Gottlieb Groups of Function Spaces

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Abstract. We analyze the Gottlieb groups of function spaces. Our results lead to explicit decompositions of the Gottlieb groups of many function spaces $\mathrm{map}(X, Y)$—including the (iterated) free loop space of Y—directly in terms of the Gottlieb groups of Y. More generally, we give explicit decompositions of the generalized Gottlieb groups of $\mathrm{map}(X, Y)$ directly in terms of generalized Gottlieb groups of Y. Particular cases of our results relate to the torus homotopy groups of Fox. We draw some consequences for the classification of T-spaces and G-spaces. For X, Y finite and Y simply connected, we give a formula for the ranks of the Gottlieb groups of $\mathrm{map}(X, Y)$ in terms of the Betti numbers of X and the ranks of the Gottlieb groups of Y. Under these hypotheses, the Gottlieb groups of $\mathrm{map}(X, Y)$ are finite groups in all but finitely many degrees.

1. Introduction: Description of Results

Let $\mathrm{aut}_1 Y$ denote the component of the function space of (unbased) self-homotopy equivalences of Y that consists of self maps (freely) homotopic to the identity map of Y. For Y a based space, evaluation at the basepoint of Y gives the *evaluation map* $\omega\colon \mathrm{aut}_1 Y \to Y$, which induces a homomorphism on nth homotopy groups

$$\omega_\#\colon \pi_n\big(\mathrm{aut}_1 Y\big) \to \pi_n(Y).$$

The *nth Gottlieb group of Y*, denoted $G_n(Y)$, is the image of $\omega_\#$ in $\pi_n(Y)$ [7].

Because $\omega\colon \mathrm{aut}_1 Y \to Y$ may be identified with the connecting map of the universal fibration for fibrations with fibre Y, the Gottlieb groups are important universal objects that feature in a variety of contexts. Recently, for example, they have appeared as objects of interest in results in symplectic topology [10, Lem.2.2], and in string topology [3, Th.2]. Unfortunately, it has proved difficult to calculate Gottlieb groups. In part, this may be due to a lack of functoriality: a map $f\colon X \to Y$ in general does not satisfy $f_\#(G_n(X)) \subseteq G_n(Y)$.

Our goal in this paper is to analyze the Gottlieb groups of function spaces, including important cases such as the free loop space. The prototype of our results is the following. Let $\Lambda Y = \mathrm{map}(S^1, Y; 0)$ denote the (null component of the) free loop space of Y.

Theorem 1. *For $n \geq 1$, we have*

$$G_n(\Lambda Y) \cong G_n(Y) \oplus G_{n+1}(Y).$$

Note that Gottlieb groups are abelian; the sum in the above is the direct sum of abelian groups. This basic result may be generalized greatly. We indicate the various generalizations in this introduction; the actual theorems in the body of the paper are more general than the samples we now give.

In fact we are able to give a similar decomposition for the (generalized) Gottlieb groups of a general function space. Let $\mathrm{map}(X, Y)$ denote the component of the null map in the function space of (unbased) maps from X to Y. We will often write this as $\mathrm{map}(X, Y; 0)$, to emphasize that we are dealing only with null components here. In what follows, our hypotheses on the spaces X and Y are very mild: they are specified at the start of Section 2. We assume X and Y are connected, but do not require any higher connectivity or finiteness hypotheses. For a suspension ΣA, denote by $[\Sigma A, Z]$ the group of based homotopy classes of based maps $\Sigma A \to Z$. Then post-composition with the evaluation map $\omega\colon \mathrm{aut}_1 Z \to Z$ induces a homomorphism of groups of homotopy classes

$$\omega_*\colon [\Sigma A, \mathrm{aut}_1 Z] \to [\Sigma A, Z],$$

whose image we denote by $\mathcal{G}(\Sigma A, Y)$, and call a *generalized Gottlieb group* (see [14]). Because $\mathrm{aut}_1 Z$ is an H-space, the group $[\Sigma A, \mathrm{aut}_1 Z]$ is abelian. Therefore, this latter description makes clear that $\mathcal{G}(\Sigma A, Z)$ is an abelian subgroup of $[\Sigma A, Z]$. For the space A, which always appears suspended, we allow it to be disconnected, so that, for example, we may obtain a (bouquet of) circle(s) in ΣA. Notice that, by taking $A = S^{n-1}$ for $n \geq 1$, we obtain $\mathcal{G}(\Sigma S^{n-1}, Z) = G_n(Z)$, the ordinary Gottlieb group.

Theorem 2 (Corollary 2.4). *For any A, we have an isomorphism of abelian groups*

$$\mathcal{G}\big(\Sigma A, \mathrm{map}(X, Y; 0)\big) \cong \mathcal{G}(\Sigma A, Y) \oplus \mathcal{G}(\Sigma(A \wedge X), Y).$$

In particular, for each $n \geq 1$, we have an isomorphism of abelian groups

$$G_n\big(\mathrm{map}(X, Y; 0)\big) \cong G_n(Y) \oplus \mathcal{G}(\Sigma^n X, Y).$$

Theorem 1 follows from this result by setting $X = S^1$. Another interesting special case is given by setting X a wedge of spheres, which yields the following identity (included in Theorem 2.5):

Corollary. *For $X = S^{i_1} \vee \cdots \vee S^{i_k}$, and for $n \geq 1$, we have an isomorphism*

$$G_n\big(\mathrm{map}(X, Y)\big) \cong G_n(Y) \oplus \bigoplus_{r=1,\ldots,k} G_{n+i_r}(Y).$$

For X a single sphere, of course, we have $G_n\big(\mathrm{map}(S^p, Y; 0)\big) \cong G_n(Y) \oplus G_{n+p}(Y)$.

For X a product of spaces, Theorem 2 may be applied directly. However, in this case, it is also possible to use the result iteratively, so as to decompose the right-hand sides of the identity into simpler pieces. This approach leads, in particular, to an explicit decomposition of the Gottlieb groups of function spaces of the form

$$\mathrm{map}\big((S^{a_1} \vee \cdots \vee S^{a_k}) \times \overset{N\text{-times}}{\cdots} \times (S^{b_1} \vee \cdots \vee S^{b_l}), Y; 0\big)$$

directly in terms of the Gottlieb groups of Y. Also, in these cases, some interesting combinatorial expressions arise. We give one such decomposition here, which is included in Corollary 3.1. Section 3 gives more general results, along similar lines.

Theorem 3. *For $N \geq 1$, let $\Lambda^N(Y)$ denote the iterated free loop space of Y (see Section 3 for a definition). Then, for each $n \geq 1$, we have*

$$G_n\big(\Lambda^N(Y)\big) \cong \bigoplus_{j=0}^{N} \binom{N}{j} G_{n+j}(Y).$$

In these formulas, a notation of the form kG, for $k \geq 1$ an integer and G an abelian group, denotes the direct sum of k copies of G. These decompositions are quite remarkable, given the apparent difficulty of identifying Gottlieb groups for "small" spaces such as manifolds, or cell complexes with few cells, including spheres (see [6]).

The iterative use of Theorem 2 depends on the exponential law, whereby we we may identify the function spaces $\mathrm{map}(A, \mathrm{map}(B, Y))$ and $\mathrm{map}(A \times B, Y)$. In the case of the iterated free loop space $\Lambda^N(Y)$, repeated use of this allows us to identify $\Lambda^N(Y)$ with the function space $\mathrm{map}(T^N, Y; 0)$, where T^N denotes the N-torus $S^1 \times \cdots \times S^1$. So the identity of Theorem 3 gives $G_n\big(\mathrm{map}(T^N, Y; 0)\big)$ in terms of $G_*(Y)$. In this case, there is a relation between our identities here, and similar identities noticed by Fox and others, concerning the so-called *Fox torus homotopy groups* [4], and the *Fox-Gottlieb groups* introduced in [5]. We discuss the connection in Section 3.

In a final generalization of our basic result, we relativize Theorem 1. This begins with replacing the free loop space ΛY with a pullback of the free loop fibration $\Lambda Y \to Y$ over a map $f\colon X \to Y$, which we denote $L_f Y$. In this setting, we obtain a relation that involves the so-called *relative Gottlieb groups*, or the *evaluation subgroups of a map*. We review the definition of these groups now, before indicating our result.

Let $f\colon X \to Y$ be a based map. Denote by $\mathrm{map}(X, Y; f)$ the path component of the space of (unbased) maps $X \to Y$ that consists of maps (freely) homotopic to f. Evaluation at the basepoint of X gives the *evaluation map* $\omega\colon \mathrm{map}(X, Y; f) \to Y$. The *nth evaluation subgroup of* f, denoted $G_n(Y, X; f)$, is the image of $\omega_\#$ in $\pi_n(Y)$ (cf. [7, p.731]). The Gottlieb group $G_n(Y)$ occurs as the special case in which $X = Y$ and $f = 1_Y$.

From the construction of $L_f Y$, we obtain a canonical "whisker" map $\phi\colon \Lambda X \to L_f Y$. Our basic result in this relative setting, which may be viewed as a relative version of the prototypical Theorem 1, is the following.

Theorem 4 (Theorem 4.1)**.** *For each $n \geq 2$, we have an isomorphism of abelian groups*

$$G_n\big(L_f Y, \Lambda X; \phi\big) \cong G_n(X) \oplus G_{n+1}(Y, X; f).$$

As with Theorem 2, it is possible to handle iterates of this relativized construction, although we do not develop this direction here.

Our results lead to a strong consequence for the global structure of the Gottlieb groups of a function space. Let $\beta_i(X)$ denote the ith Betti number of X, and $\gamma_i(Y)$—which we propose calling the ith *Gottlieb number* of Y—denote the rank (as an abelian group) of $G_i(Y)$. We deduce the following formula:

Theorem 5. *Let X and Y be finite complexes, with Y simply connected. Then we have*

$$\gamma_n\big(\mathrm{map}(X, Y; 0)\big) = \sum_{i=0}^{\dim X} \beta_i(X)\gamma_{n+i}(Y),$$

for $n \geq 1$, where $\dim X$ is the dimension of X.

It follows (Corollary 5.4) that, under these hypotheses, $G_n\big(\mathrm{map}(X, Y; 0)\big)$ *is a finite group for all but finitely many n.*

The paper is organized as follows. In Section 2 we set hypotheses and show our basic results, including Theorem 2. In Section 3 we focus on the iterated bouquet spaces and establish several explicit decompositions, such as that of Theorem 3. This section also includes our discussion of the Fox torus homotopy groups. In Section 4, we carry out the relativization of Theorem 1 and obtain Theorem 4. In Section 5 we deduce several consequences of our results, including Theorem 5 above.

2. Generalized Gottlieb Groups of Function Spaces

We begin by establishing notation and carefully specifying our hypotheses. We use X, Y and A to denote based spaces, with x_0 the basepoint of X. Our hypotheses on these spaces are driven by the proofs in this section: we wish to make several identifications of—based and unbased—function spaces using the exponential law, and we want evaluation maps to be fibrations—actually, to induce Barratt-Puppe sequences of homotopy sets. To this end, we assume that spaces X, Y, and A are locally compact and Hausdorff. The function space $\mathrm{map}(X, Y)$ has the compact-open topology. Then X, Y, and $\mathrm{map}(X, Y)$ are compactly generated spaces, and the identifications we use are given in Theorems 5.6 and 5.12 of [13]. Note that Hausdorff-ness of $\mathrm{map}(X, Y)$ is inherited from Y. Under our hypotheses, there is no need to adjust the topologies on the products or function spaces that arise here, in order to apply the results of [13]. Also, we assume that spaces X, Y, and A are well-pointed, from which it follows that the evaluation maps that we use are (Hurewicz) fibrations [12, Th.2.8.2]. We suppose that spaces X and Y are connected but, as mentioned in the introduction, a space A (or B) that appears suspended may be disconnected. Note, in particular, that we do not require any finiteness hypotheses on our spaces X, Y, or A, and neither do we require that X or Y be simply connected, or that any of these spaces be (of the homotopy type of) a CW complex. For based spaces, ΣA denotes the reduced suspension, and $A \wedge X$ the smash product. We also use the notation and vocabulary introduced before Theorem 2 of the Introduction.

Generally speaking, we are interested in identifying generalized Gottlieb groups of a function space, that is, the image of a homomorphism

$$[\Sigma A, \mathrm{aut}_1 \mathrm{map}(X, Y; 0)] \to [\Sigma A, \mathrm{map}(X, Y; 0)]$$

induced by the evaluation map of $\mathrm{map}(X, Y; 0)$. To this end, we have the following general result. We emphasize that, here, $\mathrm{map}(X, Y)$ denotes the null component $\mathrm{map}(X, Y; 0)$.

Lemma 2.1. *Let $\mathrm{ev}\colon \mathrm{map}(X, Y) \to Y$ be the evaluation fibration given by $\mathrm{ev}(g) = g(x_0)$. Then $\omega_Y\colon \mathrm{aut}_1 Y \to Y$ is a retract (as a map) of $(\omega_Y)_*\colon \mathrm{map}(X, \mathrm{aut}_1 Y) \to \mathrm{map}(X, Y)$, which in turn is a retract of $\omega_{\mathrm{map}(X,Y)}\colon \mathrm{aut}_1 \mathrm{map}(X, Y) \to \mathrm{map}(X, Y)$.*

Proof. For consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{aut}_1 Y \arrow[r, "\sigma_{\mathrm{aut}_1 Y}"'] \arrow[d, "\omega_Y"'] & \mathrm{map}(X, \mathrm{aut}_1 Y) \arrow[l, bend right, "\mathrm{ev}_{\mathrm{aut}_1 Y}"'] \arrow[r, "\Phi"'] \arrow[d, "(\omega_Y)_*"'] & \mathrm{aut}_1 \mathrm{map}(X, Y) \arrow[l, bend right, "r"'] \arrow[d, "\omega_{\mathrm{map}(X,Y)}"] \\
Y \arrow[r, "\sigma_Y"] & \mathrm{map}(X, Y) \arrow[l, bend left, "\mathrm{ev}_Y"] \arrow[r, equal] & \mathrm{map}(X, Y)
\end{tikzcd}$$

The vertical maps are evaluation (at the basepoint) maps, or the map induced by the evaluation map in the case of the middle one. In the left-hand square, the retractions and their sections are the usual evaluation maps of the form $\mathrm{ev}_Y \colon \mathrm{map}(X, Y; 0) \to Y$, and $\sigma_Y \colon Y \to \mathrm{map}(X, Y; 0)$, defined by evaluation at the base point of X, and the null section, respectively. The latter means $\sigma_Y(y) = C_y$, with $C_y(x) = y$, the null map at y. Note that the null map $X \to \mathrm{aut}_1 Y$ maps each point of X to the *identity* of Y, which is the base point in $\mathrm{aut}_1 Y$.

The maps Φ and r may be described as follows. The exponential law gives homeomorphisms

$$\mathrm{map}(X, \mathrm{aut}_1 Y; 0) \equiv \mathrm{map}(X \times Y, Y; \pi_2)$$

and

$$\mathrm{aut}_1 \mathrm{map}(X, Y; 0) \equiv \mathrm{map}(\mathrm{map}(X, Y; 0) \times X, Y; \mathrm{EV}).$$

Here, $\mathrm{EV} \colon \mathrm{map}(X, Y; 0) \times X \to Y$ denotes the "big" evaluation map, given by $\mathrm{EV}(f, x) = f(x)$. For the spaces on the right-hand sides, we have an obvious section and retraction

$$\begin{tikzcd}
\mathrm{map}(X \times Y, Y; \pi_2) \arrow[r, "\big(T\circ(\mathrm{ev}_Y \times 1)\big)^*"'] & \mathrm{map}(\mathrm{map}(X, Y; 0) \times X, Y; \mathrm{EV}) \arrow[l, bend right, "\big((1\times\sigma_Y)\circ T\big)^*"']
\end{tikzcd}$$

where $X \times Y \underset{\equiv}{\overset{T}{\longleftrightarrow}} Y \times X$ denotes the switching map. Via the preceding adjunctions, we may define Φ and r, then, as

$$\Phi(g)(f)(x) := g(x)\big(f(x_0)\big) \qquad \text{and} \qquad r(\psi)(x)(b) := \psi(C_b)(x),$$

for $g \in \mathrm{map}(X, \mathrm{aut}_1 Y)$, $f \in \mathrm{map}(X, Y)$, $x \in X$, and $\psi \in \mathrm{aut}_1 \mathrm{map}(X, Y)$, $x \in X$, $b \in Y$.

A straightforward check shows that both squares commute in either direction, and that we have the desired retractions. $\square$

Remark 2.2. Collapsing the two retractions from Lemma 2.1 yields a retraction (of maps)

$$\begin{tikzcd}
\mathrm{aut}_1 Y \arrow[r, "\iota"] \arrow[d, "\omega_Y"'] & \mathrm{aut}_1 \mathrm{map}(X, Y) \arrow[r, "(\mathrm{ev}_Y)_* \circ (\sigma_Y)^*"] \arrow[d, "\omega_{\mathrm{map}(X,Y)}"] & \mathrm{aut}_1 Y \arrow[d, "\omega_Y"] \\
Y \arrow[r, "\sigma_Y"'] & \mathrm{map}(X, Y) \arrow[r, "\mathrm{ev}_Y"'] & Y
\end{tikzcd}$$

In place of the sectioned fibration $\mathrm{ev}_Y \colon \mathrm{map}(X, Y) \to Y$, we could consider a general sectioned fibration $p \colon E \to B$, with section $\sigma \colon B \to E$ (so E dominates B).

In this case, we do have a diagram

$$\begin{tikzcd}
\mathrm{aut}_1 B \arrow[r, dotted] \arrow[d, "\omega_B"'] & \mathrm{aut}_1 E \arrow[rr, "p_* \circ \sigma^*"] \arrow[d, "\omega_E"] & & \mathrm{aut}_1 B \arrow[d, "\omega_B"] \\
B \arrow[r, "\sigma"'] & E \arrow[rr, "p"'] & & B.
\end{tikzcd}$$

The right-hand square here yields the standard fact that, since p has a right homotopy inverse, we have $p_{\#}\big(G_*(E)\big) \subseteq G_*(B)$. On the other hand, the lack of a filler $\mathrm{aut}_1 B \to \mathrm{aut}_1 E$ in the left-hand square, in general, reflects the fact that $G_n(-)$ fails to be a functor. This makes clear that we are relying on the particular properties of the evaluation fibration $\mathrm{ev}_Y \colon \mathrm{map}(X, Y) \to Y$ in order to obtain our results.

We now give our main result.

Theorem 2.3. *For X, Y, and A satisfying the hypotheses above, there is a split short exact sequence of abelian groups*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\Sigma(A \wedge X), Y) \arrow[r] & \mathcal{G}\big(\Sigma A, \mathrm{map}(X, Y; 0)\big) \arrow[r, "(\mathrm{ev}_Y)_*"'] & \mathcal{G}(\Sigma A, Y) \arrow[l, bend right, "(\sigma_Y)_*"'] \arrow[r] & 0.
\end{tikzcd}$$

Proof. Consider the following commutative diagram.

(1)

$$\begin{tikzcd}
& \mathrm{aut}_1 \mathrm{map}(X, Y) \arrow[d, shift left, "r"] & \\
\mathrm{map}_*(X, \mathrm{aut}_1 Y) \arrow[r, "j_{\mathrm{aut}_1 Y}"'] \arrow[d, "(\omega_Y)_*"'] & \mathrm{map}(X, \mathrm{aut}_1 Y) \arrow[u, shift left, "\Phi"] \arrow[r, "\mathrm{ev}_{\mathrm{aut}_1 Y}"'] \arrow[d, "(\omega_Y)_*"'] & \mathrm{aut}_1 Y \arrow[l, bend right, "\sigma_{\mathrm{aut}_1 Y}"'] \arrow[d, "\omega_Y"] \\
\mathrm{map}_*(X, Y) \arrow[r, "j_Y"'] & \mathrm{map}(X, Y) \arrow[r, "\mathrm{ev}_Y"] & Y \arrow[l, bend left, "\sigma_Y"]
\end{tikzcd}$$

The horizontal rows are fibre sequences of evaluation fibrations (restricted to the null component), each with a section. The vertical maps are the evaluation map and its induced maps. The maps Φ and r are those from Lemma 2.1.

Now apply $[\Sigma A, -]$ to this diagram. Then the lower part of the diagram yields a ladder of split extensions

$$\begin{tikzcd}
{[\Sigma(A \wedge X), \mathrm{aut}_1 Y]} \arrow[r, tail, "j"'] \arrow[d, "(\omega_Y)_*"'] & {[\Sigma A, \mathrm{map}(X, \mathrm{aut}_1 Y)]} \arrow[r, two heads, "\mathrm{ev}_*"'] \arrow[d, "(\omega_{\mathrm{map}(X,Y)} \circ \Phi)_*"'] & {[\Sigma A, \mathrm{aut}_1 Y]} \arrow[l, bend right, "\sigma_*"'] \arrow[d, "(\omega_Y)_*"] \\
{[\Sigma(A \wedge X), Y]} \arrow[r, tail, "j"'] & {[\Sigma A, \mathrm{map}(X, Y)]} \arrow[r, two heads, "(\mathrm{ev})_*"] & {[\Sigma A, Y]} \arrow[l, bend left, "(\sigma)_*"]
\end{tikzcd}$$

All maps are those induced from (1), and we have omitted subscripts off the horizontal ones. We have used the exponential law for based mapping spaces to write the left-hand terms in this way: the identification

$$[\Sigma A, \mathrm{map}_*(X, Z)] = \pi_0\big(\mathrm{map}_*(\Sigma A, \mathrm{map}_*(X, Z))\big) = \pi_0\big(\mathrm{map}_*(\Sigma A \wedge X, Z)\big) = [\Sigma A \wedge X, Z]$$

is natural with respect to maps induced by maps of Z. Now consider the image of the upper split extension in the lower. Generally, the image of an exact sequence in an exact sequence fails to be exact. However, here we have the image of one split, short exact sequence in another. One easily checks—a standard diagram chase—that the sequence of image subgroups is also split short exact. Furthermore, the image subgroups are abelian, since $\mathrm{aut}_1 Y$ is an H-space, and hence the groups in the upper sequence are abelian. The first and last image subgroups are $\mathcal{G}(\Sigma(A \wedge X), Y)$ and $\mathcal{G}(\Sigma A, Y)$ by definition. The middle image subgroup is $\mathcal{G}\big(\Sigma A, \mathrm{map}(X, Y; 0)\big)$ because of the factorization $(\omega_Y)_* = (\omega_{\mathrm{map}(X,Y)} \circ \Phi)_* \colon \mathrm{map}(X, \mathrm{aut}_1 Y) \to \mathrm{map}(X, Y)$ as in (1), together with the fact that the sequences are split short exact. $\square$

Corollary 2.4. *We have an isomorphism of abelian groups*

$$\mathcal{G}\big(\Sigma A, \mathrm{map}(X, Y; 0)\big) \cong \mathcal{G}(\Sigma A, Y) \oplus \mathcal{G}(\Sigma(A \wedge X), Y).$$

In particular, for each $n \geq 1$, we have an isomorphism of abelian groups

$$G_n\big(\mathrm{map}(X, Y; 0)\big) \cong G_n(Y) \oplus \mathcal{G}(\Sigma^n X, Y). \quad \square$$

Now specialize to the case in which $X = S^{i_1} \vee \cdots \vee S^{i_k}$ or, more generally, in which we have $\Sigma X = S^{i_1+1} \vee \cdots \vee S^{i_k+1}$.

Theorem 2.5. *If $\Sigma X = S^{i_1+1} \vee \cdots \vee S^{i_k+1}$, then we have an isomorphism*

$$G_n\big(\mathrm{map}(X, Y)\big) \cong G_n(Y) \oplus \bigoplus_{r=1,\dots,k} G_{n+i_r}(Y),$$

for $n \geq 1$. More generally, given a product decomposition $X = X' \times X''$, for which we have $\Sigma X' = S^{i_1+1} \vee \cdots \vee S^{i_k+1}$, then for $n \geq 1$ we have an isomorphism

$$G_n\big(\mathrm{map}(X, Y)\big) \cong G_n\big(\mathrm{map}(X'', Y)\big) \oplus \bigoplus_{r=1,\dots,k} G_{n+i_r}\big(\mathrm{map}(X'', Y)\big).$$

Proof. In the first case, we have $\Sigma^n X \simeq S^{n+i_1} \vee \cdots \vee S^{n+i_k}$, and hence $[\Sigma^n X, Z] \cong \oplus_{r=1,\dots,k}[S^{n+i_r}, Z]$ for any Z. It follows that $\mathcal{G}(\Sigma^n X, Y) \cong \oplus_{r=1,\dots,k} G_{n+i_r}(Y)$, and the first assertion follows immediately from Corollary 2.4. For the more general statement, simply write $\mathrm{map}(X, Y)$ as $\mathrm{map}(X', \mathrm{map}(X'', Y))$ and apply the first statement. This contains the first statement as the case in which $X'' = *$. $\square$

3. Explicit Decompositions: Gottlieb Groups of Iterated Free Bouquet Spaces

In this section, we consider the particular cases in which X is a product, each factor of which splits as a wedge of spheres after one suspension. Here, the first isomorphism of Theorem 2.5 may be applied recursively, so as to express the Gottlieb groups of $\mathrm{map}(X, Y)$ directly in terms of the Gottlieb groups of Y. This yields the formulas of Theorem 1 and Theorem 3 from Section 1, which are included in the next result, as well as their more general versions.

For $m \geq 1$, denote by $\Lambda_m Y$ the *free m-bouquet space of Y*, i.e., the space $\mathrm{map}(S_m, Y)$, where $S_m = S^1 \vee \cdots \vee S^1$ is a bouquet of m circles. Thus $\Lambda_1 Y = \Lambda Y$, and the free 2-bouquet space of Y, $\Lambda_2 Y$, might also be called the space of "free figure-eights" in Y. Iterating this construction, whereby we set $\Lambda_m^1 Y = \Lambda_m Y$, and $\Lambda_m^N Y = \Lambda_m(\Lambda_m^{N-1} Y)$ for $N \geq 2$, we obtain $\Lambda_m^N Y$, the *N-fold iterated free m-bouquet* *space of* Y. In the special case in which $m = 1$, we have $\Lambda_1^N Y = \Lambda^N Y$, the N-fold iterated free loop space of Y.

Corollary 3.1 (to Theorem 2.5)**.** *For $\Lambda_m^N Y$, the N-fold iterated free m-bouquet space of Y, we have*

$$G_n(\Lambda_m^N Y) \cong \bigoplus_{j=0}^{N} m^j \binom{N}{j} G_{n+j}(Y),$$

for $n \geq 1$. If $m = 1$, then we obtain Theorem 3 of the Introduction.

Proof. We work inductively over N. For $N = 1$, Theorem 2.5 gives

$$G_n(\Lambda_m Y) = G_n\big(\mathrm{map}(S_m, Y)\big) \cong G_n(Y) \oplus m\, G_{n+1}(Y).$$

Now suppose the decomposition holds for $N \leq k-1$, some $k \geq 2$. Then

$$\Lambda_m^k Y = \Lambda_m(\Lambda_m^{k-1} Y) = \mathrm{map}(S_m, \Lambda_m^{k-1} Y),$$

and so Theorem 2.5 gives

$$G_n(\Lambda_m^k Y) \cong G_n(\Lambda_m^{k-1} Y) \oplus m\, G_{n+1}(\Lambda_m^{k-1} Y).$$

By induction, we have that

$$\begin{aligned}
G_n(\Lambda_m^k Y) &\cong \bigoplus_{j=0}^{k-1} m^j \binom{k-1}{j} G_{n+j}(Y) \oplus m \bigoplus_{j=0}^{k-1} m^j \binom{k-1}{j} G_{n+j+1}(Y) \\
&= G_n(Y) \oplus \bigoplus_{j=1}^{k-1} m^j \binom{k-1}{j} G_{n+j}(Y) \\
&\qquad \oplus \bigoplus_{j=0}^{k-2} m^{j+1} \binom{k-1}{j} G_{n+j+1}(Y) \oplus m^k\, G_{n+k}(Y) \\
&= G_n(Y) \oplus \left(\bigoplus_{j=1}^{k-1} m^j \left(\binom{k-1}{j} + \binom{k-1}{j-1} \right) G_{n+j}(Y) \right) \oplus m^k\, G_{n+k}(Y).
\end{aligned}$$

Now

$$\binom{k-1}{j} + \binom{k-1}{j-1} = \binom{k}{j},$$

and the result follows. □

We may replace the bouquet of circles by a more general bouquet of spheres (of different dimensions), and obtain explicit decompositions of the Gottlieb groups of the corresponding function spaces directly in terms of the Gottlieb groups of Y, although the expressions become more cumbersome to present. We offer one example to illustrate the general idea, and then give a summary result. Denote by $\Lambda_{(2,2,2)} Y$ the space $\mathrm{map}(S^2 \vee S^2 \vee S^2, Y)$, and by $\Lambda_{(2,2,2)}^N Y$ the iterated version of the same, whereby we set $\Lambda_{(2,2,2)}^1 Y = \Lambda_{(2,2,2)} Y$, and $\Lambda_{(2,2,2)}^N Y = \Lambda_{(2,2,2)}(\Lambda_{(2,2,2)}^{N-1} Y)$ for $N \geq 2$.

Example 3.2. With the above notation, for $n \geq 1$ we have

$$G_n(\Lambda_{(2,2,2)}^N Y) \cong \bigoplus_{j=0}^{N} 3^j \binom{N}{j} G_{n+2j}(Y).$$

We summarize these explicit decompositions in the following result. Suppose X is an N-fold product $X = X_1 \times \cdots \times X_N$, each factor of which suspends to some bouquet of spheres. For each $i = 1, \ldots, N$, we write

$$\Sigma X_i = S^{(i,1)+1} \vee \cdots \vee S^{(i,n_i)+1}.$$

Here, a superscript $(i, j) + 1$ denotes the dimension of the jth sphere in ΣX_i. For example, we could just take $X_i = S^{(i,1)} \vee \cdots \vee S^{(i,n_i)}$, a plain bouquet of spheres.

Theorem 3.3. *With X as above, we have isomorphisms*

$$G_n\big(\mathrm{map}(X, Y; 0)\big) \cong G_n(Y) \oplus \bigoplus_{r=1,\ldots,N} G_{n+(i_1,m_1)+\cdots+(i_r,m_r)}(Y),$$

where the sum, for each r, runs over all possible r-tuples $1 \leq i_1 < i_2 < \cdots < i_r \leq N$ and all m_j with $1 \leq m_j \leq n_{i_j}$.

Proof. We argue by induction over N, the number of factors in X. For $N = 1$, we have

$$G_n\big(\mathrm{map}(X, Y; 0)\big) \cong G_n(Y) \oplus \bigoplus_{1\leq m\leq n_1} G_{n+(1,m)}(Y),$$

from Theorem 2.5.

Now assume the statement for a product of $N - 1$ factors, and write $X = X_1 \times X_{(2,\ldots,N)}$, where $X_{(2,\ldots,N)} = X_2 \times \cdots \times X_N$. Then we have

$$\mathrm{map}(X, Y; 0) = \mathrm{map}\big(X_1, \mathrm{map}(X_{(2,\ldots,N)}, Y)\big),$$

and hence

$$G_n\big(\mathrm{map}(X, Y; 0)\big) \cong G_n\big(\mathrm{map}(X_{(2,\ldots,N)}, Y)\big) \oplus \bigoplus_{1\leq m\leq n_1} G_{n+(1,m)}\big(\mathrm{map}(X_{(2,\ldots,N)}, Y)\big),$$

again from Theorem 2.5. Applying the induction hypothesis to each of these terms completes the induction step, and the result follows. $\square$

The decomposition of Theorem 3.3 leads to the following observation, which we develop at the end of the paper.

Remark 3.4. Let X be a space that satisfies the hypotheses of Theorem 3.3. Then, for each $n \geq 1$, $G_n(\mathrm{map}(X, Y; 0))$ may be written as a direct sum of $G_n(Y)$ and Gottlieb groups of Y of dimension greater than n.

In case X is a suspension, or product of such, that does not necessarily split after a further suspension, we may also obtain fairly explicit decompositions for the (generalized) Gottlieb groups of $\mathrm{map}(X, Y)$ in terms of (generalized) Gottlieb groups of Y. From Corollary 2.4, we obtain the following decomposition. Recall, from our hypotheses established in Section 2, that the spaces that appear suspended in the next two results, namely A, B, or B_i, need not be connected.

Corollary 3.5. *For $n \geq 1$, we have isomorphisms of abelian groups*

$$G_n\big(\mathrm{map}(\Sigma B, Y)\big) \cong G_n(Y) \oplus \mathcal{G}(\Sigma^{n+1}B, Y)$$

and

$$\mathcal{G}\big(\Sigma A, \mathrm{map}(\Sigma B, Y)\big) \cong \mathcal{G}(\Sigma A, Y) \oplus \mathcal{G}(\Sigma^2(A \wedge B), Y). \quad \square$$

Then this result may be used iteratively, to obtain decompositions for X a product of suspensions. We state one result for the case in which $k = 2$, to give the general idea. Here, the combinatorial aspects of the decompositions are not so elegant as above.

Corollary 3.6. *For $X = \Sigma B_1 \times \Sigma B_2$ we have*

$$G_n\big(\mathrm{map}(X,Y)\big) \cong$$
$$G_n(Y) \oplus \mathcal{G}(\Sigma^{n+1}B_1, Y) \oplus \mathcal{G}(\Sigma^{n+1}B_2, Y) \oplus \mathcal{G}(\Sigma^{n+2}(B_1 \wedge B_2), Y),$$

Note that there are various ways of re-writing terms of the form $\Sigma^{n+2}(B_1 \wedge B_2)$, which could be helpful in further decomposing the last term in specific cases.

3.1. **Fox Torus Groups and Fox-Gottlieb Groups.** We discuss a relation between our results above, and results on the torus homotopy groups, originally studied by Fox [4]. These non-abelian groups are defined as

$$\tau_n(Y) = \pi_1\big(\mathrm{map}(T^{n-1}, Y; 0)\big),$$

for each $n \geq 2$. Equivalently, as in [5], they may be defined as

$$\tau_n(Y) = [\Sigma(T^{n-1} \sqcup \{*\}), Y],$$

the set of based homotopy equivalence classes of maps from the (reduced) suspension $\Sigma(T^{n-1} \sqcup \{*\})$, where $T^{n-1} \sqcup \{*\}$ denotes the $(n-1)$-torus with a disjoint base point. For $n = 1$, then, we take $\tau_1(Y) = \pi_1(Y)$, the ordinary fundamental group of Y.

It seems that various relations amongst these groups, and relations between results about these groups and our results above, may be elucidated by consideration of free loop fibrations. The basic connection is that, for $n \geq 2$, we may rewrite $\mathrm{map}(T^{n-1}, Y; 0)$ as the iterated free loop space $\Lambda^{n-1}Y$, and thus we have $\tau_n(Y) = \pi_1(\Lambda^{n-1}Y)$.

First, from a free loop fibration $\Omega Z \to \Lambda Z \to Z$, which has a section, we obtain a split short exact sequence of homotopy groups

$$\begin{tikzcd} 0 \arrow[r] & \pi_i(\Omega Z) \arrow[r] & \pi_i(\Lambda Z) \arrow[r] & \pi_i(Z) \arrow[r] \arrow[l, bend right] & 1 \end{tikzcd},$$

for each $i \geq 1$. This leads to direct sums

$$\pi_i(\Lambda Z) \cong \pi_{i+1}(Z) \oplus \pi_i(Z),$$

for the higher homotopy groups $i \geq 2$ of the free loop space ΛZ, and a semi-direct product

$$\pi_1(\Lambda Z) \cong \pi_2(Z) \rtimes \pi_1(Z)$$

for the fundamental group of ΛZ. Now by applying these expressions iteratively, beginning with the free loop fibration

$$\Omega\Lambda^{N-1}Y \to \Lambda^N Y \to \Lambda^{N-1}Y,$$

we obtain a direct sum formula

$$\pi_i(\Lambda^N Y) \cong \bigoplus_{r=0}^{N} \binom{N}{r} \pi_{i+r}(Y),$$

for the higher homotopy groups $i \geq 2$ of the iterated free loop space, in terms of the homotopy groups of Y. For the fundamental group, we have a sequence of split extensions

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(\Omega\Lambda^{N-1}Y) \arrow[r] & \pi_1(\Lambda^N Y) \arrow[r] & \pi_1(\Lambda^{N-1}Y) \arrow[l, bend right] \arrow[r] & 1 , \end{tikzcd} \tag{2}$$

for $N \geq 1$, with $\Lambda^0 Y$ meaning Y.

Now we observe the following identity.

Lemma 3.7. *For any Z, we have $\Omega(\Lambda Z) = \Lambda(\Omega Z)$.*

Proof. Indeed, either may be identified by adjunction with the subspace of $\mathrm{map}(S^1 \times S^1, Z)$ that consists of maps F with $F(s_0, s) = z_0$. $\square$

From this identity, we may write the term $\pi_1(\Omega\Lambda^{N-1}Y)$ that appears in (2) as $\pi_1\big(\Lambda^{N-1}(\Omega Y)\big)$, which results in the following split short exact sequences of Fox torus homotopy groups

$$\begin{tikzcd} 0 \arrow[r] & \tau_N(\Omega Y) \arrow[r] & \tau_{N+1}(Y) \arrow[r] & \tau_N(Y) \arrow[l, bend right] \arrow[r] & 1 . \end{tikzcd}$$

Alternatively, we may write the term $\pi_1(\Omega\Lambda^{N-1}Y)$ from (2) as $\pi_2(\Lambda^{N-1}Y)$, and then by using the above direct sum expression, we may write (2) as

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{r=0}^{N-1} \binom{N}{r} \pi_{2+r}(Y) \arrow[r] & \tau_{N+1}(Y) \arrow[r] & \tau_N(Y) \arrow[l, bend right] \arrow[r] & 1 . \end{tikzcd}$$

These expressions retrieve the result of Fox, which appears in Theorems 1.2 and 1.3 in [5].

In [5, Def.2.1], the ith *Fox-Gottlieb group* is defined as the image of the homomorphism induced by the evaluation map $\omega\colon \mathrm{aut}_1\, Y \to Y$ on ith Fox torus groups. This is a homomorphism

$$\tau_n(\omega)\colon \pi_1\big(\mathrm{map}(T^{n-1}, \mathrm{aut}_1\, Y)\big) \to \pi_1\big(\mathrm{map}(T^{n-1}, Y; 0)\big),$$

whose image is denoted by $G\tau_n(Y) \subseteq \tau_n(Y)$. When $n = 1$, we retrieve $G\tau_1(Y) = G_1(Y)$, the ordinary first Gottlieb group of Y. When $n \geq 2$, we may identify the homomorphism induced by the evaluation map on nth Fox torus groups as

$$\tau_n(\omega) = (\omega_*)_\#\colon \pi_1\big(\Lambda^{n-1}\mathrm{aut}_1\, Y\big) \to \pi_1\big(\Lambda^{n-1}Y\big),$$

and thus we have

$$G\tau_n(Y) = G_1(\Lambda^{n-1}Y) \cong \bigoplus_{j=0}^{n-1} \binom{n-1}{j} G_{n-1+j}(Y) \tag{3}$$

which, omitting the middle term, retrieves Theorem 2.1 of [5]. Other identities from [5] may also be deduced from properties of the free loop fibration. Because we may identify $\mathrm{map}(T^n, Y; 0) = \mathrm{map}(T^{n-1}, \Lambda Y; 0)$, it follows from the definition that we have

$$\tau_n(Y) = \tau_{n-1}(\Lambda Y) = \cdots = \pi_1(\Lambda^{n-1}Y).$$

Likewise, we have equalities of Fox-Gottlieb groups

$$G\tau_n(Y) = G\tau_{n-1}(\Lambda Y) = \cdots = G_1(\Lambda^{n-1}Y).$$

Hence, the identities (3) for Fox-Gottlieb groups in terms of Gottlieb groups may be used to write, first of all,

$$G\tau_n(Y) = \bigoplus_{j=0}^{n-1} \binom{n-1}{j} G_{n-1+j}(Y),$$

and then, via $G\tau_n(Y) = G\tau_{n-1}(\Lambda Y)$, as

$$G\tau_n(Y) = \bigoplus_{j=0}^{n-2} \binom{n-2}{j} G_{n-2+j}(\Lambda Y)$$

Equating the two right-hand sides here gives a relation between the ordinary Gottlieb groups of Y and of ΛY that may be used recursively to retrieve our formulas for the Gottlieb groups of iterated free loop spaces.

4. Relative Free Loop Spaces

The preceding results that concern free loop and free bouquet spaces may be relativized. For the free loop space, this is as follows. Let $f\colon X \to Y$ be a based map. The pullback of the fibration $P\colon PY \to Y \times Y$, where $P(\alpha) = \big(\alpha(0), \alpha(1)\big)$, along the map $(f, f) = \Delta_Y \circ f = (f \times f) \circ \Delta_X \colon X \to Y \times Y$ results in a fibration with fibre ΩY, thus:

$$\begin{tikzcd}
\Omega Y \arrow[r, equal] \arrow[d] & \Omega Y \arrow[d] \\
L_f Y \arrow[r] \arrow[d, "\mathrm{ev}_f"'] & PY \arrow[d, "P"] \\
X \arrow[r, "{(f,f)}"'] & Y \times Y.
\end{tikzcd}$$

In case we have $f = \mathrm{id}_Y \colon Y \to Y$, then $L_f Y$ reduces to the free loop space ΛY. We discuss this space motivated in part by its appearance as an object of interest in string topology (cf. [8, 11]).

The fibration $\mathrm{ev}_f\colon L_f Y \to X$ admits a standard null-section, which we denote by $\sigma_f\colon X \to L_f Y$. It may be identified as the whisker map induced in the following pullback diagram

$$\begin{tikzcd}
X \arrow[dr, dotted, "\sigma_f"] \arrow[drr, bend left, "C_f"] \arrow[ddr, bend right, "1_X"'] & & \\
& L_f Y \arrow[r] \arrow[d, "\mathrm{ev}_f"] & PY \arrow[d, "P"] \\
& X \arrow[r, "{(f,f)}"'] & Y \times Y,
\end{tikzcd}$$

in which $C_f\colon X \to PY$ denotes the map $C_f(x) := C_{f(x)}$, the constant path in Y at $f(x)$.

We may equally well regard this *relative free loop space* $L_f Y$ as being obtained as a pullback of the free loop fibration $\mathrm{ev}\colon \Lambda Y \to Y$ along $f\colon X \to Y$. Indeed, the above pullback may be factored as a composition of pullbacks as follows:

$$\begin{tikzcd} L_f Y \arrow[r] \arrow[d] & \Lambda Y \arrow[r] \arrow[d, "\mathrm{ev}"] & PY \arrow[d, "P"] \\ X \arrow[r, "f"'] & Y \arrow[r, "\Delta_Y"'] & Y \times Y. \end{tikzcd}$$

Then we obtain a canonical map $(\mathrm{ev}, f_*)\colon \Lambda X \to L_f Y$, determined by $f\colon X \to Y$, as the whisker map induced in the following diagram:

$$\begin{tikzcd} \Lambda X \arrow[dr, dotted] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "\mathrm{ev}"'] & & \\ & L_f Y \arrow[r] \arrow[d] & \Lambda Y \arrow[d, "\mathrm{ev}"] \\ & X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Recall from the Introduction the relative version of the Gottlieb groups, namely

$$G_n(Y, X; f) = \mathrm{im}\{\omega_\#\colon \pi_n\big(\mathrm{map}(X, Y; f)\big) \to \pi_n(Y)\},$$

for each $n \geq 1$, where $\omega\colon \mathrm{map}(X, Y; f) \to Y$ evaluates a map at the base point. Our results here will be expressed in terms of these relative Gottlieb groups.

Our basic result is a direct generalization of Theorem 1 of the Introduction.

Theorem 4.1. *For $n \geq 1$, we have a split extension*

$$\begin{tikzcd} 0 \arrow[r] & G_{n+1}(Y, X; f) \arrow[r] & G_n\big(L_f Y, \Lambda X; (\mathrm{ev}, f_*)\big) \arrow[r] & G_n(X) \arrow[r] \arrow[l, bend right] & 1\ . \end{tikzcd}$$

For each $n \geq 2$, this is a split short exact sequence of abelian groups, and we have an isomorphism of abelian groups

$$G_n\big(L_f Y, \Lambda X; (\mathrm{ev}, f_*)\big) \cong G_n(X) \oplus G_{n+1}(Y, X; f).$$

Proof. The map $f\colon X \to Y$ induces $f_*\colon \mathrm{aut}_1 X \to \mathrm{map}(X, Y; f)$, which we may use to construct $L_{f_*}\mathrm{map}(X, Y; f)$ as the pullback of $P\colon P\mathrm{map}(X, Y; f) \to \mathrm{map}(X, Y; f) \times \mathrm{map}(X, Y; f)$ along $\Delta \circ f_*\colon \mathrm{aut}_1 X \to \mathrm{map}(X, Y; f) \times \mathrm{map}(X, Y; f)$. The resulting fibre sequence, displayed as the left-hand column in the following pullback diagram

$$\begin{tikzcd} \Omega\mathrm{map}(X, Y; f) \arrow[r, equal] \arrow[d] & \Omega\mathrm{map}(X, Y; f) \arrow[d] \\ L_{f_*}\mathrm{map}(X, Y; f) \arrow[r] \arrow[d] & P\mathrm{map}(X, Y; f) \arrow[d, "P"] \\ \mathrm{aut}_1 X \arrow[r, "\Delta \circ f_*"'] & \mathrm{map}(X, Y; f) \times \mathrm{map}(X, Y; f), \end{tikzcd}$$

fits into a ladder of fibre sequences, displayed as the rows in the following diagram

$$
\begin{tikzcd}
 & & \mathrm{map}\big(\Lambda X, L_f Y; (\mathrm{ev}, f_*)\big) \arrow[d, shift left, "r"] & \\
\Omega \mathrm{map}(X, Y; f) \arrow[rr, "j_{f_*}"] \arrow[d, "\Omega_{\mathrm{ev}}"'] & & L_{f_*} \mathrm{map}(X, Y; f) \arrow[u, shift left, "\Phi"] \arrow[r, "\mathrm{ev}_{f_*}"'] \arrow[d, "(\omega_B)_*"'] & \mathrm{aut}_1 X \arrow[l, bend right, "\sigma_{f_*}"'] \arrow[d, "\omega_X"] \\
\Omega Y \arrow[rr, "j_f"'] & & L_f Y \arrow[r, "\mathrm{ev}_f"] & X. \arrow[l, bend left, "\sigma_f"]
\end{tikzcd}
$$

We describe the retraction $L_{f_*} \mathrm{map}(X, Y; f) \to \mathrm{map}\big(\Lambda X, L_f Y; (\mathrm{ev}, f_*) \to L_{f_*} \mathrm{map}(X, Y; f)$. As a first step, we identify

$$
L_{f_*} \mathrm{map}(X, Y; f) \equiv \mathrm{map}(X, L_f Y; \sigma_f).
$$

This follows from an identification of pullback squares

$$
\begin{tikzcd}
 & \mathrm{map}(X, L_f Y; \sigma_f) \arrow[rr] \arrow[dd, "(\mathrm{ev}_f)_*"] & & \mathrm{map}(X, PY) \arrow[dd, "P_*"] \\
L_{f_*} \mathrm{map}(X, Y; f) \arrow[ur] \arrow[rr] \arrow[dd, "\omega_\#"'] & & P\,\mathrm{map}(X, Y; f) \arrow[ur, equal] \arrow[dd, "\omega_\#", near start] & \\
 & \mathrm{map}(X, X; 1_X) \arrow[rr, "(f,f)_*"'] \arrow[dl, equal] & & \mathrm{map}(X, Y \times Y) \arrow[dl, equal] \\
\mathrm{aut}_1 X \arrow[rr, "(f_*, f_*)"'] & & \mathrm{map}(X, Y; f) \times \mathrm{map}(X, Y; f) &
\end{tikzcd}
$$

Then the argument is a direct generalization of that used in the proof of Theorem 2.3. Notice that, in degrees ≥ 2, all groups concerned are abelian: the ordinary Gottlieb groups are always abelian, and $G_r(Y, X; f) \subseteq \pi_r(Y)$, which is abelian for $r \geq 2$. $\square$

We may iterate this construction, and obtain decompositions that are relative versions of those in Section 3. We indicate this direction, but do not attempt any great generality. To iterate, we regard $L_f Y$ as the pullback of $\mathrm{ev}\colon \Lambda Y \to Y$ along $f\colon X \to Y$. Then the next step is to form $L_f^2 Y$ as the pullback of $\mathrm{ev}\colon \Lambda L_f Y \to L_f Y$ along $(\mathrm{ev}, f_*)\colon \Lambda X \to L_f Y$. We obtain a map $(\mathrm{ev}, f_*)^2\colon \Lambda^2 X \to L_f^2 Y$ as the whisker map in the following pullback diagram:

$$
\begin{tikzcd}
\Lambda^2 X \arrow[dr, dotted] \arrow[drr, bend left, "\Lambda(\mathrm{ev}, f_*)"] \arrow[ddr, bend right, "\mathrm{ev}"'] & & \\
 & L_f^2 Y \arrow[r] \arrow[d] & \Lambda L_f Y \arrow[d, "\mathrm{ev}"] \\
 & \Lambda X \arrow[r, "(\mathrm{ev}, f_*)"'] & L_f Y.
\end{tikzcd}
$$

We state, without proof, the decomposition for the abelian relative Gottlieb groups of the iterated construction:

Corollary 4.2. *For each $n \geq 2$, we have an isomorphism of abelian groups*

$$G_n\big(L_f^2 Y, \Lambda^2 X; (\mathrm{ev}, f_*)^2\big) \cong G_n(X) \oplus 2G_{n+1}(X) \oplus G_{n+2}(Y, X; f). \quad \square$$

If $f\colon X \to Y$ is the identity, then this decomposition and that of the previous result reduce to those in Section 3. We mention also that, as we did with the free loop space, we may extend this construction to a free bouquet space, and also iterate it, so as to obtain what we could call *iterated, relative free bouquet spaces.* In the first instance, the effect would be to introduce a coefficient of m before the $G_{n+1}(Y, X; f)$ term in Theorem 4.1.

5. Consequences and Concluding Remarks

It is well-known that, if Y is an H-space, then we have $G_*(Y) = \pi_*(Y)$. A space that satisfies $G_*(Y) = \pi_*(Y)$ is known as a *G-space.* Work of Ganea and others has produced separating examples that illustrate, in general, one may have a G-space that is not an H-space. In [1], Aguadé introduced the notion of a *T-space*, namely a space for which the free loop fibration is fibre-homotopically trivial. Again, an H-space is a T-space, but a separating example that illustrates that, generally, a T-space need not be an H-space is given in [1]. Also, any T-space must be a G-space ([16, Th.2.2]—a stronger statement holds: see in the proof below). We are unaware of a separating example between T-spaces and G-spaces. Theorem 2.5 leads to the following observations in this area.

Corollary 5.1. (1) *If Y is a T-space, then so is* $\mathrm{map}(X, Y; 0)$.

(2) *Suppose that ΣX splits as some wedge of spheres. If Y is a G-space, then so is* $\mathrm{map}(X, Y; 0)$.

Proof. (1) In [16, Th.2.2], it is shown that Y is a T-space if, and only if, we have $\mathcal{G}(\Sigma B, Y) = [\Sigma B, Y]$, for any suspension ΣB. Consider the (split) short exact sequence of image subgroups, included into the (split) short exact sequence of homotopy sets, from which we obtained Theorem 2.3. Namely, from the proof of that result, we have the following inclusion of one short exact sequence in another:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\Sigma(A \wedge X), Y) \arrow[r] \arrow[d, hook] & \mathcal{G}\big(\Sigma A, \mathrm{map}(X, Y; 0)\big) \arrow[r] \arrow[d, hook] & \mathcal{G}(\Sigma A, Y) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & {[\Sigma(A \wedge X), Y]} \arrow[r] & {[\Sigma A, \mathrm{map}(X, Y; 0)]} \arrow[r] & {[\Sigma A, Y]} \arrow[r] & 0
\end{tikzcd}$$

Since Y is a T-space, the left and right inclusions are equalities, which implies the middle inclusion is an equality. From the criterion of [16, Th.2.2], we obtain that $\mathrm{map}(X, Y; 0)$ is a T-space.

(2) Now suppose that Y is a G-space. We take $A = S^{n-1}$, for $n \geq 1$, in the above diagram and obtain the following inclusion of one short exact sequence in another:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\Sigma^n X, Y) \arrow[r] \arrow[d, hook] & G_n\big(\mathrm{map}(X, Y; 0)\big) \arrow[r] \arrow[d, hook] & G_n(Y) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & {[\Sigma^n X, Y]} \arrow[r] & \pi_n\big(\mathrm{map}(X, Y; 0)\big) \arrow[r] & \pi_n(Y) \arrow[r] & 0
\end{tikzcd}$$

Our assumption on ΣX implies that the left-hand inclusion decomposes into a product of inclusions of Gottlieb groups of Y into the corresponding homotopy groups of Y. Since Y is a G-space, left and right inclusions are equalities, and it follows that the middle inclusion is an equality. That is, $\mathrm{map}(X, Y; 0)$ is a G-space. $\square$

Remark 5.2. It is easily seen that the property of being a G-space, respectively a T-space, is inherited by retracts (cf. [16, Cor.2.10] for the latter). Since $\mathrm{map}(X, Y; 0)$ retracts onto Y, the statements in Corollary 5.1 may be replaced by two-way implications. Notice that the first includes [16, Th.2.12]. But the interest in Corollary 5.1 comes from the fact that we are able to deduce properties of the function space $\mathrm{map}(X, Y)$ from properties of Y alone, rather than the other way around. This is the point of view developed in our remaining observations.

As we have seen in Theorem 2.5, if ΣX splits as a bouquet of spheres, then Corollary 2.4 yields a decomposition of $G_*(\mathrm{map}(X, Y; 0))$ in terms of the Gottlieb groups of Y.

Example 5.3. Suppose we take $X = S^5 \cup_\alpha e^{10}$, with $\alpha \in \pi_9(S^5) \cong \mathbb{Z}_2$ the non-zero element. Since $\pi_{10}(S^6) = 0$, we have that $\Sigma X \simeq S^6 \vee S^{10}$. From Corollary 2.4, we obtain

$$G_n(\mathrm{map}(S^5 \cup_\alpha e^{10}, Y; 0)) \cong G_n(Y) \oplus G_{n+5}(Y) \oplus G_{n+10}(Y),$$

for $n \geq 1$.

Likewise, if we have a space X for which some suspension $\Sigma^n X$ splits as a wedge of spheres, then all but finitely many of the Gottlieb groups of $\mathrm{map}(X, Y)$ may be expressed as direct sums of the Gottlieb groups of Y.

Pursuing this line somewhat leads to a strong consequence for the global structure of Gottlieb groups of function spaces. Suppose Y is a simply connected, finite complex. Then a result of Félix-Halperin (cf. [2, Prop.28.8]) implies that the Gottlieb groups of Y are finite groups in all but a finite number of degrees.

The function spaces that we consider here are neither simply connected, nor finite complexes in general. But from our results above, we deduce the following intriguing fact: if X and Y are finite complexes, and Y is simply connected, then $G_n\big(\mathrm{map}(X, Y; 0)\big)$ is a finite group in all but a finite number of degrees. We first prove Theorem 5 of the Introduction. This result explicitly identifies the rational Gottlieb groups of $\mathrm{map}(X, Y)$ in terms of the rational homology of X and the rational Gottlieb groups of Y, which are both fairly amenable to calculation.

Proof of Theorem 5. Corollary 2.4 gives an isomorphism of abelian groups

$$G_n\big(\mathrm{map}(X, Y; 0)\big) \cong G_n(Y) \oplus \mathcal{G}(\Sigma^n X, Y),$$

for each $n \geq 1$. Hence $\gamma_n\big(\mathrm{map}(X, Y; 0)\big) = \gamma_n(Y) + \dim_{\mathbb{Q}}\big(\mathcal{G}(\Sigma^n X, Y) \otimes \mathbb{Q}\big)$. Here, $\dim_{\mathbb{Q}}(-)$ refers to the dimension as a rational vector space. We focus on this last term. Let $l\colon Y \to Y_{\mathbb{Q}}$ denote the rationalization of Y. The homomorphism induced by l rationalizes the group $[\Sigma^n X, Y]$. That is, we have $l_*\big([\Sigma^n X, Y]\big) \cong [\Sigma^n X, Y] \otimes \mathbb{Q}$ as abelian groups. This restricts to generalized Gottlieb groups, and gives $l_*\big(\mathcal{G}(\Sigma^n X, Y)\big) \cong \mathcal{G}(\Sigma^n X, Y) \otimes \mathbb{Q}$. Furthermore, because Y is finite, we may identify $l_*\big(\mathcal{G}(\Sigma^n X, Y)\big) = \mathcal{G}(\Sigma^n X, Y_{\mathbb{Q}})$, (see [9, Th.2.2], and [15, Th.3.2]).

Now $\Sigma^n X$ has the same rational homotopy type as some wedge of spheres (see [2, Th.24.5]). Since a rational homotopy equivalence preserves Betti numbers, we must have a rational homotopy equivalence

$$\theta\colon \bigvee_{i=1}^{\dim X} \beta_i(X) S^{n+i} \to \Sigma^n X.$$

Here, for b a non-negative integer, bS^n denotes the n-fold wedge $S^n \vee \cdots \vee S^n$. But then the isomorphism $\theta^*\colon [\Sigma^n X, Y_{\mathbb{Q}}] \to [\vee_{i=1}^{\dim X} \beta_i(X) S^{n+i}, Y_{\mathbb{Q}}]$ restricts to an isomorphism $\theta^*\colon \mathcal{G}(\Sigma^n X, Y_{\mathbb{Q}}) \to \mathcal{G}(\vee_{i=1}^{\dim X} \beta_i(X) S^{n+i}, Y_{\mathbb{Q}})$, and we may write

$$\mathcal{G}(\bigvee_{i=1}^{\dim X} \beta_i(X) S^{n+i}, Y_{\mathbb{Q}}) \cong \oplus_{i=1}^{\dim X} \beta_i(X) \mathcal{G}(S^{n+i}, Y_{\mathbb{Q}}) = \oplus_{i=1}^{\dim X} \beta_i(X) G_{n+i}(Y_{\mathbb{Q}}).$$

Once again, since Y is finite, we have $G_{n+i}(Y) \otimes \mathbb{Q} \cong G_{n+i}(Y_{\mathbb{Q}})$, and so we have

$$\dim_{\mathbb{Q}}\big(\mathcal{G}(\Sigma^n X, Y) \otimes \mathbb{Q}\big) = \sum_{i=1}^{\dim X} \beta_i(X) \gamma_{n+i}(Y).$$

Combining this with the $G_n(Y)$ term gives the result. □

Corollary 5.4. *Let X and Y be finite complexes, with Y simply connected. Then $G_n\big(\mathrm{map}(X, Y; 0)\big)$ is a finite group for all but finitely many n. If N is the highest degree in which $G_N(Y)$ has positive rank, then N is also the highest degree in which $G_N\big(\mathrm{map}(X, Y; 0)\big)$ has positive rank, and $\gamma_N\big(\mathrm{map}(X, Y; 0)\big) = \gamma_N(Y)$.*

Proof. The result of Félix-Halperin [2, Prop.28.8] mentioned above guarantees there is some N that is the highest degree in which $G_N(Y)$ has positive rank. The remaining assertions follow from Theorem 5 □

Finally, we observe that Theorem 2.5 could, in the right circumstances, provide a necessary condition for a space to be a function space of the kind considered there. For example, suppose that we have a fibration $\Omega Y \to E \to Y$ that admits a section, for some space Y. One might ask whether E is of the homotopy type of ΛY? Theorem 2.5 requires that $G_*(E) \cong G_*(Y) \oplus G_{*+1}(Y)$ for this to be possible.

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