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On The Structure of Co-Kähler Manifolds

Giovanni Bazzoni
Instituto de Ciencias Matemáticas

John Oprea
Cleveland State University, J.OPREA@csuohio.edu

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1 Recollections on co-Kähler manifolds In [13], H. Li recently gave a structure result for compact co-Kähler manifolds stating that such a manifold is always a Kähler mapping torus (see Sect. 6). In this paper, using Li's characterization, we give another type of structure theorem for co-Kähler manifolds based on classical results in [5,16,17,20]. As such, much of this paper is devoted to showing how the interplay between the known geometry and the known topology of co-Kähler manifolds

creates beautiful structure. Basic results on co-Kähler manifolds themselves come from [4] (see also [8]).[1]

Let $(M^{2n+1}, J, \xi, \eta, g)$ be an *almost contact metric manifold* given by the conditions

$$J^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(JX, JY) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

for vector fields X and Y, I the identity transformation on TM and g a Riemannian metric. Here, ξ is a vector field as well, η is a 1-form and J is a tensor of type $(1, 1)$. A local J-basis $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n, \xi\}$ may be found with $\eta(X_i) = 0$ for $i = 1, \ldots, n$. The *fundamental 2-form* on M is given by

$$\omega(X, Y) = g(JX, Y),$$

and if $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \eta\}$ is a local 1-form basis dual to the local J-basis, then

$$\omega = \sum_{i=1}^{n} \alpha_i \wedge \beta_i.$$

Note that $\iota_\xi \omega = 0$.

Definition 1.1 The geometric structure $(M^{2n+1}, J, \xi, \eta, g)$ is a **co-Kähler** structure on M if

$$[J, J] + 2\, d\eta \otimes \xi = 0 \text{ and } d\omega = 0 = d\eta$$

or, equivalently, J is parallel with respect to the metric g.

A crucial fact that we use in our result is that, on a co-Kähler manifold, the vector field ξ is Killing and parallel and the 1-form η is harmonic. This fact is well known, but we were not able to find a direct proof in the literature, so we give one here.

Lemma 1.2 *On a co-Kähler manifold, the vector field ξ is Killing and parallel. Furthermore, the 1-form η is a harmonic form.*

Proof The normality condition implies that $L_\xi J = 0$ (see [3]); in particular, $[\xi, JX] = J[\xi, X]$ for every vector field X on M. Compatibility of the metric g with J is expressed by the right-hand relation in (1); with $\omega(X, Y) = g(JX, Y)$, it yields

$$g(X, Y) = \omega(X, JY) + \eta(X)\eta(Y). \tag{2}$$

By definition,

$$(L_\xi g)(X, Y) = \xi g(X, Y) - g([\xi, X], Y) - g(X, [\xi, Y]). \tag{3}$$

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[1] The authors of [4] use the term *cosymplectic* for Li's co-Kähler because they view these manifolds as odd-dimensional versions of symplectic manifolds—even as far as being a convenient setting for time-dependent mechanics [7]. Li's characterization, however, makes clear the true underlying Kähler structure, so we have chosen to follow his terminology.

Substituting (2) in (3), we obtain

$$
\begin{aligned}
(L_\xi g)(X, Y) = \ & \xi\omega(X, JY) + \xi(\eta(X)\eta(Y)) - \omega([\xi, X], JY) - \eta([\xi, X])\eta(Y) \\
& - \omega(X, J[\xi, Y]) - \eta(X)\eta([\xi, Y]) \\
= \ & \xi\omega(X, JY) - \omega([\xi, X], JY) - \omega(X, [\xi, JY]) + (\xi\eta(X))\eta(Y) \\
& + \eta(X)(\xi\eta(Y)) - \eta([\xi, X])\eta(Y) - \eta(X)\eta([\xi, Y]) \\
= \ & (L_\xi \omega)(X, JY) + \eta(X)(\xi\eta(Y) - \eta([\xi, Y])) \\
& + \eta(Y)(\xi\eta(X) - \eta([\xi, X])) \\
= \ & \eta(X)(d\eta(\xi, Y) + Y\eta(\xi)) + \eta(Y)(d\eta(\xi, X) + X\eta(\xi)) = \ 0.
\end{aligned}
$$

The last equalities follow from these facts:

- since ω is closed and $\iota_\xi \omega = 0$, $L_\xi \omega = 0$ by Cartan's magic formula;
- $d\eta = 0$;
- as $\eta(\xi) \equiv 1$, one has $X\eta(\xi) = Y\eta(\xi) = 0$.

This proves that ξ is a Killing vector field. In order to show that ξ is parallel, we use the following formula for the covariant derivative ∇ of the Levi–Civita connection of g; for vector fields X, Y, Z on M, one has

$$
\begin{aligned}
2g(\nabla_X Y, Z) = \ & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\
& + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X). \quad (4)
\end{aligned}
$$

Setting $Y = \xi$ in (4) and recalling that, on any almost contact metric manifold, $g(X, \xi) = \eta(X)$, we obtain

$$
\begin{aligned}
2g(\nabla_X \xi, Z) = \ & Xg(\xi, Z) + \xi g(X, Z) - Zg(X, \xi) + g([X, \xi], Z) \\
& + g([Z, X], \xi) - g([\xi, Z], X) = \ \xi g(X, Z) \\
& - g([\xi, X], Z) - g([\xi, Z], X) + X\eta(Z) \\
& - Z\eta(X) - \eta([X, Z]) = \ (L_\xi g)(X, Z) + d\eta(X, Z) = \ 0.
\end{aligned}
$$

Since X and Z are arbitrary it follows that $\nabla\xi = 0$.

To prove that η is harmonic, we rely on the following result: a vector field on a Riemannian manifold (M, g) is Killing if and only if the dual 1-form is co-closed. For a proof, see for instance [9, page 107]. Applying this to ξ, we see that η co-closed; since it is closed, it is harmonic. □

Lemma 1.2 will be a key point in our structure theorem below. In fact, in [13], it is shown that we can replace η by a harmonic integral form η_θ with dual parallel vector field ξ_θ and associated metric g_θ, (1, 1)-tensor J_θ and closed 2-form ω_θ with $i_{\xi_\theta}\omega_\theta = 0$. Then we have the following (see Sect. 6 for definitions).

Theorem 1.3 ([13]) *With the structure $(M^{2n+1}, J_\theta, \xi_\theta, \eta_\theta, g_\theta)$, there is a compact Kähler manifold (K, h) and a Hermitian isometry $\psi : K \to K$ such that M is diffeomorphic to the mapping torus*

$$
K_\psi = \frac{K \times [0, 1]}{(x, 0) \sim (\psi(x), 1)}
$$

with associated fibre bundle $K \to M = K_\psi \to S^1$.

An important ingredient in Li's theorem is a result of Tischler (see [19]) stating that a compact manifold admitting a non-vanishing closed 1-form fibres over the circle. The above result indicates that co-Kähler manifolds are very special types of manifolds. However it can be very difficult to see whether a manifold is such a mapping torus. In this paper, we will give another characterization of co-Kähler manifolds which we hope will allow an easier identification.

2 Parallel vector fields

From now on, when we write a co-Kähler structure $(M^{2n+1}, J, \xi, \eta, g)$, we shall mean Li's associated integral and parallel structures. Let's now employ an argument that goes back to [20], but which was resurrected in [17]. Consider the parallel vector field ξ and its associated flow ϕ_t. Because ξ is Killing, each ϕ_t is an isometry of (M, g). Therefore, in the isometry group $\text{Isom}(M, g)$, the subgroup generated by ξ, C, is singly generated. Since M is compact, so is $\text{Isom}(M, g)$ and this means that C is a torus. Using harmonic forms and the Albanese torus, Welsh [20] actually shows that there is a subtorus $T \subseteq C$ such that $M = T \times_G F$ where $G \subset T$ is finite and F is a manifold. Following Sadowski [17], we can modify the argument as follows.

Let $S^1 \subseteq C \subset \text{Isom}(M, g)$ have associated vector field Y. Because S^1 acts on (M, g) by isometries, the vector field Y is Killing. Now, we can choose Y as close to ξ as we like, so at some point $x_0 \in M$, since $\eta(\xi)(x_0) \neq 0$, then $\eta(Y)(x_0) \neq 0$ as well. But η is harmonic and Y is Killing, so this means that $\eta(Y)(x) \neq 0$ for all $x \in M$. Hence, we may take $\eta(Y)(x) > 0$ for all $x \in M$. Now let σ be an orbit of the S^1 action. Then

$$\int_\sigma \eta = \int_0^1 \eta \left(\frac{d\sigma}{dt} \right) dt = \int \eta(Y) \, dt > 0.$$

This says that the orbit map $\alpha \colon S^1 \to M$ defined by $g \mapsto g \cdot x_0$ induces a non-trivial composition of homomorphisms

$$H_1(S^1; \mathbb{R}) \xrightarrow{\alpha_*} H_1(M; \mathbb{R}) \xrightarrow{\eta} H_1(S^1; \mathbb{R}),$$

where $d\eta = 0$ defines an integral cohomology class $\eta \in H^1(M; \mathbb{Z}) \cong [M, S^1]$. Here we use the standard identification of degree 1 cohomology with homotopy classes of maps from M to S^1. Since $H_1(S^1; \mathbb{Z}) = \mathbb{Z}$, this means that the integral homomorphism $\alpha_* \colon H_1(S^1; \mathbb{Z}) \to H_1(M; \mathbb{Z})$ is *injective*. Such an action is said to be *homologically injective* (see [5]). Hence, we have

Proposition 2.1 *A co-Kähler manifold* $(M^{2n+1}, J, \xi, \eta, g)$ *with integral structure supports a smooth homologically injective* S^1 *action.*

In fact, it can be shown that there is a homologically injective T^k action on M, where T^k is Welsh's torus T. However, we shall focus on the S^1-case since this will allow a connection to Li's mapping torus result.

3 Sadowski's transversally equivariant fibrations

Homologically injective actions were first considered by Conner and Raymond in [5] (also see [12]) and were shown to lead to topological product splittings up to finite cover (also

see [16]). Homological injectivity for a circle action is very unusual and this points out the extremely special nature of co-Kähler manifolds. Here we want to make use of the results in [17] to achieve smooth splittings for co-Kähler manifolds up to a finite cover. We will state the results of [17] only for the case we are interested in: namely, a mapping torus bundle $M \to S^1$.

Let's begin by recalling that a bundle map $p\colon M \to S^1$ is a *transversally equivariant fibration* if there is a smooth S^1-action on M such that the orbits of the action are transversal to the fibres of p and $p(t \cdot x) - p(x)$ depends on $t \in S^1$ only. This latter condition is simply the usual equivariance condition if we take an appropriate action of S^1 on itself (see [17, Remark 1.1]). Sadowski's key lemma is the following.

Lemma 3.1 [17, Lemma 1.3] *Let $p\colon M \to S^1$ be a smooth S^1-equivariant bundle map. Then the following are equivalent;*

(1) *The orbits of the S^1-action are transversal to the fibres of p;*
(2) *$p_* \circ \alpha_* \colon \pi_1(S^1) \to \pi_1(S^1)$ is injective, where $\alpha\colon S^1 \to M$ is the orbit map;*
(3) *One orbit of the S^1-action is transversal to a fibre of p at a point $x_0 \in M$.*

Remark 3.2 Note the following.

(1) Lemmas 1.1 and 1.2 of [17] show that, in the situation of Proposition 2.1, $\eta\colon M \to S^1$ is a transversally equivariant bundle map.
(2) Note also that, because $\pi_1(S^1) \cong H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$, the second condition of Lemma 3.1 is really saying that the action is homologically injective.

As pointed out in [13], *every smooth fibration $K \to M \xrightarrow{p} S^1$ can be seen as a mapping torus of a certain diffeomorphism $\varphi\colon K \to K$*, (also see Proposition 6.4 below). The following is a distillation of [17, Proposition 2.1 and Corollary 2.1] in the case of a circle action.

Theorem 3.3 *Let $M \xrightarrow{p} S^1$ be a smooth bundle projection from a smooth closed manifold M to the circle. The following are equivalent:*

(1) *The structure group of p can be reduced to a finite cyclic group $G = \mathbb{Z}_m \subseteq \pi_1(S^1)/(\mathrm{Im}(p_* \circ \alpha_*))$ (i.e. the diffeomorphism φ associated to the mapping torus $M \xrightarrow{p} S^1$ has finite order);*
(2) *The bundle map p is transversally equivariant with respect to an S^1-action on M, $A\colon S^1 \times M \to M$.*

Moreover, assuming (1) and (2), there is a finite G-cover $K \times S^1 \to M$ given by the action $(k, t) \mapsto A_t(k)$, where G acts diagonally and by translations on S^1.

Sketch of Proof ([17]) $(1 \Rightarrow 2)$ The bundle is classified by a map $S^1 \to BG$ or, equivalently, by an element of $\pi_1(BG) = G = \mathbb{Z}_m$ (since G is abelian). Now M may be written as a mapping torus K_φ for some diffeomorphism $\varphi \in \mathrm{Diffeo}(K)$ of order m. (So G is the structure group of a mapping torus bundle). Define an S^1-action by $A\colon S^1 \times M \to M$, $A(t, [k, s]) = [k, s + mt]$. (Geometrically, the action is simply winding around the mapping torus m times until we are back to the identity φ^m). Clearly, the action is transversally equivariant.

$(2 \Rightarrow 1)$ Let $A_t\colon M \to M$ be the S^1-action such that p is transversally equivariant. Let K be the fibre of p and let

$$G = \{g \in S^1 \mid A_g(K) = K\}.$$

Now, because orbits of the action are transversal to the fibre, G is a proper closed subgroup of S^1. Hence, $G = \mathbb{Z}_m = \langle g \mid g^m = 1 \rangle$ for some positive integer m. Also note that the transversally equivariant condition saying $p(A_t(x)) - p(x)$ only depends on t implies that the action carries fibres of p to fibres of p. Moreover, fibres are then mapped back to themselves by G. Hence, letting G act diagonally on $K \times S^1$ and by translations on S^1, we see that the action is free and its restriction $A| \colon K \times S^1 \to M$ is a finite G-cover. Now, if we take the piece of the orbit from $x_0 \in K$ to $A_g(x_0)$ for fixed x_0 and $g \in G$, the projection to S^1 gives an element in $\pi_1(S^1) = \mathbb{Z}$. Because the full orbit is strictly longer than this piece, we see that the corresponding element in $\pi_1(S^1) = \mathbb{Z}$ can only be in $\mathrm{Im}(p_* \circ \alpha_*)$ if $g = 1$. Hence, $G \subseteq \pi_1(S^1)/(\mathrm{Im}(p_* \circ \alpha_*))$ which is finite due to homological injectivity. □

We then have the following consequence for co-Kähler manifolds from Proposition 2.1 and Theorem 3.3.

Theorem 3.4 *A compact co-Kähler manifold* $(M^{2n+1}, J, \xi, \eta, g)$ *with integral structure and mapping torus bundle* $K \to M \to S^1$ *splits as* $M \cong S^1 \times_{\mathbb{Z}_m} K$, *where* $S^1 \times K \to M$ *is a finite cover with structure group* $\mathbb{Z}_m$ *acting diagonally and by translations on the first factor. Moreover, M fibres over the circle* $S^1/(\mathbb{Z}_m)$ *with finite structure group.*

Note that Theorem 3.4 provides the following.

Corollary 3.5 *For a compact co-Kähler manifold* $(M^{2n+1}, J, \xi, \eta, g)$ *with integral structure and mapping torus bundle* $K \to M \to S^1$, *there is a commutative diagram of fibre bundles:*

$$
\begin{array}{ccccc}
K & \longrightarrow & S^1 \times K & \longrightarrow & S^1 \\
{\scriptstyle =}\big\downarrow & & {\scriptstyle \times m}\big\downarrow & & {\scriptstyle \times m}\big\downarrow \\
K & \longrightarrow & K_\psi & \longrightarrow & S^1.
\end{array}
$$

where $K_\psi \cong M$ *according to Theorem 1.3 and the notation* $\times m$ *denotes an* $\mathbb{Z}_m$-*covering.*

Remark 3.6 Although we have used the very special results of [17] above, observe that a version of Theorem 3.4 may be proved in the continuous case using the Conner-Raymond Splitting Theorem [5]. In this case, we obtain a finite cover $S^1 \times Y \to M$, where $Y \to K$ is a homotopy equivalence. This type of result affords a possibility of weakening the stringent assumptions on co-Kähler manifolds with a view towards homotopy theory rather than geometry.

4 Betti numbers

A main result of [4] was the fact that the Betti numbers of co-Kähler manifolds increase up to the middle dimension: $b_1 \leq b_2 \leq \cdots \leq b_n = b_{n+1}$ for M^{2n+1}. The argument in [4] was difficult, involving Hodge theory and a type of Hard Lefschetz Theorem for co-Kähler manifolds. In [13], the mapping torus description of co-Kähler manifolds yielded the result topologically through homology properties of the mapping torus. Here, we would like to see the Betti number result as a natural consequence of Theorem 3.4. Recall a basic result from covering space theory.

Lemma 4.1 *If* $\overline{X} \to X$ *is a finite G-cover, then*

$$
H^*(X; \mathbb{Q}) = H^*(\overline{X}; \mathbb{Q})^G,
$$

where the designation H^G denotes the fixed algebra under the action of the covering transformations G.

In order to see the Betti number relations, we need to know that the "Kähler class" on K is invariant under the covering transformations. The following result guarantees that such a class exists.

Lemma 4.2 *There exists a class $\bar{\omega} \in H^2(K; \mathbb{R})^G \subset H^2(S^1 \times K; \mathbb{R})$ which pulls back to $\omega \in H^2(K; \mathbb{R})$ via the inclusion $K \to S^1 \times K$ contained in Corollary 3.5.*

Proof Let $\theta : S^1 \times K \to M$ denote the $G = \mathbb{Z}_m$- cover of Theorem 3.4 and Corollary 3.5. Then $\theta^*\omega = \eta \times \alpha + \bar{\omega}$, where η generates $H^1(S^1; \mathbb{R})$, $\alpha \in H^1(K; \mathbb{R})$ and $\bar{\omega} \in H^2(K; \mathbb{R})$. Note that $\bar{\omega}$ pulls back to $\omega \in H^2(K; \mathbb{R})$. Also, $\theta^*\omega$ is G-invariant, so for each $g \in G$, we have

$$\alpha \times \eta + \bar{\omega} = g^*(\alpha \times \eta + \bar{\omega})$$
$$= g^*(\alpha) \times g^*(\eta) + g^*(\bar{\omega})$$
$$= g^*(\alpha) \times \eta + g^*(\bar{\omega}),$$

using the fact that G acts on $K \times S^1$ diagonally and homotopically trivially on S^1. We then get

$$(\alpha - g^*(\alpha)) \times \eta = g^*(\bar{\omega}) - \bar{\omega}.$$

This also means that $g^*(\bar{\omega}) - \bar{\omega} \in H^2(K; \mathbb{R})$ and $(\alpha - g^*(\alpha)) \times \eta \in H^1(K; \mathbb{R}) \otimes H^1(S^1)$. Thus, the only way the equality above can hold is that both sides are zero. Hence, $\bar{\omega}$ is G-invariant. □

Theorem 4.3 *If $(M^{2n+1}, J, \xi, \eta, g)$ is a compact co-Kähler manifold with integral structure and splitting $M \cong K \times_{\mathbb{Z}_m} S^1$, then*

$$H^*(M; \mathbb{R}) = H^*(K; \mathbb{R})^G \otimes H^*(S^1; \mathbb{R}),$$

where $G = \mathbb{Z}_m$. Hence, the Betti numbers of M satisfy:

(1) $b_s(M) = \bar{b}_s(K) + \bar{b}_{s-1}(K)$, where $\bar{b}_s(K)$ denotes the dimension of G-invariant cohomology $H^s(K; \mathbb{R})^G$;
(2) $b_1(M) \leq b_2(M) \leq \cdots \leq b_n(M) = b_{n+1}(M)$.

Proof Lemma 4.1 and the fact that G acts by translations (so homotopically trivially) on S^1 produce $H^*(M; \mathbb{R}) = H^*(K; \mathbb{R})^G \otimes H^*(S^1; \mathbb{R})$. If we denote the Betti numbers of the G-invariant cohomology by $\bar{b}$, then the tensor product splitting gives

$$b_s(M) = \bar{b}_s(K) + \bar{b}_{s-1}(K),$$

using the fact that $\widetilde{H}^1(S^1; \mathbb{R}) = \mathbb{R}$ and vanishes otherwise.

Let $\{\alpha_1, \ldots, \alpha_k\}$ be a basis for $H^{s-2}(K; \mathbb{R})^G$. According to Lemma 4.2, the class $\omega \in H^2(M; \mathbb{R})$, which comes from $H^2(K; \mathbb{R})$, provides a G-invariant class in $H^2(K; \mathbb{R})$. Furthermore, since K is compact Kähler, $H^*(K; \mathbb{R})$ obeys the Hard Lefschetz Property with respect to ω. Namely, for $j \leq n$, multiplication by powers of ω,

$$\cdot \omega^{n-j} : H^j(K; \mathbb{R}) \to H^{2n-j}(K; \mathbb{R}),$$

is an isomorphism. In particular, this means that multiplication by each power ω^s, $s \leq n - j$, must be injective. Therefore, for any $s \leq n$, we have an injective homomorphism

$\cdot\, \omega \colon H^{s-2}(K; \mathbb{R}) \to H^s(K; \mathbb{R})$. Thus, since $\omega \in H^2(K; \mathbb{R})^G$, we obtain a linearly independent set $\{\omega\alpha_1, \ldots, \omega\alpha_k\} \subset H^s(K; \mathbb{R})^G$. But then we see that, for all $s \leq n$,

$$\overline{b}_{s-2}(K) \leq \overline{b}_s(K).$$

Now, let's compare Betti numbers of M. We obtain

$$\begin{aligned}
b_s(M) - b_{s-1}(M) &= \overline{b}_s(K) + \overline{b}_{s-1}(K) - \overline{b}_{s-1}(K) - \overline{b}_{s-2}(K) \\
&= \overline{b}_s(K) - \overline{b}_{s-2}(K) \geq 0,
\end{aligned}$$

by the argument above. Hence, the Betti numbers of M increase up to the middle dimension. $\qquad\square$

In [4] it was shown that the first Betti number of a co-Kähler manifold is always odd. (Indeed, it was shown later that, for M co-Kähler, $S^1 \times M$ is Kähler, so this also follows by Hard Lefschetz). Here, we can infer this as a simple consequence of our splitting. Now, K is a Kähler manifold, so $\dim(H^1(K; \mathbb{R}))$ is even and there is a non-degenerate skew symmetric bilinear (i.e. symplectic) form $b \colon H^1(K; \mathbb{R}) \otimes H^1(K; \mathbb{R}) \to H^{2n}(K; \mathbb{R}) \cong \mathbb{R}$ defined by

$$b(\alpha, \beta) = \alpha \cdot \beta \cdot \omega^{n-1}.$$

Let $G = \mathbb{Z}_m = \langle \varphi \mid \varphi^m = 1 \rangle$, note that invariance of ω implies $\varphi^*\omega = \omega$ and compute:

$$\begin{aligned}
\varphi^*(b)(\alpha, \beta) &= b(\varphi^*\alpha, \varphi^*\beta) \\
&= \varphi^*\alpha \cdot \varphi^*\beta \cdot \omega^{n-1} \\
&= \varphi^*\alpha \cdot \varphi^*\beta \cdot \varphi^*\omega^{n-1} \\
&= \varphi^*(\alpha \cdot \beta \cdot \omega^{n-1}) \\
&= \alpha \cdot \beta \cdot \omega^{n-1} \\
&= b(\alpha, \beta),
\end{aligned}$$

where the second last line comes from the fact that $\alpha \cdot \beta \cdot \omega^{n-1} = k \cdot \omega^n$ and $\varphi^*\omega^n = \omega^n$. Hence, φ^* is a symplectic linear transformation on the symplectic vector space $H^1(K; \mathbb{R})$. But now the symplectic eigenvalue theorem says that the eigenvalue $+1$ occurs with even multiplicity. Thus $\overline{b}_1(K) = \dim(H^1(K; \mathbb{R})^G)$ is even. Hence, by Theorem 4.3 (1), we have the following result.

Corollary 4.4 *The first Betti number of a compact co-Kähler manifold is odd.*

5 Fundamental groups of co-Kähler manifolds

An important question about compact Kähler manifolds is exactly what groups arise as their fundamental groups. For instance, every finite group is the fundamental group of a Kähler manifold, while a free group on more than one generator cannot be the fundamental group of a Kähler manifold (see [1] for more properties of these groups). Li's mapping torus result shows that the fundamental group of a compact co-Kähler manifold is always a semidirect product of the form $H \rtimes_\psi \mathbb{Z}$, where H is the fundamental group of a Kähler manifold. As an alternative, because the finite cover of Theorem 3.4 corresponds to the subgroup $\mathrm{Ker}(\pi_1(M) \to \mathbb{Z}_m)$, Theorem 3.4 implies the following.

Theorem 5.1 *If $(M^{2n+1}, J, \xi, \eta, g)$ is a compact co-Kähler manifold with integral structure and splitting $M \cong K \times_{\mathbb{Z}_m} S^1$, then $\pi_1(M)$ has a subgroup of the form $H \times \mathbb{Z}$, where H is the fundamental group of a compact Kähler manifold, such that the quotient*

$$\frac{\pi_1(M)}{H \times \mathbb{Z}}$$

is a finite cyclic group.

5.1 Co-Kähler manifolds with transversally positive definite Ricci tensor

Now let's see how to use our general approach to recover a result of De León and Marrero ([6]) concerning compact co-Kähler manifolds with transversally positive definite Ricci tensor. Let $(M^{2n+1}, J, \xi, \eta, g)$ be an almost contact metric manifold and let $\mathcal{F}$ be the codimension 1 foliation $\ker(\eta)$. Let $T\mathcal{F}$ be the vector subbundle of the tangent bundle of M consisting on vectors that are tangent to $\mathcal{F}$: at a point $x \in M$, then

$$T_x\mathcal{F} = \{v \in T_xM \mid \eta_x(v) = 0\}.$$

Let S be the Ricci curvature tensor of M. S is called *transversally positive definite* if S_x is positive definite on $T_x\mathcal{F}$ for all $x \in M$. In [6], the authors prove the following result.

Theorem 5.2 ([6, Theorem 3.2]) *If M is a compact co-Kähler manifold with transversally positive definite Ricci tensor, then $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.*

Their result relies, in turn, on the following theorem of Kobayashi ([11]).

Theorem 5.3 ([11, Theorem A]) *A compact Kähler manifold with positive definite Ricci tensor is simply connected.*

We now give an alternative proof of Theorem 5.2 from our viewpoint.

Proof Let $(M^{2n+1}, J, \xi, \eta, g)$ be a co-Kähler manifold and let $\mathcal{F}$ be the foliation given by $\ker(\eta)$. Assume that the Ricci curvature tensor is transversally positive. Using Li's approach, we can pass to an integer co-Kähler structure and this process uses the flow of the Reeb vector field ξ to deform the leaves of $\mathcal{F}$ into the Kähler submanifold K. Now recall that ξ is Killing on a co-Kähler manifold, so its flow consists of isometries of M. In particular, if S is transversally positive definite on $\mathcal{F}$, then K is a Kähler manifold with positive definite Ricci tensor. By Theorem 5.3, K is simply connected. Therefore $\pi_1(M)$ is the semi-direct product of the trivial group with $\mathbb{Z}$, hence isomorphic to $\mathbb{Z}$. □

5.2 Co-Kähler manifolds with solvable fundamental group

There has been much work done in the past 20 years regarding the question of whether Kähler solvmanifolds are tori. In [10], for instance, it is shown that such a manifold is a finite quotient of a complex torus which is also the total space of a complex torus bundle over a complex torus. In [8], Hasegawa's result was applied to show the following.

Theorem 5.4 *A solvmanifold has a co-Kähler structure if and only if it is a finite quotient of torus which has a structure of a torus bundle over a complex torus. As a consequence, a solvmanifold $M = G/\Gamma$ of completely solvable type has a co-Kähler structure if and only if it is a torus.*

Note that we have changed the terminology of [8] to match ours. We can use Theorem 3.4 to contribute something in this vein.

Theorem 5.5 *Let $(M^{2n+1}, J, \xi, \eta, g)$ be an aspherical co-Kähler manifold with integral structure and suppose $\pi_1(M)$ is a solvable group. Then M is a finite quotient of a torus.*
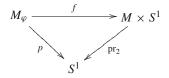
Proof We know that every aspherical solvable Kähler group contains a finitely generated abelian subgroup of finite index (see [2, section 1.5] for instance). Now, if $M = K_\varphi$ is the Li mapping torus description of M, we see that K is Kähler and aspherical with solvable fundamental group (as a subgroup of $\pi_1(M)$). Hence, K is finitely covered by a torus. By Theorem 3.4, there is a finite $\mathbb{Z}_m$-cover $K \times S^1 \to M$ and this then displays M itself as a finite quotient of a torus. $\qquad\square$

6 Automorphisms of Kähler manifolds

In this section, we connect our results above with certain facts about compact Kähler manifolds and their automorphisms. In order to do this, we first need some general results about mapping tori. Let M be a smooth manifold and let $\varphi \colon M \to M$ be a diffeomorphism. Let M_φ denote the mapping torus of φ. We have the following result.

Proposition 6.1 *The mapping torus M_φ is trivial as a bundle over S^1 (i.e. $M_\varphi \cong M \times S^1$ over S^1) if and only if $\varphi \in \mathrm{Diff}_0(M)$, where $\mathrm{Diff}_0(M)$ denotes the connected component of the identity of the group $\mathrm{Diff}(M)$.*

Proof First assume the mapping torus is trivial over S^1. We have the following commutative diagram with top row a diffeomorphism.

$$
\begin{array}{ccc}
M_\varphi & \xrightarrow{\quad f \quad} & M \times S^1 \\
& {}_{p}\searrow \quad \swarrow_{\mathrm{pr}_2} & \\
& S^1 &
\end{array}
$$

where $\mathrm{pr}_2(f([x, t])) = [t] = p([x, t])$. This means that f maps level-wise, so we have $f([x, t]) = (g_t(x), t)$, where each $g_t \colon M \to M$ is a diffeomorphism. The mapping torus relation $(k, 0) \sim (\varphi(k), 1)$ gives

$$(g_0(x), [0]) = f([x, 0]) = f(\varphi(x), 1) = (g_1(\varphi(x)), [1]) = (g_1(\varphi(x)), [0]),$$

and then we have $g_0(x) = g_1(\varphi(x))$.

Define an isotopy $F \colon M \times I \to M$ by $F(x, t) = g_0^{-1} g_t(\varphi(x))$. Then $F(x, 0) = g_0^{-1} g_0(\varphi(x)) = \varphi(x)$ and $F(x, 1) = g_0^{-1} g_1(\varphi(x)) = g_0^{-1} g_0(x) = x$. Hence, φ is isotopic to the identity.

Conversely, suppose that $\varphi \in \mathrm{Diff}_0(M)$. Then there exists a smooth map $H \colon M \times [0, 1] \to M$ such that

$$H(m, 0) = m \quad \text{and} \quad H(m, 1) = \varphi(m)$$

and $H(\cdot, t)$ is a diffeomorphism for all $t \in [0, 1]$; in particular, for all $t \in [0, 1]$, there exists a diffeomorphism $H^{-1}(\cdot, t)$. Define a map $f \colon M \times S^1 \to M_\varphi$ by

$$f(m, [t]) = [H(m, t), t];$$

where we identify $M \times S^1 = \frac{M \times [0,1]}{(\cdot,\ )_{m\,0} \sim_{m\,1} (\cdot,\ )}$. It is enough to check that f is well defined, as it is clearly smooth, but this is guaranteed by our definition of H. Next we define an inverse $g \colon M_\varphi \to M \times S^1$ by setting

$$g([m, t]) = (H^{-1}(m, t), [t]).$$

Again, g is smooth, and we must prove that it is well defined. Indeed, we have

$$g([m, 0]) = (H^{-1}(m, 0), [0]) = (m, [0])$$

and

$$g([\varphi(m), 1]) = (H^{-1}(\varphi(m), 1), [1]) = (\varphi^{-1}(\varphi(m)), [1]) = (m, [1]).$$

But $[m, [0]] = [m, [1]]$ in $M \times S^1$, so g is well-defined and is an inverse for f. □

Remark 6.2 For reference, we make the simple observation that, for a diffeomorphism $\varphi \in \text{Diff}_0(M)$, which is isotopic to the identity, the induced map on cohomology $\varphi^* \colon H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{Z})$ is the identity map.

The proposition suggests that, in order to obtain non-trivial examples of mapping tori, one should consider diffeomorphisms that do not belong to the identity component of the group of diffeomorphisms. It is then interesting to look at the groups $\text{Diff}(M)/\text{Diff}_0(M)$ or $\text{Diff}_+(M)/\text{Diff}_0(M)$, the latter in case one is interested in orientation-preserving diffeomorphisms.

Remark 6.3 In case M is a compact complex manifold, one can replace $\text{Diff}(M)$ by the group $\text{Aut}(M)$ of holomorphic diffeomorphisms of M. Further, when M is compact Kähler, one may consider the subgroup $\text{Aut}_\omega(M)$ of elements which preserve the Kähler class (but not necessarily the Kähler form). In each case, the corresponding mapping torus is trivial if and only if the automorphism belongs to the identity component.

Now let's consider the structure group of a mapping torus. Let M be a smooth manifold and let $\varphi \colon M \to M$ be a diffeomorphism. Then the mapping torus M_φ is a fibre bundle over S^1 with fibre M. In general, the structure group of a fibre bundle $F \to E \to B$ is a subgroup G of the homeomorphism group of F such that the transition functions of the bundle take values in G.

Proposition 6.4 *The structure group G of a mapping torus M_φ is the cyclic group $\langle \varphi \rangle \subset \text{Diff}(M)$.*

Sketch of Proof (see [18, Section 18]) The mapping torus M_φ is a fibre bundle over S^1 with fiber the manifold M. We can cover S^1 by two open sets U, V such that $U \cap V = \{U_0, U_1\}$ consists of two disjoint open sets. Then $M_\varphi|_U = M \times U$ and $M_\varphi|_V = M \times V$, and the mapping torus is trivial over U and V. To describe M_φ it is sufficient to give the transition function $g \colon U \cap V \to \text{Diff}(M)$. We can assume that g is the identity on U_0 and $g = \varphi$ on U_1. Then φ generates G. □

Remark 6.5 Another way to describe the mapping torus of a diffeomorphism $\varphi \colon M \to M$ is as the quotient of $M \times \mathbb{R}$ by the group $\mathbb{Z}$ acting on $M \times \mathbb{R}$ by

$$(m, (p, t)) \mapsto (\varphi^m(p), t - m).$$

It is then clear that the structure group of M_φ is isomorphic to the group generated by φ.

Let (K, h, ω) be a compact Kähler manifold, where h denotes the Hermitian metric and ω is the Kähler form. A *Hermitian isometry* is a holomorphic map $\varphi \colon K \to K$ such that $\varphi^* h = h$, where h is the Hermitian metric of K. Note that φ preserves both the Riemannian metric and

the symplectic form associated to h. Let $\mathrm{Isom}(K, h) \subseteq \mathrm{Aut}(K)$ denote the group of Hermitian isometries of K and let $\psi \in \mathrm{Isom}(K, h)$. Then ψ is a holomorphic diffeomorphism of K which preserves the Hermitian metric h. In particular, $\psi^* \omega = \omega$. Li's theorem [13] says that the mapping torus of ψ, denoted by K_ψ is a compact co-Kähler manifold and, conversely, compact co-Kähler manifolds are always such mapping tori. We say that a mapping torus is a *Kähler mapping torus* if it is a mapping torus K_φ of a Hermitian isometry $\varphi \colon K \to K$ of a Kähler manifold K. If K_ψ is non-trivial, then according to Proposition 6.1, ψ defines a non-zero element in

$$H := \mathrm{Isom}(K, h)/\mathrm{Isom}_0(K, h).$$

Our results prove that, up to a finite covering, $K_\psi \cong K \times_{\mathbb{Z}_m} S^1$ (Theorem 3.4), and the $\mathbb{Z}_m$ action is by translations on the S^1 factor. Furthermore, we get a fibre bundle $K_\psi \to S^1$ with structure group the finite group $\mathbb{Z}_m$. Notice that when we display K_ψ as a fibre bundle with fibre K, the structure group of this bundle is $\langle \psi \rangle$, the cyclic group generated by ψ in H. We then have the following theorem.

Theorem 6.6 *If K is a Kähler manifold, then all elements of the group H have finite order.*

Proof Pick an element $\psi \in H$ and form the mapping torus K_ψ. The discussion above proves that ψ has finite order in H. Since ψ is arbitrary, the result follows. □

Indeed, Lieberman [14] proves a much more general result, but in a much harder way.

Theorem 6.7 ([14, Proposition 2.2]) *Let K be a Kähler manifold and let $\mathrm{Aut}_\omega(K)$ denote the group of automorphisms of K preserving a Kähler class (but not necessarily the Kähler form). Let $\mathrm{Aut}_0(K)$ be the identity component. Then the quotient*

$$\mathrm{Aut}_\omega(K)/\mathrm{Aut}_0(K)$$

is a finite group.

Remark 6.8 In [13], Li also shows that the almost cosymplectic manifolds of [4] arise as symplectic mapping tori. That is, if M is almost cosymplectic in the terminology of [4], then there is a symplectic manifold S and a symplectomorphism $\varphi \colon S \to S$ such that $M \cong S_\varphi$. Li calls these manifolds *co-symplectic*. By the discussion in Sect. 3 and the results above, we see that there is a version of Theorem 3.4 for Li's co-symplectic manifolds *when the defining symplectomorphism φ is of finite order in*

$$\mathrm{Symp}(S)/\mathrm{Symp}_0(S).$$

Thus, knowledge about when this can happen would be very interesting.

In general, one can not expect a non-zero element in $\mathrm{Symp}(S)/\mathrm{Symp}_0(S)$ to have finite order. As an example, consider the torus T^2 with the standard symplectic structure and let $\varphi \colon T^2 \to T^2$ be the diffeomorphism covered by the linear transformation $A \colon \mathbb{R}^2 \to \mathbb{R}^2$ with matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Then φ is an area-preserving diffeomorphism of T^2, hence a symplectomorphism. Notice that the action of φ on $H^1(T^2; \mathbb{R})$, which is represented by the matrix A, is nontrivial. Hence the symplectic mapping torus T^2_φ is not diffeomorphic to $T^3 = T^2 \times S^1$; according to Proposition 6.1, φ is non-zero in $\mathrm{Symp}(T^2)/\mathrm{Symp}_0(T^2)$. Clearly φ has infinite order.

7 Examples

The first example of a compact co-Kähler manifold that is not homeomorphic to the global product of a Kähler manifold and S^1 was given in [4]. This example was generalized to every odd dimension in [15]. Each of these examples is a *solvmanifold* (i.e. a compact quotient of a solvable Lie group by a lattice) and can be described as a mapping torus of a suitable Hermitian isometry of the torus T^{2n}. Although the examples were constructed in every dimension $2n + 1$, it was not clear whether they could be the product of some compact Kähler manifold of dimension $2n$ and a circle. Of course, from what we have said above, they *are* products *up to a finite cover*.

In this section we analyze these examples from both Li's mapping torus and our finite cover splitting points of view. We also show that these examples are never the global product of a compact manifold and a circle, thus producing, in every odd dimension, examples of compact co-Kähler manifolds that are not products.

Let us begin with the CDM example. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in $GL(\mathbb{Z}, 2)$ and note that it defines a Kähler isometry of T^2 which we can write as $A(x, y) = (y, -x)$. Li's approach says to form the mapping torus

$$T_A^2 = \frac{T^2 \times [0, 1]}{(x, y, 0) \sim (A(x, y), 1)},$$

and then T_A^2 is a co-Kähler manifold with associated fibre bundle $T^2 \to T_A^2 \to S^1$ given by the projection

$$[x, y, t] \mapsto [t].$$

Now, A has order 4, so the picture is quite simple: namely, a central circle winds around the mapping torus 4 times before closing up. Therefore, we see that we have a circle action on T_A^2 given by

$$S^1 \times T_A^2 \to T_A^2, \qquad ([s], [x, y, t]) \mapsto [x, y, t + 4s].$$

When the orbit map $S^1 \to T_A^2, [s] \mapsto [x_0, y_0, 0]$ is composed with the projection map $T_A^2 \to S^1$, we get

$$S^1 \to S^1, \qquad [s] \mapsto [4s]$$

which induces multiplication by 4 on $H_1(S^1; \mathbb{Z})$. Hence, the S^1-action is homologically injective and Theorem 3.4 then gives a finite cover of T_A^2 of the form $T^2 \times S^1$. Hence, T_A^2 is finitely covered by a torus. Now let's look at the Betti numbers of T_A^2 using Theorem 4.3.

The diffeomorphism A acts on $H^1(T^2; \mathbb{R})$ by the matrix $P_* = A^t$, $P_*(x, y) = (-y, x)$, and on $H^2(T^2; \mathbb{R})$ by the identity; hence the Kähler class is invariant (as we know in general). Otherwise, there are no invariant classes in degrees greater than zero. To see this, suppose $P_*(ax + by) = -ay + bx = ax + by$. Thus, $a = b$ and $a = -b$, so $a = b = 0$. Now we have the following.

- $b_1(T_A^2) = \bar{b}_1(T^2) + 1 = 0 + 1 = 1$;
- $b_2(T_A^2) = \bar{b}_2(T^2) + \bar{b}_1(T^2) = 1 + 0 = 1$;
- $b_3(T_A^2) = \bar{b}_3(T^2) + \bar{b}_2(T^2) = 0 + 1 = 1$.

As noted in [4], this shows that T_A^2 is not a global product. For, as an orientable 3-manifold with first Betti number 1, there is no other choice but $S^1 \times S^2$ and this is ruled out since the fibre bundle $T^2 \to T_A^2 \to S^1$ shows that T_A^2 is aspherical.

The CDM example also fits in the scope of Theorem 5.1. To see this, we compute the fundamental group of T_A^2 explicitly. The fibre bundle $T^2 \to T_A^2 \to S^1$ shows that we have a short exact sequence of groups

$$0 \to \mathbb{Z}^2 \to \Gamma \to \mathbb{Z} \to 0,$$

where $\Gamma = \pi_1(T_A^2)$. Since $\mathbb{Z}$ is free, Γ is a semidirect product $\mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$. The action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is given by the group homomorphism $\phi \colon \mathbb{Z} \to \mathrm{SL}(2, \mathbb{Z})$ sending $1 \in \mathbb{Z}$ to $\phi(1) = A \in \mathrm{SL}(2, \mathbb{Z})$. As we remarked above, T_A^2 is covered $4 : 1$ by a torus T^3 and this covering gives a map $\psi \colon \mathbb{Z}^3 \to \Gamma$. The map ψ sends $(m, n, p) \in \mathbb{Z}^3$ to $(m, n, 4p) \in \Gamma$, hence the quotient $\Gamma / \mathbb{Z}^3$ is isomorphic to $\mathbb{Z}_4$.

For any $n \geq 1$ we give an example of a compact co-Kähler manifold of dimension $(2n+1)$ which is not homeomorphic to the global product of a compact manifold of dimension $2n$ and a circle. This example was constructed by Marrero and Padrón (see [15], example B1). We describe it according to our mapping torus and splitting approach.

Let $\zeta = e^{2\pi i/6}$ and consider the lattice $\Lambda \subset \mathbb{C}$ spanned by 1 and ζ. Set $T^2 = \mathbb{C}/\Lambda$ and $T^{2n} = \underbrace{T^2 \times \cdots \times T^2}_{n \text{ times}}$. Then T^{2n} is a compact Kähler manifold, with Kähler structure inherited by $\mathbb{C}^n$. Let $B \colon T^{2n} \to T^{2n}$ be the map covered by the linear transformation $\tilde{B} \colon \mathbb{C}^n \to \mathbb{C}^n$, $\tilde{B} = \mathrm{diag}(\zeta, \ldots, \zeta)$. Then B is a Hermitian isometry of the torus T^{2n}. Let T_B^{2n} be the mapping torus of B; then T_B^{2n} is a co-Kähler manifold and the associated fibre bundle $T^{2n} \to T_B^{2n} \to S^1$ is given by the projection $[p, t] \mapsto t$, where $p \in T^{2n}$. The Hermitian isometry B has order 6, so we obtain a circle action on T_B^{2n} given by

$$S^1 \times T_B^{2n} \to T_B^{2n}, \quad ([s], [p, t]) \mapsto ([p, t + 6s]).$$

Composing the orbit map with the projection $T_B^{2n} \to S^1$, we obtain $S^1 \to S^1, s \mapsto 6s$, which induces multiplication by 6 in cohomology. The S^1-action is homologically injective, and Theorem 3.4 gives us a finite cover of T_B^{2n} of the form $T^{2n} \times S^1$. Hence T_B^{2n} is finitely covered by a torus.

The fundamental group of T_B^{2n} is the semidirect product $\Gamma = \Lambda^n \rtimes_\phi \mathbb{Z}$, where the action of $\mathbb{Z}$ on Λ^n is given by the group homomorphism $\phi \colon \mathbb{Z} \to \mathrm{SL}(\Lambda^n)$, $\phi(1) = B$. The commutator subgroup $[\Gamma, \Gamma]$ is Λ^n, so in particular, Γ is a solvable group. The first homology of T_B^{2n} is

$$H_1(T_B^{2n}; \mathbb{Z}) \cong \frac{\Gamma}{[\Gamma, \Gamma]} \cong \mathbb{Z},$$

so $b_1(T_B^{2n}) = 1$. Now assume that T_B^{2n} is the product of a compact manifold K and a circle, $T_B^{2n} \cong K \times S^1$. The fundamental group of K is solvable, being a subgroup of Γ. Applying the Künneth formula with integer coefficients to $T_B^{2n} = K \times S^1$, we see that $H_1(K; \mathbb{Z}) = 0$. Hence, $\pi_1(K) \cong [\pi_1(K), \pi_1(K)]$, but this is not possible because $\pi_1(K)$ is solvable. We conclude that T_B^{2n} is not homeomorphic to the product of a compact, $2n$-dimensional manifold and a circle.

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