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John Oprea
Cleveland State University, J.OPREA@csuohio.edu

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The propagation of non-Lefschetz type, the Gottlieb group and related questions

John Oprea

To Vladimir Arnold, who taught me symplectic geometry without knowing it

Abstract. This is a brief note which indicates how the property of being non-Lefschetz may be propagated by equivariant symplectic maps. We also discuss some questions related to the Gottlieb group and nilpotency of symplectic manifolds.

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I was first introduced to the subject of symplectic geometry in a seminar held at Ohio State University where the main text was Vladimir Arnold's *Mathematical Methods in Classical Mechanics* [Arn]. Although the cohomological nature of the symplectic form was even apparent then, I never dreamed that I would be led to work on the homotopy theory of symplectic geometry (see [LO1, LO2]). Homotopical methods have proved to be important in symplectic geometry and, in particular, when mixed with hard analysis, have led to a proof of Arnold's conjecture on fixed points of hamiltonian diffeomorphisms (see [RO], and [CLOT]). In this sense, Arnold's famous problems have been good for homotopy theory as well as geometry!

In this note, we consider the relationship between the existence of symplectic mappings between symplectic manifolds with circle action (or with appropriate Gottlieb structure) and the existence of Kähler structures (or, more generally, Lefschetz type structures of cohomology) on the manifolds. Much work has been done in recent years that clarifies the relationship between Lefschetz type and rational homotopy properties such as formality (see for instance, [Cav], [HT] and the forthcoming [FOT]). Also, more work has been done on symplectically aspherical manifolds and their fundamental groups (see [IKRT]), and this leads to several interesting questions concerning Gottlieb groups and symplectic manifolds.

The main results of this short note follow from the framework created in [LO2] for the study of symplectic circle actions on symplectic manifolds. With this in mind, we now turn to a brief review of that framework.

1. Review of the $\lambda_{\hat{\alpha}}$ -invariant

An S^1 -action and its orbit map may be generalized to produce certain basic elements in the fundamental group of M and of the function space $(M^M, 1_M)$ in the following manner. Say that an element $\alpha \in \pi_1(M)$ is in the *Gottlieb group* of M , denoted $\alpha \in G(M)$, if there is an *associated map*

$$A: S^1 \times M \rightarrow M$$

such that $A|_M = 1_M$ and $A|_{S^1} = \alpha$. From [Go1] and [Go2], we know certain things about $G(M)$. In particular, we know that

- (1) $G(M) \subseteq \mathcal{Z}(\pi_1(M))$.
- (2) If $M = K(\pi, 1)$, then $G(M) = \mathcal{Z}(\pi_1(M))$.

By the exponential law, there is a map $\hat{\alpha}: S^1 \rightarrow M^M$, $\hat{\alpha}(s)(x) = A(s, x)$, such that evaluation $\text{ev}(f) = f(p)$ of a function at a basepoint $m \in M$ satisfies $\text{ev} \circ \hat{\alpha} = \alpha$. Hence, $\text{ev}_{\#}(\hat{\alpha}) = \alpha$ where $\text{ev}_{\#}: \pi_1(M^M, 1_M) \rightarrow \pi_1(M)$. Note that a group action $A: S^1 \times M \rightarrow M$ provides a Gottlieb element $\alpha \simeq A|_{S^1}$ which may be identified with the homotopy class of the orbit map $a: S^1 \rightarrow M$. Also note that it is possible for α to be nullhomotopic, but $\hat{\alpha}$ to be essential. For this reason, *we shall be more interested in the associated map A and its adjoint $\hat{\alpha}$ than in the (possible) Gottlieb element α .*

There is a classifying space $\text{Baut}(M)$ for fibrations with fibre M which obeys $\pi_{i+1} \text{Baut}(M) = \pi_i(M^M, 1_M)$. Therefore, $\hat{\alpha} \in \pi_1(M^M, 1_M)$ corresponds to an element in $\pi_2 \text{Baut}(M)$ represented by a map $S^2 \rightarrow \text{Baut}(M)$. Pulling back the universal fibration with fibre M gives a fibration

$$M \xrightarrow{i} E \rightarrow S^2$$

with $\alpha = \partial_{\#}(1) \in \text{Im}(\partial_{\#}: \pi_2 S^2 \rightarrow \pi_1 M)$. Such a fibration has a Wang sequence associated to it,

$$\dots \rightarrow H^q(E) \xrightarrow{i^*} H^q(M) \xrightarrow{\lambda_{\hat{\alpha}}} H^{q-1}(M) \rightarrow H^{q+1}(E) \rightarrow \dots$$

The map $\lambda_{\hat{\alpha}}$, called the *Wang derivation*, is a derivation on $H^*(M)$. That is, $\lambda_{\hat{\alpha}}$ satisfies the relation $\lambda_{\hat{\alpha}}(uv) = \lambda_{\hat{\alpha}}(u)v + (-1)^{|u|}u\lambda_{\hat{\alpha}}(v)$. There is a beautiful connection between the Wang sequence and the map A . Namely, for any $u \in H^q(M)$,

$$(\dagger) \quad A^*(u) = 1 \times u + \bar{\sigma} \times \lambda_{\hat{\alpha}}(u),$$

where $\bar{\sigma} \in H^1(S^1)$ is a chosen fixed generator, $\alpha = \text{ev}_\#(\hat{\alpha})$ and \times is the external product. In case $u \in H^1(M)$, we have $\alpha^*(u) = \lambda_{\hat{\alpha}}(u)\bar{\sigma}$, with $\lambda_{\hat{\alpha}}(u) \in H^0(M) = \mathbb{Q}$, and the expression (†) may be rewritten

$$A^*(u) = 1 \times u + \bar{\sigma} \times \lambda_{\hat{\alpha}}(u) = 1 \times u + \alpha^*(u) \times 1.$$

For further details on the derivation $\lambda_{\hat{\alpha}}$, see [Go3]. We have a basic result, which follows immediately from the exactness of the Wang sequence.

Proposition 1.1. $\lambda_{\hat{\alpha}}(\omega) = 0$ if and only if there exists $\bar{\omega} \in H^2(E)$ with $i^*\bar{\omega} = \omega$.

The following fundamental result connects the $\lambda_{\hat{\alpha}}$ -invariant to symplectic geometry [LO2]:

Theorem 1.2. If $\hat{\alpha}$ comes from a symplectic S^1 -action on a compact symplectic manifold (M, ω) , then

$$\lambda_{\hat{\alpha}}(\omega) = [i(\underline{X})\omega],$$

where \underline{X} denotes the fundamental vector field associated to the symplectic S^1 -action and $i(-)$ is interior multiplication.

Because the equality $[i(\underline{X})\omega] = 0$ is precisely what symplectic geometers mean by a Hamiltonian action, this theorem provides the basis for

Definition 1.3. A closed manifold M^{2n} is *cohomologically symplectic* (often shortened to *c-symplectic*) if there exists $\omega \in H^2(M)$ with $\omega^n \neq 0$. For a c-symplectic manifold (M, ω) , an element $\hat{\alpha} \in \pi_1(M^M, 1_M)$ is said to be *cohomologically Hamiltonian*, or c-Hamiltonian, if $\lambda_{\hat{\alpha}}(\omega) = 0$. If $\hat{\alpha}$ comes from an S^1 -action and $\lambda_{\hat{\alpha}}(\omega) = 0$, then we say that the action is c-Hamiltonian.

For the moment, let M be an N -dimensional manifold, with top class $\mu \in H^N(M; \mathbb{Q})$. For applications we will specialize to a c-symplectic manifold (M^{2n}, ω) with $\mu = \omega^n$. In general, we take $\hat{\alpha} \in \pi_1(M^M, 1_M)$ and $\alpha = \text{ev}_\#(\hat{\alpha}) \in \pi_1(M)$. Recall that the Hurewicz homomorphism $h: \pi_1(M) \rightarrow H_1(M)$ is defined by $h(\gamma) = \gamma_*(\sigma)$, where σ is a generator of $H_1(S^1; \mathbb{Z})$ dual to $\bar{\sigma} \in H^1(S^1; \mathbb{Z})$ and γ_* is the map on homology induced by $\gamma: S^1 \rightarrow M$.

Our initial goal is to clarify the relationship between $\lambda_{\hat{\alpha}}$ and $h(\alpha)$. First, recall that Poincaré duality may be characterized in terms of cup, cap and Kronecker products: For $a \in H_q(M)$, $z \in H_N(M)$ a fundamental class Kronecker dual to the top class μ , the element $\hat{a} \in H^{N-q}(M)$ is *Poincaré dual* to a if

$$\langle b \cup \hat{a}, z \rangle = \langle b, \hat{a} \cap z \rangle = \langle b, a \rangle$$

for any $b \in H^q(M)$. If $u \in H^1(M)$, then $u \cup \mu = 0$ by dimensional considerations. Since $\lambda_{\hat{\alpha}}$ is a derivation, we get

$$0 = \lambda_{\hat{\alpha}}(u \cup \mu) = \lambda_{\hat{\alpha}}(u) \cdot \mu - u \cup \lambda_{\hat{\alpha}}(\mu).$$

Since u is in degree 1, we have $\alpha^*(u) = \lambda_{\hat{\alpha}}(u) \cdot \bar{\sigma}$. Hence,

$$\langle u \cup \lambda_{\hat{\alpha}}(\mu), z \rangle = \langle \lambda_{\hat{\alpha}}(u) \cdot \mu, z \rangle = \lambda_{\hat{\alpha}}(u) = \langle \alpha^*(u), \sigma \rangle = \langle u, \alpha_*(\sigma) \rangle = \langle u, h(\alpha) \rangle.$$

Our first result follows immediately from these observations.

Lemma 1.4. *Let $\mu \in H^N(M)$ be a top class and let $\hat{\alpha} \in \pi_1(M^M, 1_M)$ with $\alpha = \text{ev}_{\#}(\hat{\alpha})$.*

- (1) *For $u \in H^1(M)$, $\lambda_{\hat{\alpha}}(u) = \langle u, h(\alpha) \rangle$.*
- (2) *$\lambda_{\hat{\alpha}}(\mu)$ is Poincaré dual to $h(\alpha)$.*

Combining Lemma 1.4 with another general observation yields the following result.

Proposition 1.5. *Let $\mu \in H^N(M)$ be a top class and let $\hat{\alpha} \in \pi_1(M^M, 1_M)$ with $\alpha = \text{ev}_{\#}(\hat{\alpha})$. We have the equivalences*

$$\lambda_{\hat{\alpha}}(\mu) = 0 \Leftrightarrow \lambda_{\hat{\alpha}}(H^1(M)) = 0 \Leftrightarrow h(\alpha) = 0.$$

Proof. The first equivalence follows from a general result: If θ is a degree -1 derivation on a Poincaré duality algebra H , of top dimension N , then $\theta(H^N) = 0$ if and only if $\theta(H^1) = 0$ (see [LO1, Lemma 3.3] or [FOT, Proposition 7.60]). The second equivalence follows from Lemma 1.4(1). \square

This then gives

Proposition 1.6. *Let (M^{2n}, ω) be a c -symplectic manifold, with top class ω^n , and let $\hat{\alpha} \in \pi_1(M^M, 1_M)$ with $\alpha = \text{ev}_{\#}(\hat{\alpha})$. If $\hat{\alpha}$ is c -Hamiltonian (i.e. if $\lambda_{\hat{\alpha}}(\omega) = 0$), then the three equivalent conditions of Proposition 1.5 hold.*

Proof. Since $\lambda_{\hat{\alpha}}$ is a derivation, we have $\lambda_{\hat{\alpha}}(\omega^n) = n\omega^{n-1}\lambda_{\hat{\alpha}}(\omega) = 0$. \square

Definition 1.7. A c -symplectic manifold (M^{2n}, ω) has *Lefschetz type* if the multiplication homomorphism $\omega^{n-1}: H^1(M) \rightarrow H^{2n-1}(M)$ is an isomorphism.

The Lefschetz type hypothesis allows for the implication of Proposition 1.6 to be reversed.

Theorem 1.8. *Let (M^{2n}, ω) be a c -symplectic manifold of Lefschetz type and let $\hat{\alpha} \in \pi_1(M^M, 1_M)$ with $\alpha = \text{ev}_{\#}(\hat{\alpha})$. Then the following are equivalent:*

- (1) *The element $\hat{\alpha}$ is c -Hamiltonian (i.e. $\lambda_{\hat{\alpha}}(\omega) = 0$).*
- (2) *$\lambda_{\hat{\alpha}}(\omega^n) = 0$.*
- (3) *$\lambda_{\hat{\alpha}}(H^1(M)) = 0$.*
- (4) *$h(\alpha) = 0$.*

Proof. In light of Proposition 1.6, we need only show that $h(\alpha) = 0$ implies $\lambda_{\hat{\alpha}}(\omega) = 0$. Now, by Lemma 1.4, $h(\alpha)$ is Poincaré dual to $\lambda_{\hat{\alpha}}(\omega^n) = n \cdot \lambda_{\hat{\alpha}}(\omega) \cup \omega^{n-1}$. By duality, $h(\alpha) = 0$ then implies $n \cdot \lambda_{\hat{\alpha}}(\omega) \cup \omega^{n-1} = 0$ and, by the Lefschetz type hypothesis, this can only happen if $\lambda_{\hat{\alpha}}(\omega) = 0$. \square

2. c-Symplectic maps and the $\lambda_{\hat{\alpha}}$ -invariant

In this section, let (N^{2n}, ω_N) and (M^{2m}, ω_M) be c-symplectic manifolds with S^1 -actions

$$A_N: S^1 \times N \rightarrow N, \quad A_M: S^1 \times M \rightarrow M.$$

Let $f: N \rightarrow M$ be a based homotopy equivariant map which is *c-symplectic*. We shall always require our maps between c-symplectic manifolds to be basepoint preserving, and homotopy equivariance will refer to basepoint preserving homotopies. Homotopy equivariance is expressed by a homotopy commutative diagram

$$\begin{array}{ccc} S^1 \times N & \xrightarrow{A_N} & N \\ 1 \times f \downarrow & & \downarrow f \\ S^1 \times M & \xrightarrow{A_M} & M \end{array}$$

The map f is *c-symplectic* if $f^*(\omega_M) = \omega_N$. Of course, any symplectic map is c-symplectic.

This diagram provides the link between $\lambda_{\hat{\alpha}_N}(\omega_N)$ and $\lambda_{\hat{\alpha}_M}(\omega_M)$.

Lemma 2.1. *Let $\hat{\alpha}_N \in \pi_1(N^N, 1_N)$ and $\hat{\alpha}_M \in \pi_1(M^M, 1_M)$ be the adjoints of the respective actions A_N and A_M . Then*

$$f^*(\lambda_{\hat{\alpha}_M}(\omega_M)) = \lambda_{\hat{\alpha}_N}(\omega_N).$$

Proof. We can compute two ways using homotopy equivariance:

$$\begin{aligned} (1 \times f)^* A_M^* \omega_M &= (1 \times f)^*(1 \times \omega_M + \bar{\sigma} \times \lambda_{\hat{\alpha}_M}(\omega_M)) \\ &= 1 \times f^*(\omega_M) + \bar{\sigma} \times f^*(\lambda_{\hat{\alpha}_M}(\omega_M)); \\ A_N^* f^*(\omega_M) &= 1 \times f^*(\omega_M) + \bar{\sigma} \times \lambda_{\hat{\alpha}_N}(f^*(\omega_M)) \\ &= 1 \times f^*(\omega_M) + \bar{\sigma} \times \lambda_{\hat{\alpha}_N}(\omega_N). \end{aligned}$$

By comparing the expressions, we see that $f^*(\lambda_{\hat{\alpha}_M}(\omega_M)) = \lambda_{\hat{\alpha}_N}(\omega_N)$. \square

Proposition 2.2. *With the notations above, if $\hat{\alpha}_M$ is c-hamiltonian, then so is $\hat{\alpha}_N$.*

Remark 2.3. Note that an equivariant map $f: N \rightarrow M$ satisfies the above. In particular, a symplectic map of symplectic manifolds which is equivariant with respect to S^1 -actions fits the situation. Therefore, if the action on M is Hamiltonian, then so is the action on N .

We can relate these ideas to the notion of Lefschetz type as follows.

Theorem 2.4. *Let $f: N \rightarrow M$ be a c-symplectic map of c-symplectic manifolds (N, ω_N) and (M, ω_M) which is homotopy equivariant with respect to S^1 -actions A_N and A_M respectively. For the adjoints $\hat{\alpha}_N$ of A_N and $\hat{\alpha}_M$ of A_M , if $\hat{\alpha}_N$ satisfies the equivalent conditions of Proposition 1.5, but is not c-hamiltonian, then the same is true for $\hat{\alpha}_M$. Hence, M cannot have Lefschetz type (and so, cannot be Kähler).*

Proof. Since $\hat{\alpha}_N$ satisfies the conditions of Proposition 1.5, the Hurewicz image $h(\alpha_N)$ is zero. Recall that α_N is represented by the composition

$$S^1 \xrightarrow{\hat{\alpha}_N} N^N \xrightarrow{\text{ev}} N.$$

The homotopy equivariance of the map f provides a homotopy commutative diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{\hat{\alpha}_N} & N^N & \xrightarrow{\text{ev}} & N \\ \downarrow = & & & & \downarrow \\ S^1 & \xrightarrow{\hat{\alpha}_M} & M^M & \xrightarrow{\text{ev}} & M \end{array}$$

showing that $f_{\#}(\alpha_N) = \alpha_M$, where $f_{\#}: \pi_1(N) \rightarrow \pi_1(M)$. Then by the naturality of the Hurewicz map, we have

$$\begin{array}{ccc} \pi_1(N) & \xrightarrow{f_{\#}} & \pi_1(M) \\ h \downarrow & & h \downarrow \\ H_1(N) & \xrightarrow{f_*} & H_1(M) \end{array}$$

which gives $h(\alpha_M) = f_*(h(\alpha_N)) = 0$ as well.

Now, $\hat{\alpha}_N$ is not c-hamiltonian by hypothesis, so $\lambda_{\hat{\alpha}_N}(\omega_N) \neq 0$. Then, by Lemma 2.1, $f^*(\lambda_{\hat{\alpha}_M}(\omega_M)) = \lambda_{\hat{\alpha}_N}(\omega_N)$, so $\lambda_{\hat{\alpha}_M}(\omega_M) \neq 0$ also. Therefore, $\hat{\alpha}_M$ satisfies the conditions of Proposition 1.5, but is not c-hamiltonian. By Theorem 1.8, M cannot have Lefschetz type. \square

Remark 2.5. In fact, all that we have said above for S^1 -actions also holds for maps $A: S^1 \times N \rightarrow N$ which may not be actions, but which are adjoints of elements $\hat{\alpha}_N \in \pi_1(N^N, 1_N)$. Given such maps for N and M and a homotopy commutative diagram

$$\begin{array}{ccc} S^1 \times N & \xrightarrow{A_N} & N \\ 1 \times f \downarrow & & f \downarrow \\ S^1 \times M & \xrightarrow{A_M} & M \end{array}$$

we can carry out the proofs of the results above with no changes. This will prove important in the next section.

3. c-Symplectic maps of aspherical manifolds

Let us apply the results above to the case where $(N, \omega_N) = (K(\pi, 1), \omega_\pi)$ and $(M, \omega_M) = (K(\rho, 1), \omega_\rho)$ are c-symplectic aspherical manifolds and $f: K(\pi, 1) \rightarrow K(\rho, 1)$ is a c-symplectic map between them. Here, we can do without the necessity of requiring S^1 -actions on the manifolds because it is a fact that any (effective) S^1 -action A on an aspherical manifold K has an orbit map $\alpha: S^1 \rightarrow K$ which induces

an injection $\alpha_{\#}: \pi_1(S^1) \rightarrow \pi_1(K)$ whose image, in fact, must lie in the center of $\pi_1(K)$ (which is of course $G(K)$). Therefore, in the case of aspherical manifolds, we can turn our attention from S^1 -actions to the centers of fundamental groups instead. Indeed, for an aspherical space K , it can be shown that

$$\pi_1(K^K, 1_K) \xrightarrow[\cong]{\text{ev}} \mathcal{Z}(\pi_1(K)),$$

so that $\hat{\alpha}_K$ and α_K always correspond in this case. We shall assume this identification below when we show

Theorem 3.1. *Let $(K(\pi, 1), \omega_{\pi})$ and $(K(\rho, 1), \omega_{\rho})$ be c-symplectic aspherical manifolds and let $f: K(\pi, 1) \rightarrow K(\rho, 1)$ be a c-symplectic map. Suppose there exist nontrivial $\alpha_{\pi} \in \mathcal{Z}(\pi)$ and $\alpha_{\rho} \in \mathcal{Z}(\rho)$ such that $f_{\#}(\alpha_{\pi}) = \alpha_{\rho}$. If α_{π} satisfies the equivalent conditions of Proposition 1.5, but is not c-hamiltonian, then the same is true for α_{ρ} . Hence, $K(\rho, 1)$ cannot have Lefschetz type (and so, in particular, cannot be Kähler).*

Proof. First notice that, because α_{π} is in the center of π , there is a homomorphism $\mathbb{Z} \times \pi \rightarrow \pi$ given by $A_{\pi\#}(n, g) = \alpha_{\pi}^n g$ and similarly for α_{ρ} . We then have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \times \pi & \xrightarrow{A_{\pi\#}} & \pi \\ 1 \times f_{\#} \downarrow & & \downarrow f_{\#} \\ \mathbb{Z} \times \rho & \xrightarrow{A_{\rho\#}} & \rho \end{array}$$

We can also make the computation

$$\begin{aligned} f_{\#} A_{\pi\#}(n, g) &= f_{\#}(\alpha_{\pi}^n g) = f_{\#}(\alpha_{\pi}^n) f_{\#}(g) = \alpha_{\rho}^n f_{\#}(g) \\ &= A_{\rho\#}(n, f_{\#}(g)) = A_{\rho\#}(1 \times f_{\#})(n, g). \end{aligned}$$

Because maps between aspherical spaces are classified at the fundamental group level and $\pi_1(S^1) = \mathbb{Z}$, this gives a homotopy commutative diagram

$$\begin{array}{ccc} S^1 \times K(\pi, 1) & \xrightarrow{A_{\pi}} & K(\pi, 1) \\ 1 \times f \downarrow & & \downarrow f \\ S^1 \times K(\rho, 1) & \xrightarrow{A_{\rho}} & K(\rho, 1) \end{array}$$

By Remark 2.5, we may now apply Theorem 2.4 to this situation to obtain the result. \square

It is known that nilmanifolds of Lefschetz type (e.g. Kähler nilmanifolds) are diffeomorphic to tori. The proof of this fact in [LO1, Theorem 3.1] (or in [O, Theorem 2.3.10]) may be interpreted as saying that, for a c-symplectic *non-toral* nilmanifold $K(\pi, 1)$, there always exists an element $\alpha_{\pi} \in \mathcal{Z}(\pi) \cap [\pi, \pi]$ such that $h(\alpha_{\pi}) = 0$ (since $\alpha_{\pi} \in [\pi, \pi]$ and $H_1(\pi; \mathbb{Z}) = \pi/[\pi, \pi]$), but $\lambda_{\hat{\alpha}_{\pi}}(\omega_{\pi}) \neq 0$ (i.e.

α_π is not c-hamiltonian). We can use this interpretation and the results above to propagate the non-Kählerness of nilmanifolds by c-symplectic maps as follows.

Corollary 3.2. *Let $(K(\pi, 1), \omega_\pi)$ be a c-symplectic nilmanifold and suppose that $(K(\rho, 1), \omega_\rho)$ is a c-symplectic manifold such that $f: K(\pi, 1) \rightarrow K(\rho, 1)$ is a c-symplectic map. Suppose that $f_\#(\alpha_\pi) = \alpha_\rho$, where $\alpha_\pi \in \mathcal{Z}(\pi) \cap [\pi, \pi]$ is as above and $\alpha_\rho \in \mathcal{Z}(\rho)$ is nontrivial. If $K(\rho, 1)$ has Lefschetz type, then $K(\pi, 1)$ is a torus.*

Proof. By Theorem 3.1, since α_π satisfies the conditions of Proposition 1.6, but is not c-hamiltonian, then the same holds for α_ρ . This, however, contradicts Theorem 1.8, since, by hypothesis, $K(\rho, 1)$ has Lefschetz type. Hence, the nilmanifold $K(\pi, 1)$ must be a torus since α_π always exists for non-toral nilmanifolds. \square

Remark 3.3. Corollary 3.2 would apply, in particular, to the case of a nilmanifold N c-symplectically embedded as the fibre of a Mostow fibration

$$N \rightarrow S \rightarrow T$$

where S is a solvmanifold and T is a torus. If α_π may be extended to an element in the center of $\pi_1(S)$ and S has Lefschetz type, then N must be a torus. By our remarks above about S^1 -actions on aspherical manifolds, such an extension would take place if the free S^1 -action on N given by α_π may be extended to all of S . Also note that, for completely solvable solvmanifolds, a symplectic form on S may be chosen which restricts to one on N ([BG]).

4. Some related questions

R. Gompf showed that any finitely presented group can be realized as the fundamental group of certain symplectic 4-manifolds. More recently, in [IKRT] certain restrictions were found on the groups that can arise as fundamental groups of symplectic manifolds where the symplectic cohomology class annihilates the image of the Hurewicz homomorphism. These are the so-called *symplectically aspherical* manifolds. A fair amount is known about the homotopy theory of symplectically aspherical manifolds (see [LO2] as well as the reference [IKRT] mentioned above), but these manifolds are very special. In particular, these are the manifolds that are amenable to a homotopical proof of the Arnold conjecture on fixed points of hamiltonian diffeomorphisms (see, for instance, [RO, CLOT]). These are just first steps in understanding the homotopy theory of symplectic manifolds and certain classes of symplectic manifolds.

The first homotopical question that arises concerns the Gottlieb groups of symplectically aspherical manifolds. From [LO2, Theorem 4.12, Corollary 5.13], we know the following.

Theorem 4.1. *If a circle acts on a symplectically aspherical manifold M , then the orbit map at a point, $S^1 \rightarrow M$, induces an element of infinite order in the Gottlieb group $G(M)$. Further, if M also has Lefschetz type, then the Hurewicz image of this Gottlieb element is of infinite order in $H_1(M; \mathbb{Z})$.*

Here we see that homotopy theory and geometry of M intertwine. In fact, however, this type of result is rare for Gottlieb groups. Indeed, it seems to be unknown if all finitely generated abelian groups arise as Gottlieb groups of compact manifolds. The result above, however, says that the following question may have an answer.

Question 4.2. *What are the Gottlieb groups of symplectically aspherical manifolds?*

A more fanciful line of thought asks

Question 4.3. *What restrictions, if any, are placed on the Gottlieb groups of manifolds of Lefschetz type?*

The Gottlieb group is also known to lie inside the subgroup of $\pi_1(M)$ consisting of all elements that act trivially on all higher homotopy groups (under the standard action). This brings up a general question about symplectic manifolds. Recall that a space X is *nilpotent* if $\pi_1(X)$ acts nilpotently on $\pi_j(X)$ for all $j \geq 1$. The methods of homotopy theory and, especially, rational homotopy theory, work best for nilpotent spaces. So, if we are to make use of these methods, then it would be very nice to know the answer to the following

Question 4.4. *How can nilpotent symplectic manifolds be recognized? If a symplectic manifold is a nilpotent space, what special homotopical properties are apparent? Conversely, what nilpotent spaces have symplectic or c-symplectic structures?*

This is a question which connects geometry and homotopy theory in a fundamental way. Although we do not know of any general results in this direction, here is a result that gives a slight indication of how the action of the fundamental group on higher homotopy may be recognized in the symplectic world. While the proposition holds in general, it pays to think of ω as the symplectic (or c-symplectic) class. For a symplectic manifold (M, ω) , the condition of symplectic asphericity is equivalent to the condition that $\omega = f^*(\tilde{\omega})$, where $f: M \rightarrow K(\pi_1(M), 1)$ classifies the universal cover and $\tilde{\omega}$ is some class in $H^2(K(\pi_1(M), 1); \mathbb{R})$ (see [LO2]). Hence, $p^*(\omega) = 0$, where $p: \tilde{M} \rightarrow M$ is the universal cover. On the other hand, the result below applies to the generic case of symplectic manifolds that are *not* symplectically aspherical. The proposition is a special case of one found in [LMP], but the homotopical proof is new.

Proposition 4.5. *Suppose that M is a path connected space with $a \in H^1(M; \mathbb{Q})$ and $\omega \in H^2(M; \mathbb{Q})$ obeying $a \cup \omega = 0$ and $p^*(\omega) \neq 0$, where $p: \tilde{M} \rightarrow M$ is the universal cover. Then the action of $\pi_1(M)$ on $\pi_2(M)$ is nontrivial.*

Proof. The proof and interpretation will come together in several steps.

Step 1. The condition $p^*(\omega) \neq 0$ is equivalent to saying that $\omega|_{\pi_2(M)} \neq 0$, where $\omega \in H^2(M) \cong \text{Hom}(H_2(M; \mathbb{Z}), \mathbb{R})$ is considered dual to homology and operating

on the image of Hurewicz in $H_2(M; \mathbb{Z})$. The equivalence of the conditions is implied immediately by the following commutative diagram:

$$\begin{array}{ccc} \pi_2(\widetilde{M}) & \xrightarrow[\cong]{h} & H_2(\widetilde{M}; \mathbb{Z}) \\ p\# \downarrow & & \downarrow p_* \\ \pi_2(M) & \xrightarrow{h} & H_2(M; \mathbb{Z}) \end{array}$$

The condition $\omega|_{\pi_2(M)} \neq 0$ removes the universal cover from consideration and focusses on M and ω .

Step 2. So, take $\gamma \in \pi_2(M)$ such that $\omega(h(\gamma)) \neq 0$ and $\alpha \in \pi_1(M)$ such that $a(h(\alpha)) \neq 0$. We wish to understand the action of α on γ in a classical way. Namely, the deviation of the action from being trivial is detected by the Whitehead product

$$\alpha \cdot \gamma - \gamma = [\alpha, \gamma].$$

Thus, to show that the action of $\pi_1(M)$ on $\pi_2(M)$ is nontrivial, it is sufficient to show that the Whitehead product $[\alpha, \gamma]$ is nonzero. In order to do this, we will use Steenrod's functional cup product.

Step 3. Consider the following situation: $f: X \rightarrow Y$ and $v \in H^q(Y)$. We have a commutative diagram

$$\begin{array}{ccccccccc} H^{p-1}(Y) & \xrightarrow{f^*} & H^{p-1}(X) & \xrightarrow{\delta} & H^p(Y, X) & \xrightarrow{i^*} & H^p(Y) & \xrightarrow{f^*} & H^p(X) \\ \lambda_1 \downarrow & & \lambda_2 \downarrow & & \lambda_3 \downarrow & & \lambda_1 \downarrow & & \\ H^{p+q-1}(Y) & \xrightarrow{f^*} & H^{p+q-1}(X) & \xrightarrow{\delta} & H^{p+q}(Y, X) & \xrightarrow{i^*} & H^{p+q}(Y) & \xrightarrow{f^*} & H^{p+q}(X) \end{array}$$

where $\lambda_1(y) = y \cup v$, $\lambda_2(u) = u \cup f^*v$ and $\lambda_3(z) = z \cup v$ (using relative cup product). Let $u \in \text{Ker}(f^*) \cap \text{Ker}(\lambda_1)$. Then $u \cup v = 0$. Also, because $u \in \text{Ker}(f^*)$, there exists $\beta \in H^p(Y, X)$ with $i^*\beta = u$ and

$$\lambda_1 i^* \beta = \lambda_1 u = 0 = i^* \lambda_3 \beta = i^*(\beta \cup v).$$

Hence, there exists $\tau \in H^{p+q-1}(X)$ with $\delta\tau = \beta \cup v$. Thus we can associate τ to the element u . This association $u \mapsto \tau$ provides a homomorphism

$$\sigma: \text{Ker}(f^*) \cap \text{Ker}(\lambda_1) \rightarrow H^{p+q-1}(X) / f^* H^{p+q-1}(Y) + \lambda_2 H^{p-1}(X)$$

and we define the functional cup product (as a coset) by

$$u \cup_f v := \sigma(u).$$

The following facts are found in [W]:

- Fact 0. If f is nullhomotopic, then $u \cup_f v = 0$.
- Fact 1. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, then $u \cup_{gf} v \subset g^*(u) \cup_f g^*(v)$ for $u, v \in H^*(Z)$.

- Fact 2. For the Whitehead product map $[\iota_p, \iota_q]: S^{p+q-1} \rightarrow S^p \vee S^q$ and u_p, u_q the respective generators of $H^p(S^p), H^q(S^q)$, we have

$$u_p \cup_{[\iota_p, \iota_q]} u_q = -u_{p+q-1},$$

where u_{p+q-1} generates $H^{p+q-1}(S^{p+q-1})$.

Now, for $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$, represent the Whitehead product $[\alpha, \beta]$ by

$$F: S^{p+q-1} \xrightarrow{[\iota_p, \iota_q]} S^p \vee S^q \xrightarrow{h} X,$$

where $h|_{S^p} = \alpha, h|_{S^q} = \beta$. Let $u, v \in H^*(X)$ be as above and such that $u \cup v = 0$. Now, for $p, q > 1$, $F(H^{p+q-1}(X)) = 0$ since F factors through $S^p \vee S^q$, and $F^*v = 0$ since $H^q(S^{p+q-1}) = 0$. For the case $p = 1, q > 1$, $F^*(H^q(X)) = 0$ because the Whitehead product $[\iota_p, \iota_q]$ itself induces the zero homomorphism $H^q(S^1 \vee S^q) \rightarrow H^q(S^q)$ due to the fact that $[\iota_p, \iota_q]$ attaches the top cell in $S^1 \times S^q$. Hence there is no indeterminacy in the definition of the functional cup product and we have

$$u \cup_F v \in H^{p+q-1}(S^{p+q-1}).$$

By Fact 1 above, $u \cup_F v \in h^*(u) \cup_{[\iota_p, \iota_q]} h^*(v)$.

Now suppose that $u \in H^p(X)$ is dual to a Hurewicz element $h(\alpha)$ for $\alpha \in \pi_p(X)$ and $v \in H^q(X)$ is dual to a Hurewicz element $h(\beta)$ for $\beta \in \pi_q(X)$. Then $h^*(u)(\bar{u}_p) = u(h_*(\bar{u}_p)) = u(h(\alpha)) = 1$ and similarly $h^*(v)(\bar{u}_q) = 1$. Thus,

$$u \cup_F v = h^*(u) \cup_{[\iota_p, \iota_q]} h^*(v) = u_p \cup_{[\iota_p, \iota_q]} u_q = -u_{p+q-1}$$

by Fact 2. Since $u_{p+q-1} \neq 0$, by Fact 0, $F = [\alpha, \beta] \neq 0$ as well.

Step 4. The situation above is almost exactly that of Proposition 4.5 with $\alpha \in \pi_1(M)$ and $\beta = \gamma \in \pi_2(M)$. There is a slight difference which occurs because ω is not necessarily dual to $h(\gamma)$, but only is nonzero on it. This just introduces a nonzero factor ξ into the equation

$$a \cup_{[\alpha, \gamma]} \omega = \xi u_2,$$

where u_2 generates $H^2(S^2)$, and hence is nonzero. Therefore, the Whitehead product $[\alpha, \gamma] = \alpha \cdot \gamma - \gamma \neq 0$ and the action of $\pi_1(M)$ on $\pi_2(M)$ is nontrivial. \square

Another proof of Proposition 4.5 using the minimal models of rational homotopy theory is given in [FOT, Proposition 4.100]. Other interactions between symplectic (and complex) manifolds and rational homotopy theory may be found there as well.

5. Self-maps of (symplectic) manifolds

Another subject in symplectic geometry which has not been studied extensively is that of the homotopy theory of self-maps. Here is a type of result that relates a homotopical property of a space to the more geometrical question of how the space sits as a fibre in fibrations (or bundles).

Proposition 5.1. *Suppose A is a manifold such that $H^{\text{odd}}(A; \mathbb{Q}) = 0$ and A sits as the fibre in a fibration $A \xrightarrow{i} E \rightarrow B$. If $f: A \rightarrow A$ is not rationally nullhomotopic, then $i \circ f: A \rightarrow E$ is also not rationally nullhomotopic.*

Remark 5.2. By a result of [LO1], any symplectic (or c-symplectic) homogeneous space A must be of maximal rank, and hence has $H^{\text{odd}}(A; \mathbb{Q}) = 0$.

Proof of Proposition 5.1. If $i \circ f: A \rightarrow E$ were nullhomotopic, then f would factor through the connecting map in the Puppe sequence, $\partial: \Omega B \rightarrow A$. Then Theorem 5.3 below would imply that f is rationally null, a contradiction. \square

We shall now give a new proof of the fundamental theorem of Lupton and Smith. Their original proof used minimal models, but here we use only classical homotopy theory.

Theorem 5.3 ([LuSm, Theorem 3.2]). *Let $f: A \rightarrow X$ be a map whose rationalization factors through an H -space. If X is a finite complex and $H^{\text{odd}}(A; \mathbb{Q}) = 0$, then f is rationally trivial.*

Since we are taking rationalizations in the theorem, from now on we will simply take A and X to be rational spaces. Also, by Hopf's theorem, we can take the H -space through which f factors to be a product of $K(\mathbb{Q}, j)$'s.

First, let us recall a standard result. Let X_n denote the n -th Postnikov term of X . (Also, for later use, recall that the n -connective cover of X , denoted $X\langle n \rangle$, is the homotopy fibre of $X \rightarrow X_n$.) Milnor showed that, for any space K , there is a short exact sequence

$$* \rightarrow \varprojlim^1 [SK, X_n] \rightarrow [K, X] \rightarrow \varprojlim [K, X_n] \rightarrow *,$$

where SK denotes suspension.

Lemma 5.4. *If X is a rational space, then $\varprojlim^1 [SK, X_n] = 0$.*

Proof. We always have $[SK, X_n] = [K, \Omega X_n]$ and, because of Hopf's result, we have the following homotopy commutative diagram expressing the fact that any map $K \rightarrow \Omega X_n$ can be lifted, up to homotopy, to $K \rightarrow \Omega X_{n+1}$.

$$\begin{array}{ccc} & \Omega X_{n+1} & \xrightarrow{\cong} & \prod K(\mathbb{Q}, n_j) \times K(\pi_{n+1}(X), n+1) \\ & \uparrow \alpha \times * & & \downarrow \text{projection} \\ K & \xrightarrow{\alpha} & \Omega X_n & \xrightarrow{\cong} & \prod K(\mathbb{Q}, n_j). \end{array}$$

Therefore, we see that we have surjections of groups $[SK, X_{n+1}] \rightarrow [SK, X_n]$ for all n . The tower of these groups then satisfies the Mittag-Leffler condition, so $\varprojlim^1 [SK, X_n] = 0$. \square

The result above is sometimes expressed by saying that there are no rational phantom maps. Therefore, in order to understand the homotopy class of a rational map $K \rightarrow X$, it is sufficient to understand the projections to all Postnikov pieces.

Now suppose we have a factorization $A \xrightarrow{g} H \xrightarrow{h} X$ with $f = hg$. Let $H = H_o \times H_e$, where

$$H_o = \prod K(\mathbb{Q}, 2k_j + 1) \quad \text{and} \quad H_e = \prod K(\mathbb{Q}, 2k_i)$$

by Hopf's theorem. It is always true that $[K, X \times Y] = [K, X] \times [K, Y]$, so the homotopy class of g is determined by that of the projections $A \rightarrow H_o$ and $A \rightarrow H_e$. Because $H_o = \prod K(\mathbb{Q}, 2k_j + 1)$, the class of $A \rightarrow H_o$ is given by an element in $H^{\text{odd}}(A; \mathbb{Q}) = 0$. Hence, we may restrict attention to H_e . In fact, we now focus on the restriction $h_e: H_e \rightarrow X$. Note that H_e has the property that every element in $H^*(H_e; \mathbb{Q})$ has infinite height (i.e. cup product powers never vanish).

Lemma 5.5. *Suppose Z has the property that every element in $H^*(Z; \mathbb{Q})$ has infinite height and let X be a rational space of finite category (e.g. the rationalization of a finite complex). Then $[Z, X] = *$.*

Proof. First note that $H^{\text{odd}}(Z; \mathbb{Q}) = 0$ since elements of odd degree have finite height 1. Secondly, note that a composition $Z \xrightarrow{h} X \xrightarrow{p_2} X_2 = K(\pi_2(X), 2)$ must be nullhomotopic because the homotopy class is determined by an element of $H^2(Z; \pi_2(X))$ which is pulled back from an element of $H^2(X; \pi_2(X))$ by h^* . But the element in $H^2(X; \pi_2(X))$ would then have infinite height, and this is impossible by the assumption that X has finite category (which means that the cuplength of X is finite). This begins an induction where, for any map $h: Z \rightarrow X$, we have the following homotopy commutative diagram:

$$\begin{array}{ccccc}
 & & X \langle i \rangle & \xrightarrow{\rho} & K(\pi_{i+1}(X), i+1) \\
 & \nearrow \tilde{h} & \downarrow & & \downarrow \\
 Z & \xrightarrow{h} & X & \xrightarrow{p_{i+1}} & X_{i+1} \\
 & \searrow p_i h \simeq * & \downarrow p_i & & \downarrow \tilde{p}_i \\
 & & X_i & \xrightarrow{\text{id}} & X_i
 \end{array}$$

Of course, we are able to factor through the fibres only because the map $p_i h$ is trivial.

There are two cases. If $i+1$ is odd, then the homotopy class of $\rho \tilde{h}: Z \rightarrow K(\pi_{i+1}(X), i+1)$ is an element of $H^{\text{odd}}(Z; \mathbb{Q}) = 0$. Hence, the composition $p_{i+1} h: Z \rightarrow X_{i+1}$ is also trivial and the induction continues.

If $i+1$ is even, then, by what we have just said, we have a factorization through fibres $Z \xrightarrow{\tilde{h}} X \langle i \rangle \xrightarrow{\rho} K(\pi_{i+1}(X), i+1)$. As before, the class of this map is given by an element in $H^{i+1}(Z; \pi_{i+1}(X))$ which is pulled back from an element in $H^{i+1}(X \langle i \rangle; \pi_{i+1}(X))$ by \tilde{h}^* . If this map is not nullhomotopic, then this element in $H^{i+1}(X \langle i \rangle; \pi_{i+1}(X))$ would have infinite height. But this is impossible because the i -connective covering map $X \langle i \rangle \rightarrow X$ is *injective* on homotopy groups, and the

Mapping Theorem of rational homotopy theory (see [FOT, Theorem 2.81]) then says that

$$\text{cup}_0(X\langle i \rangle) \leq \text{cat}_0(X\langle i \rangle) \leq \text{cat}_0(X) < \infty.$$

Hence, $p_{i+1}h \simeq *$. Therefore, we see inductively that $p_jh \simeq *: Z \rightarrow X_j$ for all j . By Lemma 5.4, we have $h \simeq *$. \square

Proof of Theorem 5.3. We have already seen that we can restrict to a map $h_e: H_e \rightarrow X$, where $H_e = \prod K(\mathbb{Q}, 2k_i)$. But then H_e satisfies the hypotheses of Lemma 5.5, so $[H_e, X] = *$. Therefore, we obtain

$$f \simeq hg \simeq h(g_e \times g_o) \simeq h(g_e \times *) \simeq h_e g_e \simeq *g_e \simeq *. \quad \square$$

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