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Evaluation Maps in Rational Homotopy

Yves Felix

Université Catholique de Louvain

Gregory Lupton

Cleveland State University, g.lupton@csuohio.edu

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Evaluation maps in rational homotopy

Yves Félix, Gregory Lupton

1. Introduction

Let X be a based space and let $\text{Map}(X, X)$ be the space of unbased, or free, maps from X to itself. In general, $\text{Map}(X, X)$ is disconnected; we denote by $\text{Map}(X, X; 1)$ its identity component; that is, the path component that consists of self maps that are (freely) homotopic to the identity. Then we have *the evaluation map* $\omega: \text{Map}(X, X; 1) \rightarrow X$ defined by evaluation at the basepoint of X . This map occupies a central place in the homotopy theory of fibrations (cf. [5–8]).

The evaluation map ω and its rationalization will play a distinguished role in this paper. However, our methods and results apply equally well to other “evaluation maps”. For example, consider the

space $\text{Top}(X, X; 1)$ of self-homeomorphisms of X homotopic to the identity, and the corresponding evaluation map $w: \text{Top}(X, X; 1) \rightarrow X$. Likewise, if X is a smooth manifold, then replace $\text{Top}(X, X)$ with $\text{Diff}(X, X)$, and so forth. A further example of an “evaluation map” concerns *configuration spaces*. Let $F(X, k)$ denote the configuration space that consists of ordered k -tuples of distinct points in a space X , and let (p_1, \dots, p_k) be a choice of basepoint in $F(X, k)$. Then we have an evaluation map $\theta: \text{Top}(X, X; 1) \rightarrow F(X, k)$ given by $\theta(\alpha) = (\alpha(p_1), \dots, \alpha(p_k))$.

Motivated by the preceding examples, we now make a formal definition of the evaluation maps that we consider. Recall that a strict H -space is an H -space (E, μ) with a strict unit. By an *action of E on a space X* , we mean a map $A: E \times X \rightarrow X$ that satisfies $A \circ i_2 = 1: X \rightarrow X$. We say that *the action is associative* if, in addition, we have $A \circ (\mu \times 1) = A \circ (1 \times A)$.

Definition 1.1. Given a strict H -space E and an associative action $A: E \times X \rightarrow X$, define the *generalized evaluation map* associated to A as $w = A \circ i_1: E \rightarrow X$.

Examples 1.2. (1) The action $A: \text{Map}(X, X; 1) \times X \rightarrow X$ given by $A(f, x) = f(x)$ makes $\omega: \text{Map}(X, X; 1) \rightarrow X$ a generalized evaluation map according to Definition 1.1. Similarly for all the other examples mentioned above.

(2) Suppose G is a connected topological group and $A: G \times X \rightarrow X$ is a group action. Then the *orbit map* of the action is a generalized evaluation map $G \rightarrow X$.

(3) More generally, suppose we are given a fibration $X \rightarrow Y \rightarrow B$. Then the connecting map $\partial: \Omega B \rightarrow X$ is a generalized evaluation map. This follows from the usual action of the Moore loops ΩB on the fibre X .

Revert now to the ordinary evaluation map $\omega: \text{Map}(X, X; 1) \rightarrow X$. *For the remainder of the paper, we assume that X is a finite nilpotent complex*, and denote by $X_{\mathbb{Q}}$ its rationalization. Then by [11], the evaluation map for $X_{\mathbb{Q}}$, denoted $\omega_{\mathbb{Q}}$, is the rationalization of ω . We refer to $\omega_{\mathbb{Q}}$ as *the rationalized evaluation map*. Recall that the *n th Gottlieb group of X* , $G_n(X)$, is the subgroup $\text{im } \pi_n(\omega) \subset \pi_n(X)$ [7]. An element $[f] \in \pi_n(X)$ belongs to $G_n(X)$ if and only if $f \vee 1: S^n \vee X \rightarrow X$ extends to $S^n \times X$. Recall also that, by a result of Lang [12], we have $G_n(X_{\mathbb{Q}}) \cong G_n(X) \otimes \mathbb{Q}$. These rationalized Gottlieb groups have played an important role in rational homotopy theory (cf. [1,10]). A result of Félix–Halperin [1, Th. III] implies that $G_{2i}(X_{\mathbb{Q}}) = 0$ for all i , and $\dim G_*(X_{\mathbb{Q}}) < \infty$. Suppose $\{[f_1], [f_2], \dots, [f_r]\}$ is a basis of $G_*(X_{\mathbb{Q}})$ with $f_i: S^{n_i} \rightarrow X$. Denote by F_i the extension of f_i to $S^{n_i} \times X$, and by S_X the product of the odd-dimensional rational spheres $S_{\mathbb{Q}}^{n_i}$. Then we form a map $F: S_X \times X_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$ as the composition

$$F = F_1 \circ (1 \times F_2) \circ \dots \circ (1 \times \dots \times 1 \times F_r).$$

Now set $\Gamma_X = F \circ i: S_X \rightarrow X_{\mathbb{Q}}$, where i denotes the inclusion of the product of spheres as the first r factors. We refer to Γ_X as a *total Gottlieb element of $X_{\mathbb{Q}}$* .

The preceding discussion extends naturally to generalized evaluation maps. Suppose we are given $w: E \rightarrow X$ any generalized evaluation map. In Section 2, we construct a map $\Gamma_w: S_w \rightarrow X_{\mathbb{Q}}$ such that $\text{im } \pi_*(\Gamma_w) \otimes \mathbb{Q} = \text{im } \pi_*(w) \otimes \mathbb{Q}$. As with S_X above, S_w is a finite product of odd-dimensional rational spheres.

Theorem 1.3. *Let $w: E \rightarrow X$ be any generalized evaluation map with X a nilpotent, finite complex. Suppose that $\Gamma_w: S_w \rightarrow X_{\mathbb{Q}}$ is a total Gottlieb element of $X_{\mathbb{Q}}$ with respect to w . Then S_w is a*

retract of $E_{\mathbb{Q}}$, and $w_{\mathbb{Q}}$ factors up to homotopy through Γ_w . In particular, $G_*(X_{\mathbb{Q}}) = 0$ if and only if $\omega_{\mathbb{Q}} : \text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \rightarrow X_{\mathbb{Q}}$ is null-homotopic.

We continue with a theorem related to the homotopy behaviour of the maps $\Gamma_w : S_w \rightarrow X_{\mathbb{Q}}$. Recall that a map $f : X \rightarrow Y$ is a *homotopy monomorphism in the nilpotent category* if, for any nilpotent space A , the induced map of homotopy sets $f_* : [A, X] \rightarrow [A, Y]$ is injective [3].

Theorem 1.4. *Let X be a nilpotent, finite complex and $w : E \rightarrow X$ be any generalized evaluation map. Then $\Gamma_w : S_w \rightarrow X_{\mathbb{Q}}$ is a homotopy monomorphism in the nilpotent category. In particular, a rationalized Gottlieb element $f : S^n \rightarrow X_{\mathbb{Q}}$ is a homotopy monomorphism in the nilpotent category.*

This implies that rationalized Hopf maps are homotopy monomorphisms in the nilpotent category. By contrast, the Hopf map $\eta : S^7 \rightarrow S^4$ is not a homotopy monomorphism [3].

A further consequence of [Theorem 1.4](#) is the classification, up to rational homotopy, of cyclic maps. A map $f : A \rightarrow X$ is called *cyclic* if $(f \mid 1) : A \vee X \rightarrow X$ extends to a map $A \times X \rightarrow X$ [17]. Denote by $G(A, X)$ the set of homotopy classes of cyclic maps from A into X . This is a generalization of the n th Gottlieb group of X , which we obtain by taking $A = S^n$. Upon rationalizing a cyclic map, we obtain a map $f_{\mathbb{Q}} : A \rightarrow X_{\mathbb{Q}}$ in $G(A, X_{\mathbb{Q}})$.

Theorem 1.5. *Let X be a nilpotent, finite complex and let A be any nilpotent space. Then there are bijections of sets*

$$G(A, X_{\mathbb{Q}}) \cong [A, S_X] \cong \bigoplus_r \text{Hom}(H_r(A; \mathbb{Q}), G_r(X_{\mathbb{Q}})).$$

Proof. The first bijection is given by $(\Gamma_X)_* : [A, S_X] \rightarrow G(A, X_{\mathbb{Q}})$. This is a bijection by [Theorems 1.3](#) and [1.4](#). Now remark that S_X has the homotopy type of a product of rational Eilenberg–Mac Lane spaces, $S_X = \prod_{i=1}^r K(\mathbb{Q}, n_i)$. By taking cohomology classes, we thus obtain a bijection $[A, S_X] \xrightarrow{\cong} \bigoplus_{i=1}^r H^{n_i}(A; \mathbb{Q})$. \square

Together with Sam Smith, the second-named author has studied cyclic maps from the rational homotopy point of view in [14]. Many of the results of [14] may be placed in context with the above classification result. For instance, we retrieve [14, Th. 3.2]: If $H^{\text{odd}}(A; \mathbb{Q}) = 0$, then any map $g : A \rightarrow S_w$ must be null-homotopic.

Our last topic is the (co)homological behavior of generalized evaluation maps which, for the ordinary evaluation map, has been studied by Gottlieb [9], Oprea [15,16] and Halperin [10]. Let $h_X : \pi_*(X) \otimes \mathbb{Q} \rightarrow H_*(X; \mathbb{Q})$ denote the rationalized Hurewicz homomorphism. We generalize [15, Th. 1] by proving:

Theorem 1.6. *Let $w : E \rightarrow X$ be any generalized evaluation map with X a finite, nilpotent complex. Then, $\dim \text{im}(h_X \circ (w_{\#} \otimes \mathbb{Q})) = r$ if and only if $\dim \text{im} H_*(w; \mathbb{Q}) = 2^r$. In this case, there is a rational homotopy equivalence $X \simeq_{\mathbb{Q}} S \times Y$, with S a product of r odd-dimensional spheres.*

As $\chi(S \times Y) = 0$, we obtain the following sharpening of [9, Th.3].

Corollary 1.7. *Suppose that $\chi(X) \neq 0$. Then, for every generalized evaluation map $w : E \rightarrow X$, we have $\tilde{H}_*(w; \mathbb{Q}) = 0 : \tilde{H}_*(E; \mathbb{Q}) \rightarrow \tilde{H}_*(X; \mathbb{Q})$.*

On the other hand, the cohomology algebra structure of a symplectic manifold M , or more generally a c -symplectic space [13], does not allow a decomposition of M as $S^{2n+1} \times Y$. Hence, we get:

Corollary 1.8. *Let M be a simply connected, symplectic manifold. Then every generalized evaluation map $w: E \rightarrow M$ is trivial on rational homology. Consequently, if G is a connected Lie group and $a: G \rightarrow M$ is the orbit map of any G -action on M , we have $\widetilde{H}_*(a; \mathbb{Q}) = 0$.*

Finally, and directly from [Theorems 1.3](#) and [1.6](#) we have the following:

Corollary 1.9. *Let $w: E \rightarrow X$ be an evaluation map with X a nilpotent, finite complex. The following are equivalent:*

- (1) *The homomorphism $H_*(w): H_*(E; \mathbb{Q}) \rightarrow H_*(X; \mathbb{Q})$ is surjective;*
- (2) *$\Gamma_w: S_w \rightarrow X$ is a rational homotopy equivalence.*

The text is divided into four parts. In [Section 2](#), we present the factorization results. [Section 3](#) contains the monomorphism theorem. The homological behaviour of generalized evaluation maps is discussed in [Section 4](#).

We assume the reader's familiarity with rational homotopy theory and use the standard notation and terminology for minimal models as presented in [\[2\]](#). The basic facts that we use are as follows: each nilpotent space X has a unique Sullivan minimal model (\mathcal{M}_X, d) in the category of commutative DG (differential graded) algebras over \mathbb{Q} . This DG algebra (\mathcal{M}_X, d) is of the form $\mathcal{M}_X = \wedge V$, a free graded commutative algebra generated by a positively graded vector space V of finite type. The differential d is decomposable, in that $d(V) \subseteq \wedge^{\geq 2} V$, and V admits a basis $\{v_\alpha\}$ indexed by a well-ordered set such that $d(v_\alpha) \in \wedge(\{v_\beta\}_{\beta < \alpha})$. An H_0 -space has a minimal model with zero differential. Each map $f: X \rightarrow Y$ also has a Sullivan minimal model which is a DG algebra map $\mathcal{M}_f: \mathcal{M}_Y \rightarrow \mathcal{M}_X$. The Sullivan minimal model is a complete rational homotopy invariant. We have natural isomorphisms $H(\mathcal{M}_X, d) \cong H^*(X; \mathbb{Q})$ and $Q(\mathcal{M}_X) \cong \text{Hom}(\pi_*(X), \mathbb{Q})$ where $Q(\mathcal{M}_X) \cong V$ denotes the vector space of indecomposable elements in $\wedge V$. If $f, g: X \rightarrow Y$ are maps of *rational spaces*, then f and g are homotopic if and only if \mathcal{M}_f and \mathcal{M}_g are homotopic in an algebraic sense.

2. Factorization of an evaluation fibration

We start this section with a natural generalization of the Félix–Halperin result on Gottlieb groups [\[1, Theorem 3\]](#).

Proposition 2.1. *Let X be a nilpotent space and $p: E \rightarrow X$ be any map with E an H_0 -space. If X has finite rational category, that is, if $\text{cat}_0(X) = r < \infty$, then $p_\#(\pi_{\text{even}}(E_{\mathbb{Q}})) = 0$ and $\dim p_\#(\pi_{\text{odd}}(E_{\mathbb{Q}})) \leq r$.*

Proof. For the first assertion, suppose that $\beta \in \pi_{2i}(E_{\mathbb{Q}})$ satisfies $p_\#(\beta) \neq 0$. We identify $K(\mathbb{Q}, 2i) \simeq \Omega\Sigma S_{\mathbb{Q}}^{2i}$. Let $\epsilon: X \rightarrow \Omega\Sigma X$ denote the adjoint of the (suspension of the) identity. Since $E_{\mathbb{Q}}$ is an H -space, we may choose a retraction $r: \Omega\Sigma E_{\mathbb{Q}} \rightarrow E_{\mathbb{Q}}$ of $\epsilon: E_{\mathbb{Q}} \rightarrow \Omega\Sigma E_{\mathbb{Q}}$ so that $r \circ \epsilon = 1$ and the following diagram commutes:

$$\begin{array}{ccc}
 \Omega\Sigma S_{\mathbb{Q}}^{2i} & \xrightarrow{\Omega\Sigma\beta} & \Omega\Sigma E_{\mathbb{Q}} \\
 \uparrow \epsilon & & \uparrow \epsilon \Big) r \\
 S_{\mathbb{Q}}^{2i} & \xrightarrow{\beta} & E_{\mathbb{Q}}.
 \end{array}$$

That is, β extends to a map $\tilde{\beta} = r \circ \Omega\Sigma\beta$ such that $p \circ \tilde{\beta}: K(\mathbb{Q}, 2i) \rightarrow X_{\mathbb{Q}}$ is injective in homotopy. But then, the mapping theorem of [1] implies that $\infty = \text{cat}_0(K(\mathbb{Q}, 2i)) \leq \text{cat}_0(X) = r$, which is a contradiction. Therefore, we have $p_{\#}(\pi_{\text{even}}(E_{\mathbb{Q}})) = 0$. For the second assertion, consider any finite, linearly independent subset $\{\alpha_1, \dots, \alpha_k\}$ of $\pi_{\text{odd}}(X_{\mathbb{Q}})$ in the image of $p_{\#}$. Choose a $\beta_i \in \pi_{n_i}(E_{\mathbb{Q}})$ with $p_{\#}(\beta_i) = \alpha_i$ for each i . Using the multiplication of $E_{\mathbb{Q}}$, we extend the map $\bigvee_i S_{\mathbb{Q}}^{n_i} \rightarrow E_{\mathbb{Q}}$, defined as β_i on each summand into a map $\tilde{\Gamma}_p: S_p = \prod_{i=1}^k S_{\mathbb{Q}}^{n_i} \rightarrow E_{\mathbb{Q}}$. Since $p \circ \tilde{\Gamma}_p: S_p \rightarrow X_{\mathbb{Q}}$ is injective in homotopy groups, by the mapping theorem we have $k = \text{cat}_0(S_p) \leq r$. The second assertion follows. \square

In order to study generalized evaluation maps $w: E \rightarrow X$, we first present a global structure result concerning maps between H_0 -spaces.

Proposition 2.2. *Let $f: X \rightarrow Y$ be a map between H_0 -spaces. Up to homotopy equivalence, $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ decompose as products, $X_{\mathbb{Q}} = A \times B$, $Y_{\mathbb{Q}} = A \times C$ such that*

- (a) $p_A \circ f_{\mathbb{Q}} = p_A: A \times B \rightarrow A$, $p_C \circ f_{\mathbb{Q}}: A \times B \rightarrow C$ is zero in homotopy groups, and $f_{\mathbb{Q}} \circ i_A = i_A: A \rightarrow A \times C$. In particular, if $(f_{\mathbb{Q}})_*$ is surjective, then C is rationally a point.
- (b) If $f_{\mathbb{Q}}$ is an H -map, then $p_C \circ f_{\mathbb{Q}}$ itself is null-homotopic.

Proof. To prove the result, we translate the above into minimal models and prove

- (a) The map f admits a Sullivan minimal model of the form $\varphi: (\wedge(V \oplus R), 0) \rightarrow (\wedge(V \oplus S), 0)$ with $\varphi(v) = v$ for $v \in V$, and such that $\varphi(R) \in \wedge^{\geq 2}(V \oplus S) \cap \wedge V \otimes \wedge^+(S)$;
- (b) If $f_{\mathbb{Q}}$ is an H -map, then f admits a model of the form $\varphi: (\wedge(V \oplus K), 0) \rightarrow (\wedge(V \oplus S), 0)$ with $\varphi(v) = v$ for $v \in V$ and $\varphi(K) = 0$.

(a) We denote by V a maximal subspace of T such that $Q(\varphi): V \rightarrow W$ is injective. Denote by $R \subseteq T$ a complement of V , and by $S \subseteq W$ a complement of $\text{im } Q(\varphi)$ in W . Let $\{v_i\}_{i \in I}$ be a graded basis for V . Then the elements $\varphi(v_i)$ are linearly independent indecomposable elements in $\wedge W$. Denote by $\{r_j\}_{j \in J}$ a graded basis for R , and by $\{s_k\}_{k \in K}$ a graded basis for S . With respect to the generators $\{v_i, r_j\}$ for $\wedge T$ and $\{v'_i = \varphi(v_i), s_k\}$ for $\wedge W$, the map φ satisfies $\varphi(v_i) = v'_i$ and $\varphi(R) \subset \wedge^{\geq 2}(W)$. We can thus suppose that $\varphi(v) = v$, and that $\varphi(R)$ is decomposable. We now change generators in R so that $\varphi(R)$ also belongs to the ideal generated by S . Suppose that this is true for $R^{<n}$, and let r be a generator in R^n . If $\varphi(r) = a + b$ with $a \in \wedge V$ and b in the ideal generated by S , we change the generator to $r' = r - a$. The result follows by induction.

(b) Here, we apply the previous step to write $\varphi: \wedge(V \oplus K) \rightarrow \wedge(V \oplus S)$ with $\varphi(v) = v$ for $v \in V$ and $\varphi(k)$ both decomposable and in $\wedge V \otimes \wedge^+(S)$ for $k \in K$. We now prove by induction that φ is zero on K .

The existence of multiplications on $X_{\mathbb{Q}}$ and $Y_{\mathbb{Q}}$ is reflected in their Sullivan models by morphisms of algebras $\Delta_1: \wedge T \rightarrow \wedge T \otimes \wedge T$ and $\Delta_2: \wedge W \rightarrow \wedge W \otimes \wedge W$ that satisfy $\Delta_1(v) - (v \otimes 1 + 1 \otimes v) \in \wedge^+ T \otimes \wedge^+ T$, and likewise for Δ_2 . Furthermore, since $f_{\mathbb{Q}}$ is an H -map, then $(\varphi \otimes \varphi) \circ \Delta_1 = \Delta_2 \circ \varphi$. Assume inductively that we have $\varphi(K^{\leq n}) = 0$, and suppose that $u \in K^{n+1}$. We write

$$\varphi(u) = \varphi_r(u) + \varphi_{r+1}(u) + \dots + \varphi_m(u)$$

with $\varphi_r(u) \in \wedge^r(V \oplus S)$. By the definition of K , we have $r \geq 2$. Consider a term in $\varphi_r(u)$ that is of minimal length q in $\wedge S$, for some $1 \leq q \leq r$, $s_{i_1} s_{i_2} \dots s_{i_q} v$ with $v \in \wedge^{r-q} V$. Then $\Delta_2 \varphi(u)$ contains a contribution $s_{i_1} \otimes s_{i_2} \dots s_{i_q} v$, and this term will appear uniquely as such in $\Delta_2 \varphi(u) - (1 \otimes \varphi(u) +$

$\varphi(u) \otimes 1$). On the other hand, $\Delta_1(u) - (1 \otimes u + u \otimes 1) \in \wedge^+(V \oplus K^{\leq n}) \otimes \wedge^+(V \oplus K^{\leq n})$, and so $(\varphi \otimes \varphi)\Delta_1(u) - (1 \otimes \varphi(u) + \varphi(u) \otimes 1)$ cannot contain any occurrence of a term such as $s_{i_1} \otimes s_{i_2} \cdots s_{i_q} v$, by our induction hypothesis. In summary, if $\varphi_r(u)$ contains some non-zero term, then we cannot have $(\varphi \otimes \varphi)\Delta_1(u) = \Delta_2\varphi(u)$, which is a contradiction. It follows by induction that $\varphi(K) = 0$. \square

We remark in passing that [Proposition 2.2](#) implies the following results:

Corollary 2.3. *Let $f: X \rightarrow Y$ be a map between H_0 -spaces that is an H -map after rationalization. If $(f_{\mathbb{Q}})_{\#}$ is zero, then f is rationally null-homotopic. \square*

Corollary 2.4. *Let $g: X \rightarrow Y$ and $r: Y \rightarrow Z$ be maps between H_0 -spaces. If $g_{\mathbb{Q}}$ is an H -map and $(r_{\mathbb{Q}})_{\#}$ is surjective, then their composition $r \circ g$ admits a Sullivan minimal model of the form $\varphi: (\wedge(V \oplus K), 0) \rightarrow (\wedge(V \oplus W), 0)$ with $\varphi(v) = v$ for $v \in V$ and $\varphi(K) = 0$. \square*

So now suppose that $p: E \rightarrow X$ is any map from an H_0 -space E to a nilpotent, finite space X . The image of p in rational homotopy groups is of finite dimension, and we may pick a finite basis $\{[f_1], \dots, [f_k]\}$ in $\pi_{\text{odd}}(X_{\mathbb{Q}})$ for this image. We denote by \tilde{f}_i a lifting of f_i to $E_{\mathbb{Q}}$, and by $\tilde{\Gamma}_p: S_p \rightarrow E_{\mathbb{Q}}$ the product of the \tilde{f}_i . Then set $\Gamma_p = p_{\mathbb{Q}} \circ \tilde{\Gamma}_p: S_p \rightarrow X_{\mathbb{Q}}$. This construction gives a commutative diagram

$$\begin{array}{ccc} & E_{\mathbb{Q}} & \\ \tilde{\Gamma}_p \nearrow & \downarrow p_{\mathbb{Q}} & \\ S_p & \xrightarrow{\Gamma_p} & X_{\mathbb{Q}} \end{array}$$

in which Γ_p is both injective and onto the image of p in rational homotopy groups.

Definition 2.5. The map $\Gamma_p: S_p \rightarrow X_{\mathbb{Q}}$ is called a *total Gottlieb element* for $X_{\mathbb{Q}}$ with respect to p .

In general, there may be many choices of total Gottlieb elements with respect to p , and different lifts of each. We keep the notation $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$ for a total Gottlieb element with respect to the ordinary evaluation fibration $\omega: \text{Map}(X, X; 1) \rightarrow X$.

Theorem 2.6. *Let*

$$F \xrightarrow{j} E \xrightarrow{p} X$$

be a fibration sequence of nilpotent spaces in which F and E are H_0 -spaces and X is a nilpotent, finite space. Let $\Gamma_p: S_p \rightarrow X_{\mathbb{Q}}$ be any total Gottlieb element for $X_{\mathbb{Q}}$ with respect to p , and let $\tilde{\Gamma}_p$ be any lift of Γ_p through $p_{\mathbb{Q}}$. Assume there is an action $A: F_{\mathbb{Q}} \times E_{\mathbb{Q}} \rightarrow E_{\mathbb{Q}}$ of $F_{\mathbb{Q}}$ on $E_{\mathbb{Q}}$ that satisfies $A \circ i_1 = j_{\mathbb{Q}}$ and $p_{\mathbb{Q}} \circ A = p_{\mathbb{Q}} \circ p_2: F_{\mathbb{Q}} \times E_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$. Then there is a retraction $r: E_{\mathbb{Q}} \rightarrow S_p$ of $\tilde{\Gamma}_p$ such that $p_{\mathbb{Q}} = \Gamma_p \circ r: E_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$.

Proof. From [Proposition 2.2](#), we assume an identification $E_{\mathbb{Q}} \simeq Y \times Z$, with Y and Z rational H -spaces, together with maps $i: Y \rightarrow E_{\mathbb{Q}}$ and $\phi: Y \rightarrow F_{\mathbb{Q}}$, with $i_{\#}$ an injection onto $\text{im}(j_{\mathbb{Q}})_{\#}$ and $j_{\mathbb{Q}} \circ \phi = i$.

Now consider the following commutative diagram:

$$\begin{array}{ccc}
 Y \times S_p & \xrightarrow{p_2} & S_p \\
 A \circ (\phi \times \tilde{\Gamma}_p) \downarrow \simeq \nearrow H & & \downarrow \Gamma_p \\
 E_{\mathbb{Q}} & \xrightarrow{p_{\mathbb{Q}}} & X_{\mathbb{Q}}.
 \end{array}$$

Observe that $A \circ (\phi \times \tilde{\Gamma}_p) \circ i_1 = i: Y \rightarrow E_{\mathbb{Q}}$. Furthermore, from the long exact homotopy sequence of the fibration, we find that $A \circ (\phi \times \tilde{\Gamma}_p) \circ i_2: S_p \rightarrow E_{\mathbb{Q}}$ has image in homotopy that is complementary to $\text{im}(j_{\mathbb{Q}})_{\#}$. Hence, $A \circ (\phi \times \tilde{\Gamma}_p)$ induces an isomorphism in rational homotopy, and thus is a homotopy equivalence. Consequently, there is an inverse (rational) homotopy equivalence $H: E_{\mathbb{Q}} \rightarrow Y \times S_p$ as indicated in the diagram. Now set $r = p_2 \circ H: E_{\mathbb{Q}} \rightarrow S_p$. Then we have $r \circ \tilde{\Gamma}_p = p_2 \circ H \circ A \circ (\phi \times \tilde{\Gamma}_p) \circ i_2 = 1: S_p \rightarrow S_p$, so that r is a retraction of $\tilde{\Gamma}_p$. Furthermore, since $p \circ A(\phi \times \tilde{\Gamma}_p) = \Gamma_p \circ p_2$, we have $\Gamma_p \circ r = \Gamma_p \circ p_2 \circ H = p_{\mathbb{Q}}: E_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$, which gives the desired factorization. \square

Proof of Theorem 1.3. Consider first the case of the standard evaluation map. The action

$$A: \text{Map}_*(X, X; 1) \times \text{Map}(X, X; 1) \rightarrow \text{Map}(X, X; 1),$$

defined by $A(f, g) = g \circ f$ satisfies the hypothesis of [Theorem 2.6](#). Therefore, we may apply the result to the evaluation fibration sequence

$$\text{Map}_*(X, X; 1) \rightarrow \text{Map}(X, X; 1) \xrightarrow{\omega} X.$$

Suppose now that $w: E \rightarrow X$ is any evaluation map. Then there is an action $A: E \times X \rightarrow X$ that restricts to w . The adjoint $g: E \rightarrow \text{Map}(X, X; 1)$ of this action is a lift of w through $\omega: \text{Map}(X, X; 1) \rightarrow X$. Since the action is associative, the adjoint g is an H -map. Upon rationalizing, we obtain the commutative diagram

$$\begin{array}{ccccc}
 E_{\mathbb{Q}} & \xrightarrow{g_{\mathbb{Q}}} & \text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) & \xrightarrow{r} & S_X \\
 & \searrow w_{\mathbb{Q}} & \downarrow \omega_{\mathbb{Q}} & \swarrow \Gamma_X & \\
 & & X_{\mathbb{Q}} & &
 \end{array}$$

in which $r: \text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \rightarrow S_X$ is a retraction of $\tilde{\Gamma}_X: S_X \rightarrow \text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1)$. Since r is a retraction, $r_{\#}$ is surjective. So by [Corollary 2.4](#), a model of $r \circ g: E \rightarrow S_X$ has the form $\varphi: (\wedge(V \oplus K), 0) \rightarrow (\wedge(V \oplus W), 0)$, with $\varphi(v) = v$ for $v \in V$ and $\varphi(K) = 0$. Thus, φ factors in the form

$$\wedge(V \oplus K) \xrightarrow{\text{proj}} \wedge V \xleftarrow{\text{incl}} \wedge(V \oplus W)$$

together with the evident retraction of the inclusion $\wedge V \rightarrow \wedge(V \oplus W)$, as indicated. When translated into spaces, this implies that $r \circ g_{\mathbb{Q}}$ factors rationally through a rational H -space Y , $E_{\mathbb{Q}} \xrightarrow{q} Y \xrightarrow{j} S_X$. Notice that $j: Y \rightarrow S_X$ has as its minimal model the projection $\wedge(V \oplus K) \rightarrow \wedge V$. Furthermore, q has

a right inverse i . Now consider the commutative diagram

$$\begin{array}{ccc}
 & E_{\mathbb{Q}} & \\
 q \swarrow & & \downarrow w_{\mathbb{Q}} \\
 Y & \xrightarrow{\Gamma_X \circ j} & X_{\mathbb{Q}} \\
 & \nearrow i & \\
 & &
 \end{array}$$

The map $\Gamma_X \circ j: Y \rightarrow X_{\mathbb{Q}}$ is a total Gottlieb element for $X_{\mathbb{Q}}$ with respect to w . Since we have a retraction q of i , which here serves as our lift of $\Gamma_X \circ j$ through $w_{\mathbb{Q}}$, this total Gottlieb element satisfies the conclusion of the theorem.

The conclusion now follows for every total Gottlieb element. For, suppose we are given another total Gottlieb element $\Gamma'_p: S'_p \rightarrow X_{\mathbb{Q}}$ for $X_{\mathbb{Q}}$ with lift $\tilde{\Gamma}'_p: S'_p \rightarrow E_{\mathbb{Q}}$. Then the map $h = q \circ \tilde{\Gamma}'_p: S'_p \rightarrow Y$ is a homotopy equivalence. Therefore, we may define $r' = h^{-1} \circ q: E_{\mathbb{Q}} \rightarrow S'_p$, which is easily checked to be a retraction of $\tilde{\Gamma}'_p$ that satisfies $\Gamma'_p \circ r' = w_{\mathbb{Q}}$. \square

Example 2.7. The following example shows that, for the factorization of [Theorem 1.3](#), it is not sufficient to assume that E and the homotopy fiber F are H_0 -spaces. Let $q: S^3 \times S^3 \times S^3 \rightarrow S^9$ be the map obtained by pinching out all but the top cell of the product. As may be checked by a direct computation, the fiber sequence $F \xrightarrow{j} S^3 \times S^3 \times S^3 \xrightarrow{p} S^3 \times S^9$ with $p = (p_1, q)$ has a fiber that is rationally equivalent to $S^3 \times S^3 \times K(\mathbb{Q}, 8)$. Hence, the fiber inclusion j is a map of H_0 -spaces. Now p has an image of dimension 1 on rational homotopy groups. Evidently, however, p does not factor through S^3 .

3. Gottlieb groups and homotopy monomorphisms

Let $w: E \rightarrow X$ be an evaluation map. By [Theorem 1.3](#), $w_{\mathbb{Q}}$ factors as $w_{\mathbb{Q}} = \Gamma_w \circ r$, where $r: E_{\mathbb{Q}} \rightarrow S_w$ is a left inverse of $\tilde{\Gamma}_w$. As a retraction, r has $\tilde{\Gamma}_w$ as a right inverse, and so is a *homotopy epimorphism*. We will show that

$$(\Gamma_w)_*: [A, S_w] \rightarrow [A, X_{\mathbb{Q}}]$$

is injective for any nilpotent space A . In order to show this, we establish some technical points concerning Gottlieb groups and rational homotopy monomorphisms.

Suppose X has minimal model $(\wedge W, d)$. By changing generators if necessary, we assume that any element $w \in W$ that satisfies $d(w + \chi) = 0$ for some decomposable χ is itself a cocycle.

The Gottlieb group $G_*(X_{\mathbb{Q}})$ may be identified with the subspace of $\text{Hom}(W, \mathbb{Q})$ formed by those linear maps that extend to derivations of $\wedge W$ that commute with d (see [2] for a discussion of this). Denote by $\bar{\theta}_i$, $1 \leq i \leq r$, a linear basis of $G_*(X_{\mathbb{Q}})$, and by v_i elements of W with $\bar{\theta}_i(v_j) = \delta_{ij}$. We denote by θ_i an extension of $\bar{\theta}_i$ to a derivation of $\wedge W$ that satisfies $d\theta_i = (-1)^{|v_i|}\theta_i d$. We suppose, without loss of generality, that $|v_i| \leq |v_j|$ for $i < j$. Then we may – and do – suppose that $\theta_i(v_j) = 0$ for $i > j$. Other than this, however, we have very little control over how the $\bar{\theta}_i$ extend. This point is the main source of the technicalities. We denote by V the vector space generated by the v_i , and by Z a choice of complement in W .

Lemma 3.1. *With notations as above, the spaces Z and V may be chosen so that $\theta_i(Z) \subseteq \wedge V \otimes \wedge^+ Z$ for each i .*

Proof. Let \mathcal{L} denote the Lie algebra of derivations of \mathcal{M}_X generated by the derivations $\theta_1, \dots, \theta_r$. We prove by induction on k that we may choose Z and V , for which we have $\theta(Z) \subseteq \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$ for any $\theta \in \mathcal{L}$, for all k . Since V is oddly graded, taking $k > r$ establishes the result.

For $k = 1$, we choose $Z = \bigcap_{i=1}^r \ker(\bar{\theta}_i: W \rightarrow \mathbb{Q})$. We directly have the result that $\theta(Z) \subseteq \wedge^+(V \oplus Z)$ for any $\theta \in \mathcal{L}$, because $Z = \bigcap_{\theta \in \mathcal{L}} \ker(\varepsilon \circ \theta)$.

Now suppose that, for some $k \geq 1$, we have $\theta(Z) \subseteq \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$ for any $\theta \in \mathcal{L}$. For each j and for $z \in Z$ a basis element, we write

$$\theta_j(z) \equiv \sum_{i_1 < i_2 < \dots < i_k} \lambda_j^{(i_1, i_2, \dots, i_k)} v_{i_1} v_{i_2} \dots v_{i_k}$$

modulo terms in $\wedge^{\geq k+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. Then we make a change of basis for Z by replacing each basis element $z \in Z$ with z' , where

$$z' = z - \sum_{s=k+1}^r \sum_{i_1 < i_2 < \dots < i_k < s} \lambda_s^{(i_1, i_2, \dots, i_k)} v_s v_{i_1} v_{i_2} \dots v_{i_k}.$$

The effect of this basis change in Z is that we have

$$\theta_j(z') \equiv \sum_{i_1 < i_2 < \dots < i_k | i_k \geq j} \rho_j^{(i_1, i_2, \dots, i_k)} v_{i_1} v_{i_2} \dots v_{i_k} \quad (1)$$

modulo terms in $\wedge^{\geq k+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. We now claim that all the coefficients $\rho_j^{(i_1, i_2, \dots, i_k)}$ that appear in (1) are in fact zero. For, suppose that this is not the case, and let j be the least index for which some $\rho_j^{(i_1, i_2, \dots, i_k)}$ in (1) is non-zero. Denote by $n \geq j$ the maximum of the i_k with $\rho_j^{(i_1, i_2, \dots, i_k)} \neq 0$. Then $\theta_n \circ \theta_j(z') = \alpha + \beta$, with $\alpha \neq 0 \in \wedge^{k-1}(v_{i_1}, v_{i_2}, \dots, v_{n-1})$, and $\beta \in \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$. If $n = j$, then $\theta_n \circ \theta_j = \frac{1}{2}[\theta_n, \theta_j] \in \mathcal{L}$, and this contradicts the induction hypothesis. On the other hand, if $n > j$, then $\theta_j \circ \theta_n(z') = \gamma + \delta$, with γ of length $k-1$ but in $\wedge(v_{i_1}, v_{i_2}, \dots, \widehat{v}_j, \dots, v_{n-1}) \otimes \wedge^+(v_n, v_{n+1}, \dots, v_r)$, and $\delta \in \wedge^{\geq k} V \oplus (\wedge V \otimes \wedge^+ Z)$. This gives an element $[\theta_n, \theta_j] \in \mathcal{L}$ that contradicts the induction hypothesis.

To complete the inductive step, we consider $\theta \in \mathcal{L}$. For $z \in Z$, write

$$\theta(z) \equiv \sum_{i_1 < i_2 < \dots < i_k} \mu^{(i_1, i_2, \dots, i_k)} v_{i_1} v_{i_2} \dots v_{i_k}$$

modulo terms in $\wedge^{\geq k+1} V + \wedge V \otimes \wedge^+ Z$. We claim that all the coefficients $\mu^{(i_1, i_2, \dots, i_k)}$ are zero. For, suppose not, and once again, denote by n the maximum of the i_k for which some $\mu^{(i_1, i_2, \dots, i_k)} \neq 0$. The composition $\theta_n \circ \theta(z)$ then contains a non-zero term in $\wedge^{k-1} V$. On the other hand, since θ is a derivation, we have $\theta \circ \theta_n(z) \in \wedge^{\geq k+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. Therefore, $[\theta_n, \theta] \in \mathcal{L}$ contradicts the induction hypothesis. The induction is complete. \square

Proposition 3.2. *With Z chosen as in Lemma 3.1, we have:*

1. $d(W) \subseteq \wedge V \otimes \wedge^{\geq 2} Z$. In particular, the ideal generated by Z is d -stable.
2. There exists a total Gottlieb element $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$ with minimal model $\mathcal{M}_{\Gamma}: (\wedge(V \oplus Z), d) \rightarrow (\wedge V, 0)$ that satisfies $\mathcal{M}_{\Gamma}(Z) = 0$ and $\mathcal{M}_{\Gamma}(v) = v$ for $v \in V$.

Proof. (1) First, we show that $d(W) \subseteq \wedge V \otimes \wedge^+ Z$. Suppose this is not true, and that $m \geq 1$ is the minimal length for which any $d(\chi)$ contains a non-zero term in $\wedge^m V$. For such a $\chi \in \wedge V \otimes \wedge Z$, write $d(\chi) = \alpha + \beta$ with $\alpha \neq 0 \in \wedge^m V$ and $\beta \in \wedge^{\geq m+1} V \oplus (\wedge V \otimes \wedge^+ Z)$. Write $\alpha = \alpha' + \alpha'' v_s$ with

$\alpha' \in \wedge^m(v_1, \dots, v_{s-1})$, and $\alpha'' \neq 0 \in \wedge^{m-1}(v_1, \dots, v_{s-1})$. Then, $d\theta_s(\chi) = \theta_s d(\chi) = \pm\alpha'' + \theta_s(\beta)$ (recall that $\theta_i(v_j) = 0$ for $i > j$). Using [Lemma 3.1](#) and the fact that θ_s is a derivation, we also have $\theta_s(\beta) \in \wedge^{\geq m} V \oplus (\wedge V \otimes \wedge^+ Z)$. This contradicts our minimal length assumption.

Now, we show that $d(W) \subseteq \wedge V \otimes \wedge^{\geq 2} Z$. For suppose that w is an element of lowest degree to the contrary, and write $d(w) = \sum_{i=1}^q z_i \omega_i + \alpha$, with z_i linearly independent in Z , $|z_1| \leq |z_2| \leq \dots \leq |z_q|$, $\omega_i \in \wedge V$ and $\alpha \in \wedge^{\geq 2} Z \otimes \wedge V$. Choose the v_s of highest degree in V so that $\omega_q = v_s \gamma + \delta$, $\gamma \neq 0$, $\gamma, \delta \in \wedge(v_1, \dots, v_{s-1})$. We have

$$\theta_s d(w) \equiv z_q \gamma \bmod (\wedge V \otimes \wedge^{\geq 2} Z) + (\wedge V \otimes \wedge Z^{<|z_q|}) + (\wedge V \otimes \wedge(z_1, \dots, z_{q-1})).$$

Since $\theta_s d(w) = d\theta_s(w)$ and $|\theta_s(w)| < |w|$, this contradicts our lowest-degree assumption on w .

(2) We will define a map $\phi: \mathcal{M}_X \rightarrow \mathcal{M}_{S_X} \otimes \mathcal{M}_X$ whose composition with the projection onto the first factor $(1 \cdot \epsilon) \circ \phi: \mathcal{M}_X \rightarrow \mathcal{M}_{S_X} \otimes \mathcal{M}_X \rightarrow \mathcal{M}_{S_X}$ is surjective and satisfies $(1 \cdot \epsilon) \circ \phi(Z) = 0$, and whose composition with the projection onto the second factor is the identity, $(\epsilon \cdot 1) \circ \phi = 1: \mathcal{M}_X \rightarrow \mathcal{M}_X$. The morphism $\mu_\Gamma = (1 \cdot \epsilon) \circ \phi$ is then the model of a total Gottlieb element.

We write $\mathcal{M}_{S_X} \otimes \mathcal{M}_X$ as $\wedge V' \otimes \wedge V \otimes \wedge Z$, with $V' = \langle v'_1, \dots, v'_r \rangle$. First, define a sequence of maps $\phi_1, \dots, \phi_r: \mathcal{M}_X \rightarrow \mathcal{M}_{S_X} \otimes \mathcal{M}_X$ by $\phi_1(\chi) = \chi + v'_1 \theta_1(\chi)$, and

$$\phi_s(\chi) = \phi_{s-1}(\chi) + v'_s \theta_s(\phi_{s-1}(\chi))$$

for $s = 2, \dots, r$. Then we set $\phi = \phi_r$. An inductive argument shows that any ϕ so defined is a DG algebra map.

Next, we show the following: $\phi(v_1) = v_1 + v'_1$ and, for $i = 2, \dots, r$,

$$\phi(v_i) = v_i + v'_i + I(v'_1, \dots, v'_{i-1}).$$

This we do by induction on s . Our induction starts with $s = 1$, where the formulas

$$\phi_1(v_1) = v_1 + v'_1 \quad \text{and} \quad \phi_1(v_i) = v_i + v'_1 \theta_1(v_i)$$

give the result. Suppose inductively that we have $\phi_s(v_1) = v_1 + v'_1$ and

$$\phi_s(v_i) = \begin{cases} v_i + v'_i + I(v'_1, \dots, v'_{i-1}) & \text{if } i = 2, \dots, s \\ v_i + I(v'_1, \dots, v'_{i-1}) & \text{if } i = s+1, \dots, r. \end{cases}$$

We compute as follows: $\phi_{s+1}(v_1) = \phi_s(v_1) + v'_{s+1} \theta_{s+1}(v_1) = v_1 + v'_1$, since $1 < s+1$ and hence $\theta_{s+1}(v_1) = 0$. For $i = 2, \dots, s$, we have

$$\begin{aligned} \phi_{s+1}(v_i) &= \phi_s(v_i) + v'_{s+1} \theta_{s+1}(\phi_s(v_i)) \\ &= v_i + v'_i + I(v'_1, \dots, v'_{i-1}) + v'_{s+1} \theta_{s+1}(v_i + v'_i + I(v'_1, \dots, v'_{i-1})) \\ &= v_i + v'_i + I(v'_1, \dots, v'_{i-1}) \end{aligned}$$

since $i < s+1$ and thus $\theta_{s+1}(v_i) = 0$, and also the ideal $I(v'_1, \dots, v'_{i-1})$ is θ_{s+1} -stable, as $\theta_{s+1}(v'_i) = 0$. Further, $\phi_{s+1}(v_{s+1}) = \phi_s(v_{s+1}) + v'_{s+1} \theta_{s+1}(\phi_s(v_{s+1})) = v_{s+1} + I(v'_1, \dots, v'_s) + v'_{s+1} \theta_{s+1}(v_{s+1} + I(v'_1, \dots, v'_s)) = v_{s+1} + v'_{s+1} + I(v'_1, \dots, v'_s)$. Finally, for $i = s+2, \dots, r$, we have

$$\begin{aligned} \phi_{s+1}(v_i) &= \phi_s(v_i) + v'_{s+1} \theta_{s+1}(\phi_s(v_i)) \\ &= v_i + I(v'_1, \dots, v'_{i-1}) + v'_{s+1} \theta_{s+1}(v_i + I(v'_1, \dots, v'_{i-1})) \\ &= v_i + I(v'_1, \dots, v'_{i-1}) \end{aligned}$$

since $s+1 \leq i-1$. This completes the induction.

Finally, we observe that, for any $z \in Z$, we have $\phi(z) \in I(Z)$. This follows easily from the fact that Z is θ_i -stable for each i .

From these facts, it is evident that $(1 \cdot \epsilon) \circ \phi$ satisfies $(1 \cdot \epsilon) \circ \phi(v_1) = v'_1$, and $(1 \cdot \epsilon) \circ \phi(v_i) = v'_i + I(v'_1, \dots, v'_{i-1})$, for $i = 2, \dots, r$. It follows that $(1 \cdot \epsilon) \circ \phi$ is surjective. Furthermore, we have $(1 \cdot \epsilon) \circ \phi(z) = 0$. For the other projection, it is evident from the definition of ϕ that we have $(\epsilon \cdot 1) \circ \phi = 1$. \square

The main tool for the proof of [Theorem 1.4](#) is the following criterion developed by Ghorbal [4] for a map to be a homotopy monomorphism in the nilpotent category.

Proposition 3.3 (*[4, Th. 3.2.1]*). *Let $f: X \rightarrow Y$ be a map of rational spaces that admits a minimal model of the form $\gamma: (\wedge(V \oplus W), d) \rightarrow (\wedge V, \bar{d})$ such that $\gamma(W) = 0$, $\gamma(v) = v$ for $v \in V$, $d(W) \subseteq \wedge V \otimes \wedge^{\geq 2} W$, and $d(V) \subseteq \wedge V \oplus (\wedge V \otimes \wedge^{\geq 2} W)$. Then f is a homotopy monomorphism in the nilpotent category.* \square

Proof of Theorem 1.4. For the ordinary evaluation map $\omega: \text{Map}(X, X; 1) \rightarrow X$, we have that $\Gamma_X: S_X \rightarrow X_{\mathbb{Q}}$ is a homotopy monomorphism in the nilpotent category by [Propositions 3.3](#) and [3.2](#). Now suppose that $w: E \rightarrow X$ is any evaluation map. From [Theorem 1.3](#), we have the following commutative diagram of solid arrows

$$\begin{array}{ccccc}
 & & E_{\mathbb{Q}} & \xrightarrow{g} & \text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \\
 r_w \swarrow & & \downarrow & & \downarrow \\
 S_w & \xrightarrow{\tilde{\Gamma}_w} & E_{\mathbb{Q}} & & \text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1) \\
 \downarrow \Gamma_w & & \downarrow w_{\mathbb{Q}} & & \downarrow \tilde{\Gamma}_X \\
 X_{\mathbb{Q}} & \xrightarrow{\Gamma_X} & X_{\mathbb{Q}} & \xrightarrow{\omega_{\mathbb{Q}}} & X_{\mathbb{Q}} \\
 & & \downarrow & & \downarrow \\
 & & X_{\mathbb{Q}} & \xrightarrow{\Gamma_X} & X_{\mathbb{Q}}
 \end{array}$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image. The image shows a more complex diagram with dashed arrows r_w and r_X and a dashed arrow $\tilde{\Gamma}_X$ from $\text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; 1)$ to S_X . The bottom row shows $X_{\mathbb{Q}} \xrightarrow{\Gamma_X} X_{\mathbb{Q}} \xrightarrow{\omega_{\mathbb{Q}}} X_{\mathbb{Q}}$ and $X_{\mathbb{Q}} \xrightarrow{\Gamma_X} X_{\mathbb{Q}}$ with a double line between the two $X_{\mathbb{Q}}$ nodes.)

with retractions r_X and r_w of $\tilde{\Gamma}_X$ and $\tilde{\Gamma}_w$ respectively. We define $j: S_w \rightarrow S_X$ by $j = r_X \circ g \circ \tilde{\Gamma}_w$, and claim that this map admits a retraction. Since Γ_w and Γ_X are both injective in a rational homotopy and $\Gamma_X \circ j = \Gamma_w$, it follows that j is injective in a rational homotopy. Also, since S_w and S_X are (finite) products of odd-dimensional rational spheres, in terms of minimal models, we have a map $\mathcal{M}_j: (\wedge V, 0) \rightarrow (\wedge W, 0)$ with $Q(\mathcal{M}_j)$ surjective. But if $Q(\mathcal{M}_j)$ is surjective, so too is \mathcal{M}_j . Therefore, we may choose a splitting of \mathcal{M}_j which corresponds to a retraction of j . Since j admits a retraction, it is a homotopy monomorphism. Finally, it follows that Γ_w is a composition of homotopy monomorphisms, and hence is a homotopy monomorphism. \square

We remark that the fact that Γ_w is associated to an evaluation map is key in [Theorem 1.4](#). In particular, we may give the following example of a map $\gamma: S \rightarrow X$ from an H_0 -space S into X that is injective in rational homotopy but is not a homotopy monomorphism in the nilpotent category.

Example 3.4. Let $S = S_a^3 \times S^5$ and $X = S_a^3 \vee S_b^3 \cup_{\alpha} e^8$, where α is the triple Whitehead bracket $[a, [a, b]]$. Then $\gamma: S \rightarrow X$ is an extension of $(1 \mid [a, b]): S_a^3 \vee S^5 \rightarrow X$ obtained using the fact that $[a, [a, b]] = 0$ in $\pi_*(X)$. Consider two maps $h, k: S^2 \times S^3 \rightarrow S_a^3 \times S^5$. The map h is the composition

$$S^2 \times S^3 \xrightarrow{p_2} S^3 \xrightarrow{i_1} S_a^3 \times S^5$$

and k is the composition of the inclusion $S^3 \vee S^5 \rightarrow S^3 \times S^5$ with the map that consists of collapsing the cell S^2 into a point

$$S^2 \times S^3 \longrightarrow S^2 \times S^3 / S^2 = S^3 \vee S^5 \longrightarrow S^3 \times S^5.$$

Clearly, $h_{\mathbb{Q}}$ and $k_{\mathbb{Q}}$ are not homotopic, because they do not induce the same map in rational homology. However, a simple computation using minimal models shows that the compositions $f_{\mathbb{Q}} \circ h_{\mathbb{Q}}$ and $f_{\mathbb{Q}} \circ k_{\mathbb{Q}}$ are homotopic.

4. Evaluation maps and homology

With the notation of [Lemma 3.1](#), decompose V as $V = V' \oplus V''$, with $d(V') = 0$ and $\dim V' = \dim \text{im}(h_X \circ (w_{\mathbb{Q}})_{\#})$. Then we have:

Proposition 4.1. *Any cocycle of $\wedge^+(V \oplus Z)$ is in the ideal generated by $V' \oplus Z$.*

Proof. The proof is similar to that of part (1) of [Proposition 3.2](#). Suppose not, and choose a cocycle χ of the form $\alpha + \beta$, with $\alpha \neq 0 \in \wedge^{\geq m} V''$, $\beta \in I(V', Z)$, the ideal generated by $V' \oplus Z$, with m minimal amongst all cocycles of this form. Write $V'' = \langle v_1'', \dots, v_s'' \rangle$ for suitable $s \leq r$, with corresponding derivations $\theta_1'', \dots, \theta_s''$. Then write $\chi = \alpha' + \alpha'' v_t'' + \alpha''' + \beta$, with $\alpha' \in \wedge^m(v_1'', \dots, v_{t-1}'')$, $\alpha'' \neq 0 \in \wedge^{m-1}(v_1'', \dots, v_{t-1}'')$, $\alpha''' \in \wedge^{m+1} V''$. Since each θ_i commutes with the differential, $\theta_i(v')$ is a cocycle for each i . Therefore, we must have that $\theta_i(V') \subseteq \wedge^{\geq m} V'' \oplus I(V', Z)$. Now since θ_t'' commutes with the differential, $\theta_t''(\chi)$ is again a cocycle. However, we have $\theta_t''(\chi) = \alpha'' + \theta_t''(\alpha''' + \beta)$ (recall that $\theta_i(v_j) = 0$ for $i > j$). Using [Lemma 3.1](#) and the fact that θ_t'' is a derivation, we have $\theta_t''(\alpha''' + \beta) \in I(\wedge^m V'', V', Z)$. This contradicts our minimal length assumption. \square

Remark 4.2. Observe that, with the notation of [Lemma 3.1](#), $\text{im } h_X \circ (w_{\mathbb{Q}})_{\#}$ is identified with the dual of the vector space V' generated by the cocycles in V . Hence, via [[10](#), [Lemma 1.1](#)], $X_{\mathbb{Q}}$ decomposes as a product $X_{\mathbb{Q}} \simeq S \times Y$, with S a product of odd-dimensional rational spheres whose model is $(\wedge V', 0)$.

Remark 4.3. Suppose that we have a homotopy equivalence $X \simeq A \times B$. Then clearly, the evaluation map ω_X factors through the product of evaluation maps $\omega_A \times \omega_B$:

$$\begin{array}{ccc} \text{Map}(X, X; 1) & \xrightarrow{\simeq} & \text{Map}(A \times B, A; p_1) \times \text{Map}(A \times B, B; p_2) & (2) \\ \downarrow \omega_X & & \downarrow (i_1)^* \times (i_2)^* & \\ & & \text{Map}(A, A; 1_A) \times \text{Map}(B, B; 1_B) & \\ & & \downarrow \omega_A \times \omega_B & \\ X & \xrightarrow[\hbar]{\simeq} & A \times B. & \end{array}$$

Proof of Theorem 1.6. Consider $\omega: \text{Map}(X, X; 1) \rightarrow X$ as a special case first, and suppose $h_X \circ (w_{\mathbb{Q}})_{\#} = 0$. If X is an H_0 -space, then the multiplication of $X_{\mathbb{Q}}$ provides a section of $\omega_{\mathbb{Q}}$, so that $H_*(\omega; \mathbb{Q})$ is surjective. If we have $X_{\mathbb{Q}} \simeq S_{\mathbb{Q}}^{2n+1} \times Y$, then we may apply [Remark 4.3](#). As S^{2n+1} is an H_0 -space, the above observation establishes that $\omega_{S^{2n+1}}$ is surjective on rational homology. Furthermore, the map $(i_1)^*$ in diagram (2) admits a section, namely $(p_1)^*$, and so it too is surjective on rational homology. It follows

that $\text{im } H_*(\omega; \mathbb{Q})$ contains at least the $H_*(S^{2n+1}; \mathbb{Q})$ factor, and thus is non-zero. This establishes item (3) of [Theorem 1.6](#).

Next, suppose that $h_X \circ (\omega_{\mathbb{Q}})_{\#} = 0$. We deduce from [Propositions 3.2](#) and [4.1](#) that a model of Γ_X is given by

$$\mu : (\wedge(V \oplus Z), d) \rightarrow (\wedge V, 0)$$

with all cocycles of $\wedge(V \oplus Z)$ in the ideal generated by Z and $\mu(Z) = 0$. Hence, the total Gottlieb element Γ_X induces the trivial homomorphism in rational cohomology.

On the other hand, suppose that $h_X \circ (\omega_{\mathbb{Q}})_{\#}$ has an image of dimension $r > 0$. Then [Remark 4.2](#) implies that we have $X_{\mathbb{Q}} \simeq S \times Y$, where S is an r -fold product of rational spheres of odd dimensions that correspond to the image of $h_X \circ (\omega_{\mathbb{Q}})_{\#}$. Now we apply [Remark 4.3](#) and conclude that $\text{im } H_*(\omega; \mathbb{Q})$ contains the $H_*(S; \mathbb{Q})$ factor. Furthermore, we have $h_Y \circ (\omega_Y)_{\#} = 0$, otherwise the image of $h_X \circ (\omega_{\mathbb{Q}})_{\#}$ would be of dimension $> r$. Therefore, $\tilde{H}_*(\omega_Y; \mathbb{Q}) = 0$ and the image of $H_*(\omega; \mathbb{Q})$ is precisely the $H_*(S; \mathbb{Q})$ factor.

Now consider a generalized evaluation map $w: E \rightarrow X$. We suppose that $\text{im } h_X \circ (\omega_{\mathbb{Q}})_{\#}$ is of dimension r and $\text{im } h_X \circ (w_{\mathbb{Q}})_{\#}$ is of dimension s . Since w factors through ω , we have $s \leq r$. We write $X_{\mathbb{Q}} \simeq S \times Y$ as above, and we obtain a commutative diagram

$$\begin{array}{ccc} E_{\mathbb{Q}} & \xrightarrow{g} & \text{Map}(X_{\mathbb{Q}}, X_{\mathbb{Q}}; e) \\ w_{\mathbb{Q}} \downarrow & & \downarrow \omega_{\mathbb{Q}} \\ X_{\mathbb{Q}} & \xrightarrow[\cong]{h} & S \times Y \end{array}$$

where g is the H -map obtained from the definition of a generalized evaluation map. By [Remark 4.3](#), the coordinate maps $p_1 \circ \omega_{\mathbb{Q}}$ and $p_2 \circ \omega_{\mathbb{Q}}$ factor through $(\omega_S)_{\mathbb{Q}}$ and $(\omega_Y)_{\mathbb{Q}}$ respectively. Because of this factorization, and the fact that $\tilde{H}_*(\omega_Y; \mathbb{Q}) = 0$, we may make the following identifications:

$$\text{im } H_*(w_{\mathbb{Q}}; \mathbb{Q}) \cong \text{im } H_*(\omega_{\mathbb{Q}} \circ g; \mathbb{Q}) \cong \text{im } H_*(p_1 \circ \omega_{\mathbb{Q}} \circ g; \mathbb{Q}) \subseteq H_*(S; \mathbb{Q}).$$

Since the composition $p_1 \circ \omega_{\mathbb{Q}} \circ g: E_{\mathbb{Q}} \rightarrow S$ satisfies the hypotheses of [Corollary 2.4](#), it admits a minimal model of the form $\varphi: (\wedge V, 0) \rightarrow (\wedge W, 0)$ with $\varphi(V) \subseteq W$. Then, the image of $p_1 \circ \omega_{\mathbb{Q}} \circ g: E_{\mathbb{Q}} \rightarrow S$ in rational homotopy has dimension s , and we may factor its minimal model $\varphi: (\wedge V, 0) \rightarrow (\wedge W, 0)$ as the composition of a surjection and an injection $\wedge(V_s \oplus K) \rightarrow \wedge V_s \rightarrow \wedge(V_s \oplus K')$, with V_s a vector space of dimension s isomorphic to the image of $\text{im } h_X \circ (w_{\mathbb{Q}})_{\#}$. This corresponds to a factorization of $p_1 \circ \omega_{\mathbb{Q}} \circ g: E_{\mathbb{Q}} \rightarrow S$ as

$$\begin{array}{ccc} E_{\mathbb{Q}} & \xrightarrow{p_1 \circ \omega_{\mathbb{Q}} \circ g} & S \simeq S' \times S'' \\ q \searrow & & \nearrow i_1 \\ & S' & \end{array}$$

with S' a product of odd-dimensional rational spheres with minimal model $(\wedge V_s, 0)$. It is now clear that the image in homology of $w_{\mathbb{Q}}$ is isomorphic to $H_*(S'; \mathbb{Q})$. \square

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