\overline{\theta}_i(v_j) = \delta_{ij}$. We denote by θ_i an extension of $\overline{\theta}_i$ to a derivation of $\wedge W$ that satisfies $\mathrm{d}\theta_i = (-1)^{|v_i|}\theta_i \mathrm{d}$. We suppose, without loss of generality, that $|v_i| \le |v_j|$ for $i < j$. Then we may – and do – suppose that $\theta_i(v_j) = 0$ for $i > j$. Other than this, however, we have very little control over how the $\overline{\theta}_i$ extend. This point is the main source of the technicalities. We denote by V the vector space generated by the v_i, and by Z a choice of complement in W.

Lemma 3.1. *With notations as above, the spaces Z and V may be chosen so that $\theta_i(Z) \subseteq \wedge V \otimes \wedge^+ Z$ for each i.*

Proof. Let $\mathcal{L}$ denote the Lie algebra of derivations of $\mathcal{M}_X$ generated by the derivations $\theta_1, \ldots, \theta_r$. We prove by induction on k that we may choose Z and V, for which we have $\theta(Z) \subseteq \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$ for any $\theta \in \mathcal{L}$, for all k. Since V is oddly graded, taking $k > r$ establishes the result.

For $k = 1$, we choose $Z = \cap_{i=1}^r \ker(\overline{\theta}_i : W \to \mathbb{Q})$. We directly have the result that $\theta(Z) \subseteq \wedge^+(V \oplus Z)$ for any $\theta \in \mathcal{L}$, because $Z = \cap_{\theta \in \mathcal{L}} \ker(\varepsilon \circ \theta)$.

Now suppose that, for some $k \geq 1$, we have $\theta(Z) \subseteq \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$ for any $\theta \in \mathcal{L}$. For each j and for $z \in Z$ a basis element, we write

$$\theta_j(z) \equiv \sum_{i_1 < i_2 < \cdots < i_k} \lambda_j^{(i_1, i_2, \ldots, i_k)} v_{i_1} v_{i_2} \cdots v_{i_k}$$

modulo terms in $\wedge^{\geq k+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. Then we make a change of basis for Z by replacing each basis element $z \in Z$ with z', where

$$z' = z - \sum_{s=k+1}^{r} \sum_{i_1 < i_2 < \cdots < i_k < s} \lambda_s^{(i_1, i_2, \ldots, i_k)} v_s v_{i_1} v_{i_2} \cdots v_{i_k}.$$

The effect of this basis change in Z is that we have

$$\theta_j(z') \equiv \sum_{i_1 < i_2 < \cdots < i_k | i_k \geq j} \rho_j^{(i_1, i_2, \ldots, i_k)} v_{i_1} v_{i_2} \cdots v_{i_k} \tag{1}$$

modulo terms in $\wedge^{\geq k+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. We now claim that all the coefficients $\rho_j^{(i_1, i_2, \ldots, i_k)}$ that appear in (1) are in fact zero. For, suppose that this is not the case, and let j be the least index for which some $\rho_j^{(i_1, i_2, \ldots, i_k)}$ in (1) is non-zero. Denote by $n \geq j$ the maximum of the i_k with $\rho_j^{(i_1, i_2, \ldots, i_k)} \neq 0$. Then $\theta_n \circ \theta_j(z') = \alpha + \beta$, with $\alpha \neq 0 \in \wedge^{k-1}(v_{i_1}, v_{i_2}, \ldots, v_{n-1})$, and $\beta \in \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$. If $n = j$, then $\theta_n \circ \theta_j = \frac{1}{2}[\theta_n, \theta_j] \in \mathcal{L}$, and this contradicts the induction hypothesis. On the other hand, if $n > j$, then $\theta_j \circ \theta_n(z') = \gamma + \delta$, with γ of length $k-1$ but in $\wedge(v_{i_1}, v_{i_2}, \ldots, \widehat{v}_j, \ldots, v_{n-1}) \otimes \wedge^+(v_n, v_{n+1}, \ldots, v_r)$, and $\delta \in \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$. This gives an element $[\theta_n, \theta_j] \in \mathcal{L}$ that contradicts the induction hypothesis.

To complete the inductive step, we consider $\theta \in \mathcal{L}$. For $z \in Z$, write

$$\theta(z) \equiv \sum_{i_1 < i_2 < \cdots < i_k} \mu^{(i_1, i_2, \ldots, i_k)} v_{i_1} v_{i_2} \cdots v_{i_k}$$

modulo terms in $\wedge^{\geq k+1} V + \wedge V \otimes \wedge^+ Z$. We claim that all the coefficients $\mu^{(i_1, i_2, \ldots, i_k)}$ are zero. For, suppose not, and once again, denote by n the maximum of the i_k for which some $\mu^{(i_1, i_2, \ldots, i_k)} \neq 0$. The composition $\theta_n \circ \theta(z)$ then contains a non-zero term in $\wedge^{k-1} V$. On the other hand, since θ is a derivation, we have $\theta \circ \theta_n(z) \in \wedge^{\geq k+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. Therefore, $[\theta_n, \theta] \in \mathcal{L}$ contradicts the induction hypothesis. The induction is complete. $\square$

Proposition 3.2. *With Z chosen as in Lemma 3.1, we have:*

1. $d(W) \subseteq \wedge V \otimes \wedge^{\geq 2} Z$. *In particular, the ideal generated by Z is d-stable.*
2. *There exists a total Gottlieb element $\Gamma_X : S_X \to X_{\mathbb{Q}}$ with minimal model $\mathcal{M}_\Gamma : (\wedge(V \oplus Z), d) \to (\wedge V, 0)$ that satisfies $\mathcal{M}_\Gamma(Z) = 0$ and $\mathcal{M}_\Gamma(v) = v$ for $v \in V$.*

Proof. (1) First, we show that $d(W) \subseteq \wedge V \otimes \wedge^+ Z$. Suppose this is not true, and that $m \geq 1$ is the minimal length for which any $d(\chi)$ contains a non-zero term in $\wedge^m V$. For such a $\chi \in \wedge V \otimes \wedge Z$, write $d(\chi) = \alpha + \beta$ with $\alpha \neq 0 \in \wedge^m V$ and $\beta \in \wedge^{\geq m+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. Write $\alpha = \alpha' + \alpha'' v_s$ with

$\alpha' \in \wedge^m(v_1, \ldots, v_{s-1})$, and $\alpha'' \neq 0 \in \wedge^{m-1}(v_1, \ldots, v_{s-1})$. Then, $d\theta_s(\chi) = \theta_s d(\chi) = \pm\alpha'' + \theta_s(\beta)$ (recall that $\theta_i(v_j) = 0$ for $i > j$). Using Lemma 3.1 and the fact that θ_s is a derivation, we also have $\theta_s(\beta) \in \wedge^{\geq m} V \oplus (\wedge V \otimes \wedge^+ Z)$. This contradicts our minimal length assumption.

Now, we show that $d(W) \subseteq \wedge V \otimes \wedge^{\geq 2} Z$. For suppose that w is an element of lowest degree to the contrary, and write $d(w) = \sum_{i=1}^q z_i\omega_i + \alpha$, with z_i linearly independent in Z, $|z_1| \leq |z_2| \leq \cdots \leq |z_q|$, $\omega_i \in \wedge V$ and $\alpha \in \wedge^{\geq 2} Z \otimes \wedge V$. Choose the v_s of highest degree in V so that $\omega_q = v_s\gamma + \delta, \gamma \neq 0, \gamma, \delta \in \wedge(v_1, \ldots, v_{s-1})$. We have

$$\theta_s d(w) \equiv z_q\gamma \bmod (\wedge V \otimes \wedge^{\geq 2} Z) + (\wedge V \otimes \wedge Z^{<|z_q|}) + (\wedge V \otimes \wedge(z_1, \ldots, z_{q-1})).$$

Since $\theta_s d(w) = d\theta_s(w)$ and $|\theta_s(w)| < |w|$, this contradicts our lowest-degree assumption on w.

(2) We will define a map $\phi\colon \mathcal{M}_X \to \mathcal{M}_{S_X} \otimes \mathcal{M}_X$ whose composition with the projection onto the first factor $(1 \cdot \epsilon) \circ \phi\colon \mathcal{M}_X \to \mathcal{M}_{S_X} \otimes \mathcal{M}_X \to \mathcal{M}_{S_X}$ is surjective and satisfies $(1 \cdot \epsilon) \circ \phi(Z) = 0$, and whose composition with the projection onto the second factor is the identity, $(\epsilon \cdot 1) \circ \phi = 1\colon \mathcal{M}_X \to \mathcal{M}_X$. The morphism $\mu_\Gamma = (1 \cdot \varepsilon) \circ \varphi$ is then the model of a total Gottlieb element.

We write $\mathcal{M}_{S_X} \otimes \mathcal{M}_X$ as $\wedge V' \otimes \wedge V \otimes \wedge Z$, with $V' = \langle v'_1, \ldots, v'_r\rangle$. First, define a sequence of maps $\phi_1, \ldots, \phi_r\colon \mathcal{M}_X \to \mathcal{M}_{S_X} \otimes \mathcal{M}_X$ by $\phi_1(\chi) = \chi + v'_1\theta_1(\chi)$, and

$$\phi_s(\chi) = \phi_{s-1}(\chi) + v'_s\theta_s\left(\phi_{s-1}(\chi)\right)$$

for $s = 2, \ldots, r$. Then we set $\phi = \phi_r$. An inductive argument shows that any ϕ so defined is a DG algebra map.

Next, we show the following: $\phi(v_1) = v_1 + v'_1$ and, for $i = 2, \ldots, r$,

$$\phi(v_i) = v_i + v'_i + I(v'_1, \ldots, v'_{i-1}).$$

This we do by induction on s. Our induction starts with $s = 1$, where the formulas

$$\phi_1(v_1) = v_1 + v'_1 \quad \text{and} \quad \phi_1(v_i) = v_i + v'_1\theta_1(v_i)$$

give the result. Suppose inductively that we have $\phi_s(v_1) = v_1 + v'_1$ and

$$\phi_s(v_i) = \begin{cases} v_i + v'_i + I(v'_1, \ldots, v'_{i-1}) & \text{if } i = 2, \ldots, s \\ v_i + I(v'_1, \ldots, v'_{i-1}) & \text{if } i = s+1, \ldots, r. \end{cases}$$

We compute as follows: $\phi_{s+1}(v_1) = \phi_s(v_1) + v'_{s+1}\theta_{s+1}(v_1) = v_1 + v'_1$, since $1 < s+1$ and hence $\theta_{s+1}(v_1) = 0$. For $i = 2, \ldots, s$, we have

$$\begin{aligned}\phi_{s+1}(v_i) &= \phi_s(v_i) + v'_{s+1}\theta_{s+1}\left(\phi_s(v_i)\right)\\ &= v_i + v'_i + I(v'_1, \ldots, v'_{i-1}) + v'_{s+1}\theta_{s+1}\left(v_i + v'_i + I(v'_1, \ldots, v'_{i-1})\right)\\ &= v_i + v'_i + I(v'_1, \ldots, v'_{i-1})\end{aligned}$$

since $i < s+1$ and thus $\theta_{s+1}(v_i) = 0$, and also the ideal $I(v'_1, \ldots, v'_{i-1})$ is θ_{s+1}-stable, as $\theta_{s+1}(v'_i) = 0$. Further, $\phi_{s+1}(v_{s+1}) = \phi_s(v_{s+1}) + v'_{s+1}\theta_{s+1}\left(\phi_s(v_{s+1})\right) = v_{s+1} + I(v'_1, \ldots, v'_s) + v'_{s+1}\theta_{s+1}\left(v_{s+1} + I(v'_1, \ldots, v'_s)\right) = v_{s+1} + v'_{s+1} + I(v'_1, \ldots, v'_s)$. Finally, for $i = s+2, \ldots, r$, we have

$$\begin{aligned}\phi_{s+1}(v_i) &= \phi_s(v_i) + v'_{s+1}\theta_{s+1}\left(\phi_s(v_i)\right)\\ &= v_i + I(v'_1, \ldots, v'_{i-1}) + v'_{s+1}\theta_{s+1}\left(v_i + I(v'_1, \ldots, v'_{i-1})\right)\\ &= v_i + I(v'_1, \ldots, v'_{i-1})\end{aligned}$$

since $s+1 \leq i-1$. This completes the induction.

Finally, we observe that, for any $z \in Z$, we have $\phi(z) \in I(Z)$. This follows easily from the fact that Z is θ_i-stable for each i.

From these facts, it is evident that $(1 \cdot \epsilon) \circ \phi$ satisfies $(1 \cdot \epsilon) \circ \phi(v_1) = v_1'$, and $(1 \cdot \epsilon) \circ \phi(v_i) = v_i' + I(v_1', \ldots, v_{i-1}')$, for $i = 2, \ldots, r$. It follows that $(1 \cdot \epsilon) \circ \phi$ is surjective. Furthermore, we have $(1 \cdot \epsilon) \circ \phi(z) = 0$. For the other projection, it is evident from the definition of ϕ that we have $(\epsilon \cdot 1) \circ \phi = 1$. □

The main tool for the proof of Theorem 1.4 is the following criterion developed by Ghorbal [4] for a map to be a homotopy monomorphism in the nilpotent category.

Proposition 3.3 *([4, Th. 3.2.1]). Let $f: X \to Y$ be a map of rational spaces that admits a minimal model of the form $\gamma: (\wedge(V \oplus W), d) \to (\wedge V, \bar{d})$ such that $\gamma(W) = 0$, $\gamma(v) = v$ for $v \in V$, $\mathrm{d}(W) \subseteq \wedge V \otimes \wedge^{\geq 2} W$, and $\mathrm{d}(V) \subseteq \wedge V \oplus (\wedge V \otimes \wedge^{\geq 2} W)$. Then f is a homotopy monomorphism in the nilpotent category.* □

Proof of Theorem 1.4. For the ordinary evaluation map $\omega: \mathrm{Map}(X, X; 1) \to X$, we have that $\Gamma_X: S_X \to X_{\mathbb{Q}}$ is a homotopy monomorphism in the nilpotent category by Propositions 3.3 and 3.2. Now suppose that $\mathrm{w}: E \to X$ is any evaluation map. From Theorem 1.3, we have the following commutative diagram of solid arrows

$$\begin{tikzcd}
 & E_{\mathbb{Q}} \arrow[r, "g"] \arrow[dl, dashed, bend right, "r_{\mathrm{w}}"'] \arrow[dd, "\mathrm{w}_{\mathbb{Q}}"] & \mathrm{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \arrow[dd, "\omega_{\mathbb{Q}}"] \arrow[dr, dashed, "r_X"] & \\
S_{\mathrm{w}} \arrow[ur, "\widetilde{\Gamma}_{\mathrm{w}}"] \arrow[dr, "\Gamma_{\mathrm{w}}"] & & & S_X \arrow[ul, "\widetilde{\Gamma}_X"'] \arrow[dl, "\Gamma_X"] \\
 & X_{\mathbb{Q}} \arrow[r, equal] & X_{\mathbb{Q}} &
\end{tikzcd}$$

with retractions r_X and r_{w} of $\widetilde{\Gamma}_X$ and $\widetilde{\Gamma}_{\mathrm{w}}$ respectively. We define $j: S_{\mathrm{w}} \to S_X$ by $j = r_X \circ g \circ \widetilde{\Gamma}_{\mathrm{w}}$, and claim that this map admits a retraction. Since Γ_{w} and Γ_X are both injective in a rational homotopy and $\Gamma_X \circ j = \Gamma_{\mathrm{w}}$, it follows that j is injective in a rational homotopy. Also, since S_{w} and S_X are (finite) products of odd-dimensional rational spheres, in terms of minimal models, we have a map $\mathcal{M}_j: (\wedge V, 0) \to (\wedge W, 0)$ with $Q(\mathcal{M}_j)$ surjective. But if $Q(\mathcal{M}_j)$ is surjective, so too is $\mathcal{M}_j$. Therefore, we may choose a splitting of $\mathcal{M}_j$ which corresponds to a retraction of j. Since j admits a retraction, it is a homotopy monomorphism. Finally, it follows that Γ_{w} is a composition of homotopy monomorphisms, and hence is a homotopy monomorphism. □

We remark that the fact that Γ_{w} is associated to an evaluation map is key in Theorem 1.4. In particular, we may give the following example of a map $\gamma: S \to X$ from an H_0-space S into X that is injective in rational homotopy but is not a homotopy monomorphism in the nilpotent category.

Example 3.4. Let $S = S_a^3 \times S^5$ and $X = S_a^3 \vee S_b^3 \cup_\alpha e^8$, where α is the triple Whitehead bracket $[a, [a, b]]$. Then $\gamma: S \to X$ is an extension of $(1 \mid [a, b]): S_a^3 \vee S^5 \to X$ obtained using the fact that $[a, [a, b]] = 0$ in $\pi_*(X)$. Consider two maps $h, k: S^2 \times S^3 \to S_a^3 \times S^5$. The map h is the composition

$$S^2 \times S^3 \xrightarrow{p_2} S^3 \xrightarrow{i_1} S_a^3 \times S^5$$

and k is the composition of the inclusion $S^3 \vee S^5 \to S^3 \times S^5$ with the map that consists of collapsing the cell S^2 into a point

$$S^2 \times S^3 \longrightarrow S^2 \times S^3/S^2 = S^3 \vee S^5 \longrightarrow S^3 \times S^5.$$

Clearly, $h_{\mathbb{Q}}$ and $k_{\mathbb{Q}}$ are not homotopic, because they do not induce the same map in rational homology. However, a simple computation using minimal models shows that the compositions $f_{\mathbb{Q}} \circ h_{\mathbb{Q}}$ and $f_{\mathbb{Q}} \circ k_{\mathbb{Q}}$ are homotopic.

4. Evaluation maps and homology

With the notation of Lemma 3.1, decompose V as $V = V' \oplus V''$, with $\mathrm{d}(V') = 0$ and $\dim V' = \dim \operatorname{im}(h_X \circ (w_{\mathbb{Q}})_\#)$. Then we have:

Proposition 4.1. *Any cocycle of $\wedge^+(V \oplus Z)$ is in the ideal generated by $V' \oplus Z$.*

Proof. The proof is similar to that of part (1) of Proposition 3.2. Suppose not, and choose a cocycle χ of the form $\alpha + \beta$, with $\alpha \neq 0 \in \wedge^{\geq m} V''$, $\beta \in I(V', Z)$, the ideal generated by $V' \oplus Z$, with m minimal amongst all cocycles of this form. Write $V'' = \langle v''_1, \dots, v''_s\rangle$ for suitable $s \leq r$, with corresponding derivations $\theta''_1, \dots, \theta''_s$. Then write $\chi = \alpha' + \alpha'' v''_t + \alpha''' + \beta$, with $\alpha' \in \wedge^m(v''_1, \dots, v''_{t-1})$, $\alpha'' \neq 0 \in \wedge^{m-1}(v''_1, \dots, v''_{t-1})$, $\alpha''' \in \wedge^{m+1} V''$. Since each θ_i commutes with the differential, $\theta_i(v')$ is a cocycle for each i. Therefore, we must have that $\theta_i(V') \subseteq \wedge^{\geq m} V'' \oplus I(V', Z)$. Now since θ''_t commutes with the differential, $\theta''_t(\chi)$ is again a cocycle. However, we have $\theta''_t(\chi) = \alpha'' + \theta''_t(\alpha''' + \beta)$ (recall that $\theta_i(v_j) = 0$ for $i > j$). Using Lemma 3.1 and the fact that θ''_t is a derivation, we have $\theta''_t(\alpha''' + \beta) \in I(\wedge^m V'', V', Z)$. This contradicts our minimal length assumption. □

Remark 4.2. Observe that, with the notation of Lemma 3.1, $\operatorname{im} h_X \circ (w_{\mathbb{Q}})_\#$ is identified with the dual of the vector space V' generated by the cocycles in V. Hence, via [10, Lemma 1.1], $X_{\mathbb{Q}}$ decomposes as a product $X_{\mathbb{Q}} \simeq S \times Y$, with S a product of odd-dimensional rational spheres whose model is $(\wedge V', 0)$.

Remark 4.3. Suppose that we have a homotopy equivalence $X \simeq A \times B$. Then clearly, the evaluation map ω_X factors through the product of evaluation maps $\omega_A \times \omega_B$:

$$\begin{tikzcd}
\mathrm{Map}(X, X; 1) \arrow[r, "\simeq"] \arrow[dd, "\omega_X"'] & \mathrm{Map}(A \times B, A; p_1) \times \mathrm{Map}(A \times B, B; p_2) \arrow[d, "(i_1)^* \times (i_2)^*"] \\
& \mathrm{Map}(A, A; 1_A) \times \mathrm{Map}(B, B; 1_B) \arrow[d, "\omega_A \times \omega_B"] \\
X \arrow[r, "\simeq", "h"'] & A \times B.
\end{tikzcd} \tag{2}$$

Proof of Theorem 1.6. Consider $\omega\colon \mathrm{Map}(X, X; 1) \to X$ as a special case first, and suppose $h_X \circ (\omega_{\mathbb{Q}})_\# = 0$. If X is an H_0-space, then the multiplication of $X_{\mathbb{Q}}$ provides a section of $\omega_{\mathbb{Q}}$, so that $H_*(\omega; \mathbb{Q})$ is surjective. If we have $X_{\mathbb{Q}} \simeq S^{2n+1}_{\mathbb{Q}} \times Y$, then we may apply Remark 4.3. As S^{2n+1} is an H_0-space, the above observation establishes that $\omega_{S^{2n+1}}$ is surjective on rational homology. Furthermore, the map $(i_1)^*$ in diagram (2) admits a section, namely $(p_1)^*$, and so it too is surjective on rational homology. It follows that im $H_*(\omega;\mathbb{Q})$ contains at least the $H_*(S^{2n+1};\mathbb{Q})$ factor, and thus is non-zero. This establishes item (3) of Theorem 1.6.

Next, suppose that $h_X \circ (\omega_{\mathbb{Q}})_\# = 0$. We deduce from Propositions 3.2 and 4.1 that a model of Γ_X is given by

$$\mu : (\wedge(V \oplus Z), d) \to (\wedge V, 0)$$

with all cocycles of $\wedge(V \oplus Z)$ in the ideal generated by Z and $\mu(Z) = 0$. Hence, the total Gottlieb element Γ_X induces the trivial homomorphism in rational cohomology.

On the other hand, suppose that $h_X \circ (\omega_{\mathbb{Q}})_\#$ has an image of dimension $r > 0$. Then Remark 4.2 implies that we have $X_{\mathbb{Q}} \simeq S \times Y$, where S is an r-fold product of rational spheres of odd dimensions that correspond to the image of $h_X \circ (\omega_{\mathbb{Q}})_\#$. Now we apply Remark 4.3 and conclude that im $H_*(\omega;\mathbb{Q})$ contains the $H_*(S;\mathbb{Q})$ factor. Furthermore, we have $h_Y \circ (\omega_Y)_\# = 0$, otherwise the image of $h_X \circ (\omega_{\mathbb{Q}})_\#$ would be of dimension $> r$. Therefore, $\widetilde{H}_*(\omega_Y;\mathbb{Q}) = 0$ and the image of $H_*(\omega;\mathbb{Q})$ is precisely the $H_*(S;\mathbb{Q})$ factor.

Now consider a generalized evaluation map $\mathrm{w}\colon E \to X$. We suppose that $\mathrm{im}\, h_X \circ (\omega_{\mathbb{Q}})_\#$ is of dimension r and $\mathrm{im}\, h_X \circ (\mathrm{w}_{\mathbb{Q}})_\#$ is of dimension s. Since w factors through ω, we have $s \le r$. We write $X_{\mathbb{Q}} \simeq S \times Y$ as above, and we obtain a commutative diagram

$$\begin{tikzcd} E_{\mathbb{Q}} \arrow[r, "g"] \arrow[d, "\mathrm{w}_{\mathbb{Q}}"'] & \mathrm{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; e) \arrow[d, "\omega_{\mathbb{Q}}"] \\ X_{\mathbb{Q}} \arrow[r, "\simeq", "h"'] & S \times Y \end{tikzcd}$$

where g is the H-map obtained from the definition of a generalized evaluation map. By Remark 4.3, the coordinate maps $p_1 \circ \omega_{\mathbb{Q}}$ and $p_2 \circ \omega_{\mathbb{Q}}$ factor through $(\omega_S)_{\mathbb{Q}}$ and $(\omega_Y)_{\mathbb{Q}}$ respectively. Because of this factorization, and the fact that $\widetilde{H}_*(\omega_Y;\mathbb{Q}) = 0$, we may make the following identifications:

$$\mathrm{im}\, H_*(\mathrm{w}_{\mathbb{Q}};\mathbb{Q}) \cong \mathrm{im}\, H_*(\omega_{\mathbb{Q}} \circ g;\mathbb{Q}) \cong \mathrm{im}\, H_*(p_1 \circ \omega_{\mathbb{Q}} \circ g;\mathbb{Q}) \subseteq H_*(S;\mathbb{Q}).$$

Since the composition $p_1 \circ \omega_{\mathbb{Q}} \circ g\colon E_{\mathbb{Q}} \to S$ satisfies the hypotheses of Corollary 2.4, it admits a minimal model of the form $\varphi\colon (\wedge V, 0) \to (\wedge W, 0)$ with $\varphi(V) \subseteq W$. Then, the image of $p_1 \circ \omega_{\mathbb{Q}} \circ g\colon E_{\mathbb{Q}} \to S$ in rational homotopy has dimension s, and we may factor its minimal model $\varphi\colon (\wedge V, 0) \to (\wedge W, 0)$ as the composition of a surjection and an injection $\wedge(V_s \oplus K) \to \wedge V_s \to \wedge(V_s \oplus K')$, with V_s a vector space of dimension s isomorphic to the image of $\mathrm{im}\, h_X \circ (\mathrm{w}_{\mathbb{Q}})_\#$. This corresponds to a factorization of $p_1 \circ \omega_{\mathbb{Q}} \circ g\colon E_{\mathbb{Q}} \to S$ as

$$\begin{tikzcd} E_{\mathbb{Q}} \arrow[rr, "p_1 \circ \omega_{\mathbb{Q}} \circ g"] \arrow[dr, "q"'] & & S \simeq S' \times S'' \\ & S' \arrow[ur, "i_1"'] & \end{tikzcd}$$

with S' a product of odd-dimensional rational spheres with minimal model $(\wedge V_s, 0)$. It is now clear that the image in homology of $\mathrm{w}_{\mathbb{Q}}$ is isomorphic to $H_*(S';\mathbb{Q})$. $\square$

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