3-15-2012

On Fox's M-Dimensional Category and Theorems of Bochner Type

John Oprea
Cleveland State University, J.OPREA@csuohio.edu

Jeff Strom
Western Michigan University

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1. Introduction

Bochner's theorem [35] (also see [28]) asserts that, in the presence of non-negative Ricci curvature, the first Betti number of a compact manifold M is bounded above by the dimension of M; $b_1(M) \leqslant \dim(M)$. Furthermore, if $b_1(M) = \dim(M)$, then M is a flat torus. While Bochner's approach was overtly analytic in nature, it was shown in [23] that the analysis could be swept under the rug of the Cheeger–Gromoll splitting theorem to obtain a topological estimate $b_1(M) \leqslant \mathrm{cat}(M)$, where $\mathrm{cat}(-)$ is the homotopy invariant known as Lusternik–Schnirelmann category. In general, we know that $\mathrm{cat}(M) \leqslant \dim(M)$, so the new upper bound provided a refinement. Indeed, if $M = S^2 \times T^2$, then M has (a metric with) non-negative Ricci curvature, $b_1(M) = 2$, $\mathrm{cat}(M) = 3$ and $\dim(M) = 4$, so we don't have to hunt hard for examples where the category bound is better. Nevertheless, the new bound had one unsatisfactory property: it obeyed the rule that $b_1(M) = \mathrm{cat}(M)$ if and only if M is a flat torus. Now, tori are very special indeed (e.g. $\mathrm{cat}(T^m) = \dim(T^m)$), so to say that equality only holds in the toral case hints at a better estimate. Just as this fact indicated that the original Bochner dimension bound was refinable by category, we can ask if yet another refinement exists for the category bound.

It is the purpose of this paper to show that, indeed, there is such a refinement within the context of category *without* the property that only flat tori give equality in the standard inequality. In [22], it was shown that the same basic approach used in refining Bochner could be used to obtain an upper bound for the rank of the Gottlieb group of a space. Here we will show that the new categorical invariant also may be used to obtain a refined upper bound in this context. Finally, we will extend our results to the class of almost non-negatively sectionally curved manifolds using the results of [19]. In this regard, we will show how the "category invariant approach" can recover a Bochner type result of Yamaguchi [34].

In order to state the main results, we need to recall two definitions. The *Lusternik–Schnirelmann category* of a space X, denoted $\mathrm{cat}(X)$, is the smallest integer k so that X can be covered by open sets $U_0, U_1, \ldots, U_k$, each of which is contractible to a point in X. Such a covering is called a *categorical covering*. LS category is an important numerical invariant in algebraic topology, critical point theory and symplectic geometry (see, for instance, [6,8,29]). Since it is notoriously difficult to compute, many approximating invariants have been introduced in order to estimate category from below and above (see, for instance, [26]). In this paper, we will use one of these approximating invariants, $\mathrm{cat}_1(-)$, to provide new upper bounds. We prove the following.

Theorem. *(See Theorem 5.7.) Suppose M is a compact manifold with non-negative Ricci curvature and infinite fundamental group. Then*

$$b_1(M) \leqslant \mathrm{cat}_1(M)$$

where $b_1(M)$ is the first Betti number of M.

Example 5.8 then shows that it is possible to have $b_1(M) = \mathrm{cat}_1(M)$ for a non-toral compact M with non-negative Ricci curvature and infinite fundamental group.

An element $\alpha \in \pi_1(X)$ is a *Gottlieb element* if there exists an extension A (called *an associated map*) in the diagram

$$\begin{tikzcd} S^1 \vee X \arrow[r, "(\alpha, \mathrm{id}_X)"] \arrow[d] & X. \\ S^1 \times X \arrow[ur, dotted, "A"'] & \end{tikzcd}$$

The set of all Gottlieb elements in $\pi_1(X)$ is a subgroup of the center of $\pi_1(X)$ and is denoted $G_1(X)$. If the abelian group $G_1(X)$ is finitely generated, then it was shown in [22] that the rank of $G_1(X)$ is bounded above by the Lusternik–Schnirelmann category of X. A much better bound is provided by the following.

Theorem. *(See Theorem 5.4.) Writing $\mathrm{cat}_1(X)$ for Fox's 1-dimensional category, we have*

$$\mathrm{rank}\big(G_1(X)\big) \leqslant \mathrm{cat}_1(X),$$

for any normal space X with finitely generated $G_1(X)$.

Throughout the paper, we consider spaces that are of the homotopy type of CW complexes. (In particular, spaces are paracompact normal ANR's (see [6, Appendix 1]).) Also, because the paper is intended for geometers as well as topologists, we have included as many details concerning LS category as is feasible.

2. Fox's m-dimensional category

In [15], R. Fox introduced the notion of *m-dimensional category* as an approximating invariant for LS category. Say that $\mathrm{cat}_m(X) = k$ if k is the least integer so that there exists an open cover $\{U_0, \dots, U_k\}$ of X such that, for each U_j, every composition $P \to U_j \hookrightarrow X$ with $\dim(P) \leqslant m$ is nullhomotopic (where P is a polyhedron). We say that any such open set U is *m-categorical*. Immediately, we see that $\mathrm{cat}_m(X) \leqslant \mathrm{cat}(X)$ for all $m \geqslant 0$. Also note that simplicial or cellular approximation provides the following.

Lemma 2.1. *If X is n-connected, then $\mathrm{cat}_m(X) = 0$ for all $m \leqslant n$.*

We write $X \to X[m]$ for the mth Postnikov section of X and $\phi_m : X\langle m\rangle \to X$ for its homotopy fiber, known as the *m-connected cover* of X (in particular, $X\langle 1\rangle \to X$ is, up to homotopy equivalence, the universal cover of X). Svarc [31] identified $\mathrm{cat}_m(X)$ with an invariant called the *genus* of the m-connected cover fibration $\phi_m : X\langle m\rangle \to X$. In modern parlance, the genus of a fibration $F \to E \xrightarrow{p} B$ is called the *sectional category*; it is the least integer k for which there is an open cover $B = U_0 \cup U_1 \cup \cdots \cup U_k$ such that there is a partial section of p over each U_j. Thus we have the following modern formulation of Svarc's result (also see [8, Proposition 4.4]).

Proposition 2.2. *([31, Proposition 44]) If X is a CW complex, then*

$$\mathrm{cat}_m(X) = \mathrm{secat}\big(X\langle m\rangle \to X\big).$$

Sketch of proof. First suppose $\{U_0, \dots, U_s\}$ is a cover of X where each U_j has a section $s_j : U_j \to X\langle m\rangle$. Let $f : P \to U_j$ with $\dim(P) \leqslant m$. Then, by cellular approximation, s_j factors up to homotopy through the m-skeleton of $X\langle m\rangle$ and this is homotopically trivial since $X\langle m\rangle$ is m-connected. Hence $s_j|_P \simeq *$ and so $\phi_m s_j|_P \simeq *$ as well. Therefore, $\{U_0, \dots, U_s\}$ is an m-categorical cover and $\mathrm{cat}_m(X) \leqslant \mathrm{secat}(X\langle m\rangle \to X)$.

Secondly, suppose $U_0, \ldots, U_s$ is an m-categorical cover of X. Since X is normal, there is an open refinement $V_0, \ldots, V_s$ with $V_j \subset \overline{V}_j \subset U_j$ for each $j = 0, \ldots, s$. The closed sets $\overline{V}_j$ can be taken to be subcomplexes, so the m-skeleta $\overline{V}_j^m$ map nullhomotopically into X. Then we see that the obstructions to finding a partial section of $\phi_m : X\langle m\rangle \to X$ over $\overline{V}_j$ lie in $H^{t+1}(\overline{V}_j, \overline{V}_j^m; \pi_t(F))$, where $F = \mathrm{Fiber}(\phi_m)$. But these groups are zero for $t \leqslant m-1$ and for $t \geqslant m$, $\pi_t(F) = 0$ (since ϕ_m induces isomorphisms $\pi_t(X\langle m\rangle) \to \pi_t(X)$). Thus the obstructions all vanish and we obtain a partial section of ϕ_m over $\overline{V}_j$. But every subcomplex has an open neighborhood that deformation retracts onto it, so we can intersect the neighborhoods for the $\overline{V}_j$ with the U_j to obtain s open sets with partial sections of ϕ_m. Hence, $\mathrm{cat}_m(X) \geqslant \mathrm{secat}(X\langle m\rangle \to X)$. $\square$

3. Sectional category and the category of a map

Because Fox's m-dimensional category is given by the sectional category of the m-connected cover, we can hope to understand it better by recalling the properties of secat. (Most of these properties were first proved in [31]. We present them here from a modern viewpoint with simple proofs.) Although we mentioned the definition of sectional category before Proposition 2.2, for easy reference, we state it here as

Definition 3.1. Suppose $F \to E \xrightarrow{p} B$ is a fibration. Then the *sectional category* of p, denoted $\mathrm{secat}(p)$, is the least integer n such that there exists an open covering, $U_0, \ldots, U_n$, of B and, for each U_i, a map $s_i : U_i \to E$ having $p \circ s_i = \mathrm{id}_{U_i}$. (Because the U_i are subsets of B, the lift s_i is referred to as a local (or partial) section of p.)

The basic results about secat are contained in the following.

Proposition 3.2. *Let $F \to E \xrightarrow{p} B$ be a fibration. Then*:

(1) $\mathrm{secat}(p) \leqslant \mathrm{cat}(B)$.
(2) *If E is contractible, then* $\mathrm{secat}(p) = \mathrm{cat}(B)$.
(3) *If there are $x_1, \ldots, x_k \in \widetilde{H}^*(B; R)$ (any coefficient ring R) with*

$$p^* x_1 = \cdots = p^* x_k = 0 \quad \textit{and} \quad x_1 \cup \cdots \cup x_k \neq 0,$$

then $\mathrm{secat}(p) \geqslant k$.

Proof. We prove (1) and (3) and leave (2) as an exercise.

For (1), suppose $\mathrm{cat}(B) = n$ with categorical covering $U_0, \ldots, U_n$. Consider the homotopy lifting diagram

$$\begin{tikzcd} U_i \times 0 \arrow[r, "e_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ U_i \times I \arrow[r, "H"] \arrow[ur, dashed, "G"] & B \end{tikzcd}$$

where e_0 is the constant map to a chosen point in the fiber of a basepoint $b_0 \in B$ and H is a contracting homotopy with H_0 the constant map at b_0 and H_1 the inclusion $U_i \hookrightarrow B$ (which we write as incl_{U_i}). The map G exists by the homotopy lifting property; note that $G_0 = e_0$ and $p \circ G_1 = H_1 = \mathrm{incl}_{U_i}$, so G_1 is a section of p over U_i. Since this procedure works for each U_i, we have $\mathrm{secat}(p) \leqslant n = \mathrm{cat}(B)$.

For (3), suppose $\mathrm{secat}(p) = m$ and that $U_0, \ldots, U_m$ cover B with local sections $s_0, \ldots, s_m$ respectively. Suppose that cohomology classes $x_0, \ldots, x_m \in \widetilde{H}^*(B; R)$ satisfy $p^*(x_i) = 0$ for each $i = 1, \ldots, m$. Denote the obvious inclusions by $\mathrm{incl}_{U_i} : U_i \hookrightarrow B$, $q_i : B \hookrightarrow (B, U_i)$ and $q : B \hookrightarrow (B, \bigcup U_i)$. But then the condition $p^*(x_i) = 0$ gives

$$\mathrm{incl}_{U_i}^*(x_i) = s_i^*\bigl(p^*(x_i)\bigr) = 0$$

since $p \circ s_i = \mathrm{incl}_{U_i}$, and the long exact sequence in cohomology associated to the pair (B, U_i) provides an element $\bar{x}_i \in H^*(B, U_i; R)$ with $q_i^*(\bar{x}_i) = x_i$. This can be done for each i and the resulting product $\bar{x}_0 \cup \cdots \cup \bar{x}_m \in H^*(B, \bigcup U_i; R)$ satisfies $q^*(\bar{x}_0 \cup \cdots \cup \bar{x}_m) = x_0 \cup \cdots \cup x_m$. From the definition of sectional category, we have $B = \bigcup U_i$. Thus $H^*(B, \bigcup U_i; R) = 0$ and, hence, $\bar{x}_0 \cup \cdots \cup \bar{x}_m = 0$. Therefore, $x_0 \cup \cdots \cup x_m = 0$ as well and we see that any non-zero k-fold product of classes satisfying the hypotheses of (3) must have length less than or equal to $\mathrm{secat}(p)$. $\square$

Proposition 3.2 may be generalized for fibrations obtained as a pullback along a map f of a fibration with a contractible total space. In this case we can identify sectional category with the category of the map f. The *category of a map* $f : X \to Y$ is denoted $\mathrm{cat}(f)$ and is defined to be the least integer n such that X may be covered by open sets $U_0, \ldots, U_n$ with $f|_{U_i}$ nullhomotopic for each i. Such a covering is said to be *categorical* for the map f. Now let's see how sectional category relates to the category of "classifying" maps.

Proposition 3.3. *Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration arising as a pullback of a fibration $\widehat{p}: \widehat{E} \to \widehat{B}$*

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widehat{E} \arrow[d, "\widehat{p}"] \\ B \arrow[r, "f"] & \widehat{B}. \end{tikzcd}$$

(1) *In general,* $\operatorname{secat}(p) \leqslant \operatorname{secat}(\widehat{p})$.
(2) *If $\widehat{E}$ is contractible. Then* $\operatorname{secat}(p) = \operatorname{cat}(f)$.

Proof. (1) Suppose $s: \widehat{U} \to \widehat{E}$ has $\widehat{p}\,\widehat{s} = 1|_{\widehat{U}}$. Let $U = f^{-1}(\widehat{U})$ and use the pullback property to define $s: U \to E$ as follows:

$$\begin{tikzcd} U \arrow[dr, dashed, "s"] \arrow[drr, bend left, "sf"] \arrow[ddr, bend right, "j"'] & & \\ & E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widehat{E} \arrow[d, "\widehat{p}"] \\ & B \arrow[r, "f"] & \widehat{B}. \end{tikzcd}$$

The pullback property then gives $ps = j = 1_U$. Hence, a categorical cover for $\widehat{p}$ provides one for p and, consequently $\operatorname{secat}(p) \leqslant \operatorname{secat}(\widehat{p})$.

(2) We shall prove inequalities both ways, thereby establishing the equality of the invariants. Suppose $\operatorname{secat}(p) = n$ and that $U_0, \ldots, U_n$ form an open covering of B with, for each i, a section $s_i: U_i \to E$ of p. By commutativity of the pullback diagram, we have $\widehat{p}\,\widetilde{f} s_i = f p s_i = f$ since $p s_i = 1_B$. This says that the map $f|_{U_i}$ factors through the contractible space $\widehat{E}$, and so $f|_{U_i}$ is nullhomotopic. Thus $U_0, \ldots, U_n$ is categorical for f and therefore $\operatorname{cat}(f) \leqslant n = \operatorname{secat}(p)$.

Now suppose that $\operatorname{cat}(f) = n$ with categorical covering $U_0, \ldots, U_n$. For each $i = 1, \ldots, n$, consider the homotopy lifting diagram

$$\begin{tikzcd} U_i \times 0 \arrow[r, "e_0"] \arrow[d, hook] & \widehat{E} \arrow[d, "\widehat{p}"] \\ U_i \times I \arrow[r, "H"] \arrow[ur, dashed, "G"] & \widehat{B} \end{tikzcd}$$

in which $H_0 = *$, $H_1 = f|_{U_i}$ and e_0 is the constant map to a point in the fiber over $* \in \widehat{B}$. Since $\widehat{p}$ is a fibration, there is a lift G that satisfies $\widehat{p} \circ G_1 = H_1 = f|_{U_i}$ up to homotopy. Now, again since $\widehat{p}$ is a fibration, the (topological) pullback is a homotopy pullback. Therefore, for each i, we have a map $s_i: U_i \to E$ guaranteed by the (homotopy) pullback diagram

$$\begin{tikzcd} U_i \arrow[dr, dashed, "s_i"] \arrow[drr, bend left, "G_1"] \arrow[ddr, bend right, "j"'] & & \\ & E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & \widehat{E} \arrow[d, "\widehat{p}"] \\ & B \arrow[r, "f"] & \widehat{B}, \end{tikzcd}$$

in which $j: U_i \to B$ is the inclusion. Now $p \circ s_i = j$ and therefore s_i is a section of p over U_i. Hence, $\operatorname{secat}(p) \leqslant n = \operatorname{cat}(f)$. $\square$

Proposition 3.3 has immediate relevance for computing $\operatorname{cat}_m(X)$. The m-connected cover $X\langle m\rangle \to X$ arises as the fiber of the mth Postnikov section, $j_m: X \to X[m]$, so it is a homotopy pullback

$$\begin{tikzcd} X\langle m\rangle \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r, "j_m"] & X[m]. \end{tikzcd}$$

By Propositions 3.3 (2) and 2.2, we see

Theorem 3.4. *For a normal ANR X,*

$$\mathrm{cat}_m(X) = \mathrm{cat}(j_m),$$

where $j_m : X \to X[m]$ is the mth Postnikov section.

Corollary 3.5. *If $\pi_1(X) = \pi$, $B\pi = K(\pi, 1)$ and k is the maximum cup length of a product in the image of $j_1^* : H^k(B\pi; \mathcal{A}) \to H^k(X; \mathcal{A})$, then*

$$k \leqslant \mathrm{cat}_1(X).$$

Proof. This is simply a translation of the standard cup length bound for the category of a map (in this case for j_1). See [6, Exercise 1.16]. □

Cup length can be refined by the notion of *category weight* (see, for instance, [6, Definition 8.20 and Proposition 8.22]) originally due, in the non-homotopy invariant case to Fadell–Husseini and in the homotopy invariant case, independently, to Y. Rudyak and J. Strom.

The *category weight* of a non-zero cohomology class $u \in H^*(X; A)$ (for some, possibly local, coefficient ring A) is defined by

$$\mathrm{wgt}(u) \geqslant k \quad \text{if and only if} \quad \phi^*(u) = 0 \quad \text{for any } \phi : Z \to X \text{ with } \mathrm{cat}(\phi) < k.$$

The basic facts that we require about category weight are that

- if $u \in H^s(K(\pi, 1); A)$, then $\mathrm{wgt}(u) = s$;
- if $f : Y \to X$ has $f^*(u) \neq 0$ for some $u \in H^s(X; A)$, then $\mathrm{cat}(f) \geqslant \mathrm{wgt}(u)$.

The following consequence of Theorem 3.4 is implicit in the more complicated results of [12,20].

Corollary 3.6. *If $\pi_1(X) = \pi$, $B\pi = K(\pi, 1)$ and k is the maximum degree for which $j_1^* : H^k(B\pi; \mathcal{A}) \to H^k(X; \mathcal{A})$ is non-trivial (for any local coefficients $\mathcal{A}$), then*

$$k \leqslant \mathrm{cat}_1(X) \leqslant \mathrm{cat}(B\pi) = \dim(B\pi).$$

Moreover, if $X = B\pi$ and $\dim(B\pi) > 3$, then $\mathrm{cat}_1(X) = \dim(B\pi)$.[1]

Proof. Because $\mathrm{cat}_1(X) = \mathrm{cat}(j_1)$, we can use information about $j_1 : X \to B\pi$ to obtain estimates for $\mathrm{cat}_1(X)$. The category of a map is always bounded above by the categories of its range and domain, so $\mathrm{cat}_1(X) \leqslant \mathrm{cat}(B\pi) = \dim(B\pi)$. The lower bound follows from properties of category weight listed above. Namely, the category weight of any cohomology class $u \in H^k(B\pi; \mathcal{A})$ has $\mathrm{wgt}(u) = k$ and if $j_1^*(u) \neq 0$, $\mathrm{cat}(j_1) \geqslant \mathrm{wgt}(u) = k$. The last statement follows immediately from these remarks (also see the discussion before Proposition 4.2). □

Corollary 3.7. *If $\pi_1(X)$ is a non-trivial free group, then $\mathrm{cat}_1(X) = 1$.*

Proof. The only thing to check is that we cannot have $\mathrm{cat}_1(X) = 0$, but this follows because $\mathrm{cat}_1(X) = 0$ would imply that the universal covering $\widetilde{X} \to X$ has a section and this can only happen if $\pi_1(X)$ is trivial. □

In fact, it is true that $\pi_1(X)$ is free if and only if $\mathrm{cat}_1(X) = 1$. This follows from the following characterization of cat_1 established in [12] (also see [20]).

Theorem 3.8. *For a CW complex X, $\mathrm{cat}_1(X) \leqslant n$ if and only if there is an n-dimensional complex L and a map $X \to L$ which induces an isomorphism on fundamental groups.*

Now, if $\mathrm{cat}_1(X) = 1$, this then implies $\pi_1(X) \cong \pi_1(L)$ with $\dim(L) = 1$. Since the fundamental group of any 1-dimensional complex is free, we see the equivalence.

Finally, the characterization of $\mathrm{cat}_1(-)$ given in Theorem 3.4 provides an integral analogue of the famous Mapping Theorem of rational homotopy theory (see [6, Theorem 4.11] for instance). Recall that this says that if there is a map $f : X \to Y$ of simply connected spaces that induces an injection $f_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$, then $\mathrm{cat}(X_0) \leqslant \mathrm{cat}(Y_0)$, where the subscript 0 denotes rationalization.

[1] Because $B\pi = K(\pi, 1)$ is determined only up to homotopy type, we define $\dim(B\pi)$ to be the smallest dimension of a CW complex which is a $K(\pi, 1)$.

Theorem 3.9. *If $f: X \to Y$ is a map of CW complexes that induces an injection $f_*: \pi_1(X) \to \pi_1(Y)$, then $\mathrm{cat}_1(X) \leqslant \mathrm{cat}_1(Y)$.*

Proof. Consider the following commutative diagram of Postnikov sections

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\ K(\pi_1 X, 1) \arrow[r, "\bar{f}"] & K(\pi_1 Y, 1). \end{tikzcd}$$

By [6, Proposition 1.10], we know that, for CW complexes, we can use closed sets in the definition of category instead of open sets. Let $\{K_i \mid i = 1, \dots, n\}$ be a closed cover of Y with $j_Y|_{K_i} \simeq *$. Then, for $L_i = f^{-1}(K_i)$, $i = 1, \dots, n$, we have $\bar{f} j_X|_{L_i} = j_Y f|_{L_i} = j_Y|_{K_i} \simeq *$. But $\bar{f} j_X : L_i \to K(\pi_1 Y, 1)$ is determined up to homotopy by the induced map on fundamental groups (since L_i is a CW complex) and f_* is injective. Thus, $j_X|_{L_i} \simeq *$. Hence, $\{L_i \mid i = 1, \dots, n\}$ is a closed categorical cover for j_X and we have

$$\mathrm{cat}_1(X) = \mathrm{cat}(j_X) \leqslant \mathrm{cat}(j_Y) = \mathrm{cat}_1(Y). \qquad \square$$

From this we obtain a result of Fox (which can also be proved by applying the homotopy lifting property to the original definition of $\mathrm{cat}_1(-)$).

Corollary 3.10. *([15, Theorem 21.2]) If X is a CW complex and $p: \overline{X} \to X$ is a covering space, then $\mathrm{cat}_1(\overline{X}) \leqslant \mathrm{cat}_1(X)$.*

4. Products and a splitting theorem

We can also use results on open covers (see Appendix A) to give a variation of the usual proof of the product inequality for LS category.

Proposition 4.1. *If X and Y are CW (or just normal ANR's), then*

$$\mathrm{cat}_m(X \times Y) \leqslant \mathrm{cat}_m(X) + \mathrm{cat}_m(Y).$$

Proof. Let $\{U_0, U_1, \dots, U_k\}$ and $\{V_0, V_1, \dots, V_\ell\}$ be respective categorical covers for $j_m^X : X \to X[m]$ and $j_m^Y : Y \to Y[m]$. By Theorem A.2, there is a $(k+1)$-cover $\{U_0, U_1, \dots, U_{k+\ell}\}$ which is categorical for j_m^X and an $(\ell+1)$-cover $\{V_0, V_1, \dots, V_{k+\ell}\}$ which is categorical for j_m^Y. Clearly then $\{U_0 \times V_0, U_1 \times V_1, \dots, U_{k+\ell} \times V_{k+\ell}\}$ is categorical for $j_m^X \times j_m^Y : X \times Y \to X[m] \times Y[m]$, so we must only show that it is a cover of $X \times Y$. Let $(x, y) \in X \times Y$. By Lemma A.1, y is in at least $(k+1)$ of the V_j. Without loss of generality by renumbering if necessary, suppose $y \in V_0 \cap \dots \cap V_k$. Since, $\{U_0, U_1, \dots, U_{k+\ell}\}$ is a $(k+1)$-cover, x is contained in at least one of $U_0, \dots, U_k$, say U_0. Therefore, $(x, y) \in U_0 \times V_0$. Thus, $\{U_0 \times V_0, U_1 \times V_1, \dots, U_{k+\ell} \times V_{k+\ell}\}$ is a categorical cover for $j_m^X \times j_m^Y$. $\square$

Now we can prove a result we will need later about $\mathrm{cat}_1(-)$ for products $K(\pi, 1) \times N$, where $\pi_1(N) = 0$. However, we must restrict the $K(\pi, 1)$'s we consider because it is possible that there exist such spaces with $\mathrm{cd}(K(\pi, 1)) = 2$, $\mathrm{cat}(K(\pi, 1)) = 2$ and $\dim(K(\pi, 1)) = 3$, where $\mathrm{cd}(-)$ denotes cohomological dimension. Recall that

$$\mathrm{cd}\big(K(\pi, 1)\big) = \sup\big\{N \bigm| H^N\big(K(\pi, 1); A\big) \neq 0, \text{ for some } \pi\text{-module } A\big\}.$$

The *Eilenberg–Ganea conjecture* asserts it is always true that $\mathrm{cd}(K(\pi, 1)) = \mathrm{cat}(K(\pi, 1)) = \dim(K(\pi, 1))$, but this is unresolved at present. As shown by Eilenberg and Ganea [12], however, for $\dim(K(\pi, 1)) > 3$, it *is* always the case that $\mathrm{cd}(K(\pi, 1)) = \mathrm{cat}(K(\pi, 1)) = \dim(K(\pi, 1))$.

Proposition 4.2. *If the Eilenberg–Ganea conjecture holds for $K(\pi, 1)$ and N is a simply connected CW complex, then*

$$\mathrm{cat}_1\big(K(\pi, 1) \times N\big) = \dim\big(K(\pi, 1)\big).$$

Proof. Because N is simply connected, the classifying map j_1 for the universal cover of $K(\pi, 1) \times N$ is the projection $p: K(\pi, 1) \times N \to K(\pi, 1)$. By Theorem 3.4, we have $\mathrm{cat}_1(K(\pi, 1) \times N) = \mathrm{cat}(p)$. But by Corollary 3.6 and the definition of cohomological dimension, $\dim(K(\pi, 1)) \leqslant \mathrm{cat}(p) = \mathrm{cat}_1(K(\pi, 1) \times N)$. By Proposition 4.1, we have $\mathrm{cat}_1(K(\pi, 1) \times N) \leqslant \mathrm{cat}_1(K(\pi, 1)) + \mathrm{cat}_1(N)$ and we know that $\mathrm{cat}_1(N) = 0$ by Lemma 2.1. Hence, $\mathrm{cat}_1(K(\pi, 1) \times N) = \dim(K(\pi, 1))$. $\square$

These results can be used to prove a general result about covering spaces which split off a torus (which of course satisfies the Eilenberg–Ganea conjecture).

Theorem 4.3. *If X is a CW complex and $\overline{X} \to X$ is a covering such that $\overline{X} \simeq T^k \times Y$ with Y simply connected, then $k = \mathrm{cat}_1(\overline{X}) \leqslant \mathrm{cat}_1(X)$.*

Proof. By Corollary 3.10, we see that $\mathrm{cat}_1(\overline{X}) \leqslant \mathrm{cat}_1(X)$ for any covering projection $\overline{X} \to X$, so we need only show that $k \leqslant \mathrm{cat}_1(\overline{X})$. Now, $\mathrm{cat}_1(\overline{X}) = \mathrm{cat}(T^k \times Y)$, and by Proposition 4.2 we have $\mathrm{cat}_1(\overline{X}) = k$ since $\dim(T^k) = k$. □

5. Splitting off tori in homotopy theory and geometry

The theorems of Section 1 rely on the fact that we can often split tori off of a space, at least up to a covering. This is made explicit in the following results. Recall the definition of the Gottlieb group $G_1(X) \subseteq \pi_1(X)$ from Section 1.

Properties 5.1. The basic properties of Gottlieb group which we shall use are the following (see [16] or [24] for instance).

(1) $G_1(X)$ is contained in the center $\mathcal{Z}\pi_1(X)$ of the fundamental group. In fact, if $X = K(\pi, 1)$, then $G(X) = \mathcal{Z}\pi_1(X)$. Moreover, *Gottlieb's Theorem* states that, for $X = K(\pi, 1)$ a finite complex, if X has non-zero Euler characteristic, then $\mathcal{Z}\pi_1(X) = 0$.
(2) If $\alpha_1, \dots, \alpha_k \in G(X)$, then there exists $A : T^k \times X \to X$ with $A|_{S^1_i} = \alpha_i$ and $A|_X = 1_X$. To see this, note that, if $\alpha, \beta \in G_1(X)$ with associated maps $A, B : S^1 \times X \to X$ respectively, then
$$S^1 \times S^1 \times X \xrightarrow{\mathrm{id} \times B} S^1 \times X \xrightarrow{A} X$$
restricts to $\alpha \vee \beta \vee \mathrm{id} : S^1 \vee S^1 \vee X \to X$.
(3) If $p : \overline{X} \to X$ is a covering and $\alpha \in \pi_1(\overline{X})$ with $p_\#(\alpha) \in G_1(X)$, then $\alpha \in G_1(\overline{X})$.

The Gottlieb group plays an important role in many homotopical structure results.

For example, assume that $H_1(X; \mathbb{Z})$ is finitely generated, and define the *Hurewicz rank* of X to be the number of $\mathbb{Z}$-summands of $H_1(X; \mathbb{Z})$ which are contained in $h(G(X))$, where $h : \pi_1(X) \to H_1(X; \mathbb{Z})$ is the Hurewicz map. We then have the following [17,21].

Theorem 5.2. *Let X be a space with $H_1(X; \mathbb{Z})$ finitely generated. If X has Hurewicz rank k, then $X \simeq T^k \times Y$, where T^k is a k-torus.*

Corollary 5.3. *If $G_1(X)$ is finitely generated and $\mathrm{rank}(G_1(X)) = k$, then there is a covering $\overline{X} \to X$ with $\overline{X} \simeq T^k \times Y$ and Y simply connected.*

Proof. Let $\overline{X}$ be the cover corresponding to the subgroup $\mathbb{Z}^k \subseteq G_1(X) \subseteq \pi_1(X)$. By [17], $G_1(\overline{X}) = \pi_1(\overline{X}) = \mathbb{Z}^k$, so Theorem 5.2 gives the splitting. □

If we now apply Theorem 4.3 to Corollary 5.3, we obtain the following.

Theorem 5.4. *If X is a normal ANR and $G_1(X)$ is finitely generated, then*
$$\mathrm{rank}\big(G_1(X)\big) \leqslant \mathrm{cat}_1(X).$$

While this result is purely homotopical, we shall give a refinement in Corollary 6.8 in the presence of extra geometric structure.

A more geometrical splitting result is the famous theorem of Cheeger and Gromoll.

Theorem 5.5 *(Cheeger–Gromoll splitting). ([4]) If M is a compact manifold with non-negative Ricci curvature, then there is a finite cover $\overline{M}$ of M with a diffeomorphism $\overline{M} \cong T^k \times N$. Further, N is simply connected and T^k is flat.*

In Theorem 5.5, it could be the case that $k = 0$. Then, since N is simply connected and the covering is finite, $\pi_1(M)$ would have to be finite. We exclude this case below and focus only on manifolds with infinite fundamental groups. There are many extensions of this result to cases where *almost* non-negative Ricci or sectional curvature is assumed together with certain extra constraints on either injectivity radius or volume (see, for instance, [3,7,30,33]), so this type of splitting is not unusual. In Section 6, we shall consider almost non-negative sectional curvature alone using the results of [19].

The Cheeger–Gromoll splitting has the special feature that the cover $\overline{M}$ is a *finite* cover. This will allow us to link the invariants of M and $\overline{M}$ by the following well-known result.

Lemma 5.6. *Suppose that $p : \overline{X} \to X$ is a finite covering space. Then $p^* : H^*(X; \mathbb{Q}) \to H^*(\overline{X}; \mathbb{Q})$ is injective. In particular, if $\pi \subseteq G$ is a finite index subgroup, then $p^* : H^*(G; \mathbb{Q}) \to H^*(\pi; \mathbb{Q})$ is injective.*

Proof. A finite covering of degree m has associated to it a transfer homomorphism $\tau : H^*(\overline{X};\mathbb{Q}) \to H^*(X;\mathbb{Q})$ with the property that $\tau \circ p^*(\alpha) = m \cdot \alpha$. For $\mathbb{Q}$ coefficients, multiplication by m is an isomorphism, so p^* is a (split) injection. The second statement follows since $K(\pi,1) \to K(G,1)$ is a finite cover. $\square$

Now we can state and prove the main result.

Theorem 5.7. *If M is a compact manifold with non-negative Ricci curvature and infinite fundamental group, then $b_1(M) \leqslant \mathrm{cat}_1(M)$, where $b_1(M)$ is the first Betti number of M.*

Proof. By Theorem 5.5, there is a splitting $\overline{M} \cong T^k \times N$. By Lemma 5.6, we see that $b_1(M) \leqslant b_1(\overline{M}) = b_1(T^k) = k$. We now apply Theorem 4.3 to obtain the result. $\square$

The bounds $\mathrm{rank}(G_1(X)) \leqslant \mathrm{cat}(X)$ (for a manifold X) and $b_1(M) \leqslant \mathrm{cat}(M)$ from [22,23] had the property that equality only held for (flat) tori. The following example shows that this is *not* the case for the new cat_1 bound.

Example 5.8. Let $X = T^2 \times S^2$. Then X has a metric with non-negative Ricci curvature and from Proposition 4.2, $b_1(X) = 2 = \mathrm{cat}_1(X)$. But we also have $\mathrm{cat}_1(X) < \mathrm{cat}(X) = 3$ by the standard cup length lower bound for category and the standard product inequality for category (see [6]):

$$3 = \mathrm{cup}_{\mathbb{Q}}(X) \leqslant \mathrm{cat}(X) \leqslant \mathrm{cat}\big(T^2\big) + \mathrm{cat}\big(S^2\big) = 2 + 1 = 3.$$

Example 5.9. Since, under the hypotheses of Theorem 5.7, we have the inequalities

$$b_1(M) \leqslant \mathrm{cat}_1(M) \leqslant \mathrm{cat}(M) \leqslant \dim(M),$$

and $b_1(M) = \mathrm{cat}(M)$ implies $M \cong T^m$, it is tempting to conjecture that $\mathrm{cat}_1(M) = \mathrm{cat}(M)$ only when the manifold M is a $K(\pi,1)$. That this is not true is exemplified by $M = \mathbb{R}P^m$. By Corollary 3.6 applied to $\mathbb{R}P^m \xrightarrow{j_1} \mathbb{R}P^\infty = K(\mathbb{Z}/2\mathbb{Z},1)$, we know that $m \leqslant \mathrm{cat}_1(\mathbb{R}P^m)$. But we also know that $\mathrm{cat}_1(\mathbb{R}P^m) \leqslant \mathrm{cat}(\mathbb{R}P^m) \leqslant \dim(\mathbb{R}P^m) = m$, so we have $\mathrm{cat}_1(\mathbb{R}P^m) = \mathrm{cat}(\mathbb{R}P^m) = m$, but $\mathbb{R}P^m$ is not a $K(\pi,1)$. An interesting question is whether $\mathrm{cat}_1(M) = \mathrm{cat}(M)$ only when M is a $K(\pi,1)$ or *a skeleton of* a $K(\pi,1)$.

6. Manifolds of almost non-negative sectional curvature

It is not generally true that a Cheeger–Gromoll type splitting theorem holds for manifolds of *almost* non-negative sectional curvature (without extra side conditions). However, there are results which are "one step away" from producing splittings.

A closed smooth manifold M^m is said to be *almost non-negatively* (*sectionally*) *curved* (or *ANSC*) if it admits a sequence of Riemannian metrics $\{g_n\}_{n\in\mathrm{N}}$ whose sectional curvatures and diameters satisfy

$$\sec(M, g_n) \geqslant -\frac{1}{n} \quad \text{and} \quad \mathrm{diam}(M, g_n) \leqslant \frac{1}{n}.$$

ANSC manifolds generalize almost flat manifolds as well as manifolds with non-negative sectional curvature. Here is a Bochner type result for ANSC manifolds due to Yamaguchi. (Also, there are versions for almost non-negatively Ricci-curved manifolds, see [5,7].)

Theorem 6.1. *([34]) If M^m is an ANSC manifold, then*:

(1) *a finite cover of M is the total space of a fibration over a torus of dimension $b_1(M)$*;
(2) *if $b_1(M) = m$, then M^m is diffeomorphic to $T^{b_1(M)}$.*

More recently, in [19] it was shown that an ANSC manifold M^m has a finite cover that is a nilpotent space in the sense of homotopy theory and that the following *fiber bundle* result holds. (Note that this does not hold for non-negatively Ricci-curved manifolds, see [1].[2])

Theorem 6.2. *([19]) If M is an ANSC manifold, then there is a finite cover $\overline{M}$ that is the total space of a fiber bundle*

$$F \to \overline{M} \xrightarrow{p} N,$$

where $N = K(\pi,1)$ is a nilmanifold and F is a simply connected closed manifold.

[2] Thanks to Wilderich Tuschmann for this observation.

Remark 6.3. (1) In fact, the fiber F is almost non-negatively curved in a certain generalized sense. Because we will not deal with this property, we refer the interested reader to [19] for the precise definition.

(2) Because $\pi_1(F) = 0$ and $N = K(\pi, 1)$, the bundle $F \to \overline{M} \to N$ is homotopy equivalent to the classifying fibration for the universal cover, $\widetilde{M} \to \overline{M} \xrightarrow{j_1} K(\pi, 1)$. (Here, note that $\widetilde{M}$ is the universal cover of M as well as of $\overline{M}$.) Of course, π is an infinite (in fact, torsionfree nilpotent) group, so $\widetilde{M}$ is non-compact. Therefore, it seems strange on the face of it that we have $\widetilde{M} \simeq F$ with F compact, but in fact, this is not so unusual. For instance, the universal cover of $S^2 \times S^1$ is $S^2 \times \mathbb{R}$ while the fiber of $S^2 \times S^1 \to S^1$ is the compact manifold S^2 of the same homotopy type as $S^2 \times \mathbb{R}$.

Now, because of the equivalence of the KPT bundle and the universal cover fibration, we see from Theorem 3.4 that

$$\mathrm{cat}_1(\overline{M}) = \mathrm{cat}(j_1) = \mathrm{cat}(p).$$

We then obtain the following Bochner-type theorem.

Theorem 6.4. *Suppose M is an ANSC manifold with associated finite cover $\overline{M}$ and fiber bundle $F \to \overline{M} \xrightarrow{p} N$, where $N = K(\pi, 1)$ is a nilmanifold and F is a simply connected closed manifold. Then*:

(i) $b_1(M) \leqslant \dim(N) \leqslant \dim(\overline{M}) = \dim(M)$;
(ii) *if $\widetilde{M}$ has non-zero Euler characteristic, then $b_1(M) \leqslant \dim(N) \leqslant \mathrm{cat}_1(M)$.*

Proof. We are given that $\overline{M} \to M$ is a finite cover, so Lemma 5.6 gives $b_1(M) \leqslant b_1(\overline{M})$. But $H_1(\overline{M}; \mathbb{Q}) \cong H_1(\pi; \mathbb{Q}) \cong H_1(N; \mathbb{Q})$, so $b_1(\overline{M}) = b_1(N)$.

Now, N is a nilmanifold, so it has a (rational homotopy theoretic) minimal model $(\Lambda(x_1, x_2, \ldots, x_k), d)$, where each generator has degree$(x_j) = 1$ and k is the rank of the torsionfree nilpotent group π (see Appendix B or [13, Theorem 3.22]). By the general theory, the differential d is zero on $x_1, \ldots, x_s$ for some $2 \leqslant s \leqslant k$ and $k = \dim(N)$. (The case $s = k$ is a torus.) Then $b_1(N) = s \leqslant k = \dim(N)$. Since $F \to \overline{M} \xrightarrow{p} N$ is a bundle, we see that $\dim(N) \leqslant \dim(\overline{M}) = \dim(M)$. This proves (i).

For (ii), because $F \simeq \widetilde{M}$ and $\chi(\widetilde{M}) \neq 0$, the bundle $F \to \overline{M} \xrightarrow{p} N$ has a transfer map $\tau : H^*(\overline{M}; \mathbb{Z}) \to H^*(N; \mathbb{Z})$ with $\tau \circ p^*(\alpha) = \chi(F) \cdot \alpha$, for all $\alpha \in H^*(N; \mathbb{Z})$ [2]. This implies that p^* is injective on rational cohomology. Since N is orientable by the discussion on nilmanifolds following Theorem B.1, Corollary 3.6 implies that $\dim(N) \leqslant \mathrm{cat}_1(\overline{M})$. Together with Corollary 3.10, we obtain $\dim(N) \leqslant \mathrm{cat}_1(M)$. $\square$

Remark 6.5. If $\pi_1(M)$ is torsionfree, then Serre's theorem on the cohomological dimension of finite index subgroups says that $\mathrm{cd}(\pi) = \mathrm{cd}(\pi_1(M))$ since π has finite index in $\pi_1(M)$. Because $\mathrm{cd}(\pi) = \dim(K(\pi, 1)) < \infty$, we then have $\dim(K(\pi_1(M), 1)) \leqslant \mathrm{cat}_1(M)$. All of this simply points out that there are other types of invariants that we can use instead of just b_1 in Bochner-type theorems.

On the face of it, there seems to be no way to go from KPT to Yamaguchi. But, in fact, it turns out we can use $\mathrm{cat}_1(-)$ to provide a bridge from the Kapovitch–Petrunin–Tuschmann Theorem 6.2 to Yamaguchi's Theorem 6.1. Unfortunately, the method only seems to give a topological version for (2) in Theorem 6.1. Nevertheless, because this approach is so simple, it reveals an interesting relationship between the geometry of, and homotopy theory associated to, ANSC manifolds. In the following, we only assume the existence of the fiber bundle of Theorem 6.2.

Theorem 6.6. *Suppose a closed manifold M has a finite cover $\overline{M}$ that is the total space of a fiber bundle*

$$F \to \overline{M} \xrightarrow{p} N,$$

where $N = K(\pi, 1)$ is a nilmanifold and F is a simply connected closed manifold. Then:

(1) *a finite cover of M is the total space of a fibration over a torus of dimension $b_1(M)$*;
(2) *if $b_1(M) = m = \dim(M)$, then M^m is homeomorphic to $T^{b_1(M)}$.*

Proof. Now, $b_1(M) \leqslant b_1(\overline{M})$ by Lemma 5.6 and the general construction of the nilmanifold N via iterated principal S^1-bundles shows that we may start the iteration by a bundle over $T^{b_1(\overline{M})}$ or any torus of lower dimension. Thus, (1) follows since a composition of fibrations is a fibration.

Now assume $b_1(M) = m = \dim(M)$. By Theorem 6.4 (i), we see that $\dim(N) = m = \dim(\overline{M})$. Hence, $\dim(F) = 0$ and (since F is connected) we have $\overline{M} = N$. Furthermore, the proof of Theorem 6.4 (i) shows that $b_1(M) \leqslant b_1(\overline{M}) = b_1(N) \leqslant \dim(M)$, so we also have $b_1(N) = m = \dim(N)$. For a nilmanifold, this can only happen if N is a torus T^m and $\pi \cong \mathbb{Z}^m$. (By Mostow rigidity (see [14] for example), N is *diffeomorphic* to T^m.) Now, $\overline{M} = T^m$ covers M, so M is a $K(G, 1)$ where $G = \pi_1(M)$. Since M is a closed m-manifold, we have that G is torsionfree. Now, π has finite index in G and $b_1(\pi) = m = b_1(M) = b_1(G)$. By Lemma 6.7 below, we have $G \cong \mathbb{Z}^m$. Hence $M = K(\mathbb{Z}^m, 1)$ is a homotopy torus. By [14, Theorem 6.1], we know that M is then homeomorphic to T^m. $\square$

Lemma 6.7. *If $\pi \cong \mathbb{Z}^m$ is a finite index subgroup of a torsionfree group G and $b_1(G) = m$, then $G \cong \mathbb{Z}^m$.*

Proof. Note first that Lemma 5.6 implies that $H_*(\pi;\mathbb{Q}) \to H_*(G;\mathbb{Q})$ is surjective. In particular, we have a surjection on rationalized abelianizations,

$$\pi_{\mathrm{ab}} \otimes \mathbb{Q} = H_1(\pi;\mathbb{Q}) \to H_1(G;\mathbb{Q}) = G_{\mathrm{ab}} \otimes \mathbb{Q}.$$

But $b_1(\pi) = b_1(G)$, and a surjection of rational vector spaces of the same dimension is an isomorphism, so $\mathbb{Q}^m \cong \pi_{\mathrm{ab}} \otimes \mathbb{Q} \cong G_{\mathrm{ab}} \otimes \mathbb{Q}$. We have the following commutative diagram:

$$\begin{tikzcd}
\pi \cong \mathbb{Z}^m \arrow[r, "i"] \arrow[d, "\cong"'] & G \arrow[d, "p"] \\
\pi_{\mathrm{ab}} \cong \mathbb{Z}^m \arrow[r, "i_{\mathrm{ab}}"] \arrow[d, "\otimes\mathbb{Q}"'] & G_{\mathrm{ab}} \arrow[d, "\otimes\mathbb{Q}"] \\
\mathbb{Q}^m \arrow[r, "\cong"] & \mathbb{Q}^m.
\end{tikzcd}$$

Note that, because the bottom row is an isomorphism, i_{ab} is an injection. We claim that $\mathrm{Ker}(p) = 0$, so p is an isomorphism (since it is a surjection by definition). Suppose $x \in G$ and $p(x) = 0$. Now, π has finite index in G and if $x^s\pi = x^t\pi$ (for $s > t$ say), then $x^{s-t} \in \pi$, so there exists some $r \in \mathbb{N}$ such that $x^r \in \pi$. But then we have the contradiction

$$0 \neq i_{\mathrm{ab}}\big(x^r\big) = p\big(i\big(x^r\big)\big) = 0.$$

Therefore, $x^r = e$, where e is the identity of G. But G is torsionfree, so $r = 0$ and $x = e$. Hence p is injective and $p : G \to G_{\mathrm{ab}}$ is an isomorphism. Therefore, G is a finitely generated torsionfree abelian group; hence $G \cong \mathbb{Z}^m$ (since $b_1(G) = m$). □

Now we can give a result that is a combination of Theorems 5.4 and 5.7 in the presence of the special geometric structure provided by ANSC and a hypothesis on the associated nilmanifold.

Corollary 6.8. *Suppose M is an ANSC manifold with associated finite cover $\overline{M}$ and fiber bundle $F \to \overline{M} \xrightarrow{p} N$, where $N = K(\pi,1)$ is a symplectic nilmanifold and F is a simply connected closed manifold. If $\widetilde{M}$ has non-zero Euler characteristic (or more generally, p^* is injective), then*

$$\mathrm{cat}_1(\overline{M}) \geqslant b_1(\overline{M}) \geqslant \mathrm{rank}(\mathcal{Z}\pi) \geqslant \mathrm{rank}\big(G_1(\overline{M})\big),$$

where $\mathcal{Z}\pi$ denotes the center of π.

Proof. Note that $b_1(\overline{M}) = b_1(N) = b_1(\pi)$ and $G_1(\overline{M}) \subseteq \mathcal{Z}\pi_1(\overline{M}) = \mathcal{Z}\pi$ since F is simply connected. We then apply Proposition B.3 and Theorem 6.4. □

Appendix A. Generalities on open covers

The main results about open covers that we shall use are described (and proved) in [10,11], but other relevant papers include [25,18,9] as well as [6, Exercise 1.12]. We take the exact statements below from [27].

An open cover $\mathcal{W} = \{W_0,\ldots,W_{m+k}\}$ of a space X is an $(m+1)$-*cover* if every subcollection $\{W_{j_0}, W_{j_1},\ldots,W_{j_m}\}$ of $m+1$ sets from $\mathcal{U}$ also covers X. The following simple, but slippery, observation (see [25] for instance) is the basis for many arguments in this approach.

Lemma A.1. *A cover $\mathcal{W} = \{W_0, W_1,\ldots,W_{k+m}\}$ is an $(m+1)$-cover of X if and only if each $x \in X$ is contained in at least $k+1$ sets of $\mathcal{W}$.*

Proof. If $\mathcal{W}$ is an $(m+1)$-cover and $x \in X$ is only in k sets in $\mathcal{W}$, then $k+m+1-k = m+1$ sets of the cover do not contain x. These $m+1$ sets do not cover X, contradicting the supposition on $\mathcal{W}$.

Suppose each $x \in X$ is contained in at least $k+1$ sets from $\mathcal{W}$ and choose a subcollection $\mathcal{V}$ of $m+1$ sets from $\mathcal{W}$. There are only $k+m+1-(m+1) = k$ sets *not* in $\mathcal{V}$, so x must belong to at least one set in $\mathcal{V}$. Thus $\mathcal{V}$ covers X, and $\mathcal{W}$ is an $(m+1)$-cover. □

An open cover can be lengthened to a $(k+1)$-cover, while retaining certain essential properties of the sets in the cover.

Theorem A.2. *([9,10]) Let $\mathcal{U} = \{U_0,\ldots,U_k\}$ be an open cover of a normal space X. Then, for any $m = k, k+1,\ldots,\infty$, there is an open $(k+1)$-cover of X, $\{U_0,\ldots,U_m\}$, extending $\mathcal{U}$ such that for $n > k$, U_n is a disjoint union of open sets that are subsets of the U_j, $0 \leqslant j \leqslant k$.*

In Theorem A.2, because the U_n for $n > k$ are disjoint unions of subsets of the original covering sets, the U_n also possess any properties of the original cover that are inherited by disjoint unions and open subsets. In particular, if the cover $\mathcal{U}$ is categorical (or m-categorical), then the extended cover is also categorical (or m-categorical).

Appendix B. Nilmanifolds and minimal models

The following is culled from [13, Chapter 3]. A *nilmanifold* N is the quotient of a simply connected nilpotent Lie group G by a co-compact discrete subgroup π. A simply connected nilpotent Lie group is diffeomorphic to a Euclidean space, so a nilmanifold has a fundamental group π that is a finitely generated torsionfree nilpotent group and has higher order homotopy groups which are trivial. Nilmanifolds then provide prime examples of $K(\pi, 1)$-manifolds; that is, compact manifolds with the fundamental group as the only non-trivial homotopy group. Clearly, any nilmanifold is orientable. Examples are given by any torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ and the Heisenberg manifold formed by the quotient of the Lie group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

with a, b and c real numbers, by the subgroup of the corresponding matrices with integer entries.

To any nilmanifold, we can associate a rational nilpotent Lie algebra $\mathfrak{g}$ with the property that there exists a basis in $\mathfrak{g}$, $e_1, e_2, \dots, e_n$, such that the structure constants c_{ij}^k arising in brackets

$$[e_i, e_j] = \sum_k c_{ij}^k e_k$$

are rational numbers for all i, j, k. In fact, corresponding to $\mathfrak{g}$, there is a simply connected nilpotent Lie group G which admits a discrete co-compact subgroup π so that $N = G/\pi$ is a compact nilmanifold.

Let $\mathfrak{g}$ have basis $\{X_1, \dots, X_s\}$. Then the dual of $\mathfrak{g}$, $\mathfrak{g}^*$, has basis $\{x_1, \dots, x_s\}$ and there is a differential δ on the *exterior algebra* $\Lambda\mathfrak{g}^*$ given by defining it to be dual to the bracket on degree 1 elements,

$$\delta x_k(X_i, X_j) = -x_k\big([X_i, X_j]\big),$$

and then extending δ to be a graded derivation. Now, $[X_i, X_j] = \sum c_{ij}^l X_l$, where c_{ij}^l are the structure constants of $\mathfrak{g}$, so duality then gives

$$\delta x_k(X_i, X_j) = -c_{ij}^k$$

and the differential has the form (on generators)

$$\delta x_k = -\sum_{i<j} c_{ij}^k x_i \wedge x_j.$$

We note that the Jacobi identity in the Lie algebra is equivalent to the condition $\delta^2 = 0$. Therefore, we obtain a *commutative differential graded algebra* (or cdga) $(\Lambda\mathfrak{g}^*, \delta)$ associated to the Lie algebra $\mathfrak{g}$. The fundamental result here is the following.

Theorem B.1. *If $N = G/\pi$ is a nilmanifold, then the cdga $(\Lambda\mathfrak{g}^*, \delta)$ associated to $\mathfrak{g}$ is a minimal model for N and, thus computes all of the rational homotopy information about N.*

The crucial homotopy fact here is that rational homotopy theory is completely algebraic. That is, there is an equivalence between the categories of rational homotopy types and isomorphism classes of minimal cdga's. Again we refer to a general source such as [13] for specifics.

Now, the minimal model of N has the form

$$\mathcal{M}_N = \big(\Lambda(x_1, \dots x_k), d\big) \quad \text{with } |x_i| = 1,$$

where the nilpotency of $\mathfrak{g}$ converts by duality into the condition that the differential on x_j is a polynomial in x_r with $r < j$. In fact, this can be refined to say that the generators are added in stages and the generators in the jth stage have differentials that are polynomials in the generators of stages 1 through $j-1$. In particular, because $\mathfrak{g}$ is nilpotent, there is a non-trivial complement to $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ which is isomorphic to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong H^1(N; \mathbb{Q})$. Duality then says that there is some s with $2 \leqslant s \leqslant k$ such that $dx_i = 0$ for $i \leqslant s$.

The geometry behind the form of the minimal model comes from a description due to Malcev of a nilmanifold $N = G/\pi$ as an iterated sequence of principal circle bundles, one for each generator x_j, $1 \leqslant j \leqslant k$ (see [13, Chapter 3]). The condition that for some s with $2 \leqslant s \leqslant k$ we have $dx_i = 0$ for $i \leqslant s$ means that the first s principal bundles are trivial. That is, *the construction of N begins by taking a torus T^s and then proceeds by taking successive principal circle bundles.*

The minimal model $\mathcal{M}_N$ is an exterior algebra so, since degree$(x_j) = 1$ for $1 \leqslant j \leqslant k$, the top degree of a non-zero element is k and a vector space generator is $x_1 \cdot x_2 \cdots x_k$. This element is obviously a cocycle, so $H^k(N;\mathbb{Q}) = \mathbb{Q}$; thus, N is orientable and any $K(\pi,1)$ must have dimension at least k.

The minimal model $\mathcal{M}_N = (\Lambda(x_1,\dots,x_k), d)$ reflects the structure of $\mathfrak{g}$ as a nilpotent Lie algebra. In particular, the center of $\mathfrak{g}$ (corresponding to the center of π) has the property that a bracket of any of its elements with any other element of $\mathfrak{g}n$ is zero and this is reflected (by duality) in the fact that, for some $t \geqslant 2$, $x_{t+1},\dots,x_k$ do not appear in the differentials of any of the x_j generators. In this notation, we have that

$$\operatorname{rank}(\mathcal{Z}\pi) = \dim\bigl(\mathcal{Z}(\mathfrak{g})\bigr) = k - t.$$

That is, *the rank of the center of the fundamental group of a nilmanifold is the number of generators of the minimal model of the nilmanifold that do not appear in differentials of generators.*

Now, nilmanifolds can sometimes be symplectic manifolds. Rather than give the definition of a symplectic manifold here, we can make use of a facet of a theorem due to Nomizu to identify symplectic nilmanifolds as the ones with a degree 2 cohomology class whose cup product power is a non-zero top degree (i.e. the dimension of the nilmanifold) rational cohomology class. Again, see [13,32] for all of this.

Example B.2. The first example of a closed symplectic non-Kähler manifold was given by Thurston (and Kodaira earlier). This Kodaira–Thurston manifold KT is the product of a circle S^1 with the 3-dimensional Heisenberg manifold obtained as the quotient of 3×3 real upper triangular matrices with 1's on the diagonal by the discrete subgroup of such matrices with integral entries. The minimal model of KT is given by $(\Lambda(x,y,u,z), d)$ with $dx = dy = du = 0$ and $dz = xy$. A representative of the symplectic cohomology class is $\omega = xz + yu$. Note that, in order for $d\omega = 0$, it is necessary for $dx = 0$ since z does not appear in any differentials. Note that $\omega^2 = 2xzyu = 2xyuz$ using the commutativity of the minimal model. In general, it is always the case that the product of all the generators of the minimal model is a top class for the manifold. (Note that $x_j^2 = 0$ for all j by (anti-)commutativity as well.) Now, the Lie algebra $\mathfrak{g}$ in this case is given by

$$\mathfrak{g} = \bigl\langle X, Y, U, Z \bigm| [X,Y] = Z \bigr\rangle,$$

with all other brackets equal to zero. Hence, we see the center is $\langle Z, U\rangle$ and this corresponds to the generators z and u not appearing in any differential. Finally, note that $b_1(N) = 3$ since x, y and u are degree 1 cocycles and $\dim(\mathcal{Z}\mathfrak{g}) = 2$.

The following result generalizes the example of the Kodaira–Thurston manifold.

Proposition B.3. *If $N = K(\pi,1)$ is a symplectic nilmanifold $N = G/\pi$, then*

$$b_1(N) = b_1(\pi) \geqslant \operatorname{rank}(\mathcal{Z}\pi) = \dim(\mathcal{Z}\mathfrak{g}).$$

Proof. Write the minimal model as

$$\bigl(\Lambda(x_1,\dots,x_b, y_1,\dots,y_\ell, z_1,\dots,z_t), d\bigr),$$

where the x_i are the generators that are cocycles, the y_i are generators with $dy_i \neq 0$ that appear in some differential and the z_i are the generators with $dz_i \neq 0$ that *do not* appear in any differential (and so are dual to the center of the Lie algebra $\mathfrak{g}$). We also take b to be maximal in the sense that no linear combination of the y_j and z_j can be a cocycle. We first assume that every cocycle generator x_j appears in the differential of some other generator.[3]

Note that the symplectic class representative ω must include all generators of the minimal model since this is the only way a power of ω can give a top class (which is a product of all generators). Now write ω as

$$\omega = \sigma + \sum_{j=1}^{t} \alpha_j z_j + \sum_{r<s} c_{rs} z_r z_s,$$

where σ is a sum of terms that are products of x_i's and y_i's and each α_i is a linear combination of x_i's and y_i's only. Note that the final term can always be written in the form indicated (i.e. $r < s$). Using $d\omega = 0$, we see that

$$0 = d\sigma + \sum d\alpha_j z_j - \sum \alpha_j dz_j + \sum c_{rs} dz_r z_s - \sum c_{rs} z_r dz_s.$$

Because we require $r < s$ in the final sum of ω, we see that the only terms in $d\omega = 0$ involving z_t are $d\alpha_t z_t$ and $\sum c_{rt} dz_r z_t$. Because the algebra is freely generated, we have $0 = (d\alpha_t + \sum c_{rt} dz_r) z_t$. Hence we have

$$0 = d\alpha_t + \sum c_{rt} dz_r = d\Bigl(\alpha_t + \sum c_{rt} z_r\Bigr).$$

[3] Thanks to Greg Lupton for pointing out the necessity of this step.

But this implies that $\alpha_t \in \langle x_1, \dots, x_b \rangle$ and $c_{rt} = 0$ for all $r = 1, \dots, t-1$ since all degree one cocycles are in $\langle x_1, \dots, x_b \rangle$. Hence,

$$\omega = \sigma + \sum_{j=1}^{t} \alpha_j z_j + \sum_{r<s<t} c_{rs} z_r z_s.$$

Now by considering z_{t-1}, the same argument as above shows that $\alpha_{t-1} \in \langle x_1, \dots, x_b \rangle$ and $c_{r(t-1)} = 0$ for all $r = 1, \dots, t-1$. Iterating this procedure, we end with

$$\omega = \sigma + \sum_{j=1}^{t} \alpha_j z_j$$

with all $\alpha_j \neq 0$ and all $\alpha_j \in \langle x_1, \dots, x_b \rangle$. (The first condition follows since ω must contain all degree one generators.)

Now, $\omega^n \neq 0$, where $2n = b + \ell + t$ since $\omega^n = c \cdot x_1 \cdots x_b y_1 \cdots y_\ell z_1 \cdots z_t$. If we write $\omega = \sigma + \beta$, where $\beta = \sum \alpha_j z_j$, then

$$\omega^n = \sum \binom{n}{p} \sigma^p \beta^{n-p}.$$

Now, $\beta^{t+u} = 0$ for $u > 0$ since some z_j would occur with an exponent higher than one. On the other hand, the monomial ω^n must contain all z_j generators and this only happens if $\beta^t \neq 0$.[4] But we have

$$\beta^t = \left(\sum \alpha_j z_j\right)^t = t! \alpha_1 \cdots \alpha_t z_1 \cdots z_t,$$

and clearly, for $\beta^t \neq 0$, it is necessary that $\alpha_1, \dots, \alpha_t$ be linearly independent. Since $\langle \alpha_1, \dots, \alpha_t \rangle \subseteq \langle x_1, \dots, x_b \rangle$, we must have $t \leqslant b$. But $b = b_1(N)$ and, from our remarks above, $t = \dim(\mathcal{Z}\mathfrak{g}) = \operatorname{rank}(\mathcal{Z}\pi)$. Hence, the result is proved under the assumption that the x_i always appear in some differential of another generator (i.e. the x_i never represent elements in the center).

Suppose, on the other hand, that (without loss of generality) $x_1, \dots, x_h$ never appear in the differential of another generator. Then clearly, the model may be written

$$\big(\Lambda(x_1, \dots, x_h, d = 0)\big) \otimes \big(\Lambda(x_{h+1}, \dots, x_b, y_1, \dots, y_\ell, z_1, \dots, z_t), d\big),$$

corresponding to a rational splitting $N \simeq T^h \times K(\pi', 1)$. By the proof above, we have $b - h \geqslant \operatorname{rank}(\mathcal{Z}\pi')$. But each circle factor in T^h contributes one to both the Betti number of N and to the center of $\mathfrak{g}$, so we obtain $b \geqslant \operatorname{rank}(\mathcal{Z}\pi)$ and we are done. $\square$

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[4] Thanks to Yves Félix for pointing out an error in an earlier version of the argument.

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