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Usefulness of Cardiac Biomarker Score for Risk Stratification in Stable Patients Undergoing Elective Cardiac Evaluation Across Glycemic Status

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Two classes of time-inhomogeneous Markov chains: Analysis of the periodic case

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Abstract We consider the M/G/1 and GI/M/1 types of Markov chains for which their one step transitions depend on the times of the transitions. These types of Markov chains are encountered in several stochastic models, including queueing systems, dams, inventory systems, insurance risk models, etc. We show that for the cases when the time parameters are periodic the systems can be analyzed using some extensions of known results in the matrix-analytic methods literature. We have limited our examples to those relating to queueing systems to allow us a focus. An example application of the model to a real life problem is presented.

1 Introduction

In many practical systems that can be represented by stochastic models we find that the associated parameters are time-varying. For example, arrival rates are usually time-varying in most real life queueing systems such as rush hour traffic, hourly internet traffic, etc. Some of these examples are discussed in Hall (1991, Chap. 6). There have been both continuous and discrete time models developed for analyzing queues with time-varying parameters (see Margolius 2005; Breuer 2001; Ingolfsson 2005, and Alfa 1982 and references therein). The input process in a dam is another example of a time-varying quantity into a system. The processing rates could also be time-varying in all these systems. Most of these stochastic models are usually set up as Markov chains, which when properly set up result in Markov chains with time-varying parameters. Surprisingly the volume of research related to Markov chains of this class is disproportionately low compared to that of its time-invariant counterpart given the importance of the time-varying case in applications.

Even though there has been a reasonable amount of work carried out on queues with time-varying parameters—and as a consequence their Markov chains have been discussed—there has not been as much work on Markov chains with time-varying parameters specifically. Two of the few works dealing specifically with Markov chains with time-varying parameters are those of Massey and Whitt (1998) and Yin and Zhang (2005). Massey and Whitt dealt with finite space continuous-time-Markov-chains with time-varying parameters. The results of that work are limited to the cases of those Markov chains with slowly varying rates. They use the idea of uniform acceleration to approximate the time-varying state distribution. Yin and Zhang (2005) applied the two-time-scale Markov chain to a quasi-birth-death (QBD) Markov chain with time-varying parameters. Using the same terminologies as for the matrix-analytical approach, i.e. levels for the first variable and phases for the second variable of the Markov chain, they obtained results for the QBD for which the levels vary at a slower rate than the phases within the level. Both these works seem to be limited to the cases when rates vary slowly. Our interest is to remove that requirement of slow varying rates and also of the finiteness of the Markov chain. However, we limit ourselves to the special types of Markov chains that are encountered frequently, i.e. the GI/M/1 and the M/G/1 types. We study these in discrete time and use the matrix-analytic approach for the case when the rates are periodic. Our main contribution is to set these up using the matrix-analytic method formalism and then show how existing methods can easily be extended to analyze the problem.

We consider the M/G/1 and GI/M/1 types of Markov chains for which their one step transitions depend on the times of the transitions and we study their time-varying Markov chains. Later we consider these Markov chains with periodic behaviors. As practical examples of these Markov chains we consider the Geo$_n$/Geo$_n$/1, Geo$_n$/Geo$_n$/c, and Geo$_n$/Geo$_n$/c_n which are discrete time analogues of the $M_t/M_t/1$, $M_t/M_t/c$ and $M_t/M_t/c_t$ queues, respectively.

The rest of this paper is organized as follows. In Sect. 2 we develop and analyze the Markov chain of the GI/M/1 type, and give some example applications of this system. The M/G/1 type is discussed in Sect. 3.

2 Time-inhomogeneous GI/M/1 type Markov chain

Consider the discrete-time Markov chain $\{X_n, J_n\}, n \geq 0$ with the state space $\{(\{0\} \times \{1, 2, \ldots, M_0\}) \cup (\{1, 2, \ldots\} \times \{1, 2, \ldots, M\})\}$, where M_0 and M are integers with $1 \leq (M_0, M) < \infty$. X_n is referred to as the level and J_n as the phase. M_0 represents the number of phases within level 0 and M the phases within levels $1, 2, \ldots$. Let us define $P(n)_{l,k;i,j} = \Pr\{X_{n+1} = i, J_{n+1} = j | X_n = l, J_n = k\}$. The transition matrix $P(n)$ associated with this Markov chain can be written as

$$P(n) = \begin{bmatrix} B_{0,0}^{(n)} & C_0^{(n)} & & & \\ B_{1,0}^{(n)} & A_1^{(n)} & A_0^{(n)} & & \\ B_{2,0}^{(n)} & A_2^{(n)} & A_1^{(n)} & A_0^{(n)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where each block represents transition within levels. For example, if we define level $\mathbf{0} = \{X_n = 0, J_n = 1, 2, \ldots, M_0\}$ and similarly level $\mathbf{i} = \{X_n = i, J_n = 1, 2, \ldots, M\}, i = 1, 2, \ldots,$

then $B_{k,0}$ is a matrix that represents transitions from level $\mathbf{k}$ to level $\mathbf{0}$, and A_k is a matrix that represents transitions from level $\mathbf{i}$ to level $\mathbf{i-k+1}$, $\forall i \geq 1$. This Markov chain has the skip-free properties in that a transition from level $\mathbf{i}$ to level $\mathbf{i+k}$, $k > 1$, will have to go through levels $\mathbf{i+1}, \mathbf{i+2}, \ldots, \mathbf{i+k-1}$ before reaching level $\mathbf{i+k}$.

Let $\boldsymbol{x}(n)_{i,j} = \Pr\{X_n = i, J_n = j\}$ with

$$\boldsymbol{x}(n)_i = [\boldsymbol{x}(n)_{i,1}, \boldsymbol{x}(n)_{i,2}, \ldots, \boldsymbol{x}(n)_{i,M}]$$

for $i > 0$ with M replaced by M_0 when $i = 0$. The time-varying state probability vector for this system can be written as $\boldsymbol{x}(n) = [\boldsymbol{x}(n)_0, \boldsymbol{x}(n)_2, \ldots, \boldsymbol{x}(n)_i, \ldots]$. It is easy to see that

$$\boldsymbol{x}(n+1) = \boldsymbol{x}(n)P(n), \quad n \geq 0.$$

Let us now define $P(m,n)_{l,k;i,j} = \Pr\{X_{n+1} = i, J_{n+1} = j | X_m = l, J_m = k\}$. It is immediately clear that

$$P(m,m) = P(m)$$

and

$$P(m,n) = P(m) \times P(m+1) \times \cdots \times P(n-1) \times P(n)$$

for $n > m$. This matrix is of the form

$$P(m,n) = \begin{bmatrix} B_{0,0}^{(m,n)} & B_{0,1}^{(m,n)} & \ldots & B_{0,n-m}^{(m,n)} & C_{n-m}^{(m,n)} & & & \\ B_{1,0}^{(m,n)} & B_{1,1}^{(m,n)} & \ldots & B_{1,n-m}^{(m,n)} & A_1^{(m,n)} & A_0^{(m,n)} & & \\ B_{2,0}^{(m,n)} & B_{2,1}^{(m,n)} & \ldots & B_{2,n-m}^{(m,n)} & A_2^{(m,n)} & A_1^{(m,n)} & A_0^{(m,n)} & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \\ B_{k,0}^{(m,n)} & B_{k,1}^{(m,n)} & \ldots & B_{k,n-m}^{(m,n)} & A_k^{(m,n)} & \ldots & A_1^{(m,n)} & A_0^{(m,n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.1}$$

The resulting Markov chain has a structure similar to the non-skip-free system as described by Gail et al. (1997), except that our Markov chain is time-inhomogeneous. In this Markov chain a transition from level $\mathbf{i}$ to level $\mathbf{i+k}$, $k > 1$, does not have to go through levels $\mathbf{i+1}$, $\mathbf{i+2}, \ldots, \mathbf{i+k-1}$ before reaching level $\mathbf{i+k}$, if $k < m$. However, by reblocking the block matrices into blocks of size $M_0(n-m+1) \times M_0(n-m+1)$ for $B_{0,0}^{(m)}$, $M(n-m+1) \times M_0(n-m+1)$ for $B_{i,0}^{(m)}$, $i \geq 1$, $M_0(n-m+1) \times M(n-m+1)$ for $B_{0,j}^{(m)}$, $j \geq 1$, and $M(n-m+1) \times M(n-m+1)$ for A_{j-1}, and $B_{i,j}^{(m)}$ $(i,j) \geq 1$, we reduce this Markov chain to the GI/M/1 type with skip-free properties, and then analyze it using the same techniques as for the GI/M/1 type. Further we have

$$\boldsymbol{x}(n+1) = \boldsymbol{x}(m)P(m,n), \quad m \leq n.$$

First we discuss the recursion for computing the blocks of the matrix $P(m,n)$ from the blocks of the matrices $P(m,n-1)$ and $P(n)$.

$$B_{0,0}^{(m,n)} = \sum_{j=0}^{n-m-1} B_{0,j}^{(m,n-1)} B_{j,0}^{(n)} + C_{n-m-1}^{(m,n-1)} B_{n-m,0}^{(n)}, \tag{2.2}$$

$$B_{k,0}^{(m,n)} = \sum_{j=0}^{n-m-1} B_{k,j}^{(m,n-1)} B_{j,0}^{(n)} + \sum_{v=0}^{k} A_{k-v}^{(m,n-1)} B_{n-m+v,0}^{(n)}, \quad k \geq 1, \tag{2.3}$$

$$B_{0,1}^{(m,n)} = B_{0,0}^{(m,n-1)} C_0^{(n)} + \sum_{j=1}^{n-m-1} B_{0,j}^{(m,n-1)} A_j^{(n)} + C_{n-m-1}^{(m,n-1)} A_{n-m}^{(n)}, \tag{2.4}$$

$$B_{k,1}^{(m,n)} = B_{k,0}^{(m,n-1)} C_0^{(n)} + \sum_{j=1}^{n-m-1} B_{k,j}^{(m,n-1)} A_j^{(n)} + \sum_{v=0}^{k} A_{k-v}^{(m,n-1)} A_{n-m+v}^{(n)}, \quad k \geq 1, \tag{2.5}$$

$$B_{0,j}^{(m,n)} = \sum_{v=j-1}^{n-m-1} B_{0,v}^{(m,n-1)} A_{v-j+1}^{(n)} + C_{n-m-1}^{(m,n-1)} A_{n-m-j+1}^{(n)}, \quad j \geq 2, \tag{2.6}$$

$$B_{k,j}^{(m,n)} = \sum_{v=j}^{n-m} B_{k,v-1}^{(m,n-1)} A_{v-j}^{(n)} + \sum_{v=0}^{j} A_{k-v}^{(m,n-1)} A_{n-m-j+v+1}^{(n)}, \quad k \geq 1,\ j \geq 2, \tag{2.7}$$

$$C_{n-m}^{(m,n)} = C_{n-m-1}^{(m,n-1)} A_0^{(n)}, \tag{2.8}$$

$$A_k^{(m,n)} = \sum_{v=0}^{k} A_{k-v}^{(m,n-1)} A_v^{(n)}, \quad k \geq 0. \tag{2.9}$$

Of special interest is the case when the initial condition is known. For example, let us assume that $\{X_0 = i, J_0 = j\}$ with certainty, i.e. $\Pr\{X_0 = i, J_0 = j\} = 1$. Let us define $\tilde{\boldsymbol{x}}(n)_{w,v} = \Pr\{X_n = w, J_n = v | X_0 = i, J_0 = j\}$, with $\tilde{\boldsymbol{x}}(n+1) = \tilde{\boldsymbol{x}}(0) P(0,n)$. It is immediately clear that

$$\tilde{\boldsymbol{x}}(n+1) = [\mathbf{e}_j^{'} B_{i,0}^{(0,n)}, \mathbf{e}_j^{'} B_{i,1}^{(0,n)}, \ldots, \mathbf{e}_j^{'} B_{i,n}^{(0,n)}, \mathbf{e}_j^{'} A_i^{(0,n)}, \mathbf{e}_j^{'} A_{i-1}^{(0,n)}, \ldots, \mathbf{e}_j^{'} A_0^{(0,n)}, 0, 0, \ldots, 0],$$
$$i \geq 1,$$

and

$$\tilde{\boldsymbol{x}}(n+1) = [\mathbf{e}_j^{'} B_{0,0}^{(0,n)}, \mathbf{e}_j^{'} B_{0,1}^{(0,n)}, \ldots, \mathbf{e}_j^{'} B_{0,n}^{(0,n)}, \mathbf{e}_j^{'} C_n^{(0,n)}, 0, 0, \ldots, 0], \quad i = 0,$$

where $\mathbf{e}_j^{'}$ is a row vector of zeros with 1 in location j.

2.1 The case of periodic transition probabilities

Now we consider the case when there exists an integer $\tau < \infty$, such that $P(m,n) = P(m + j\tau, n + j\tau)$, $j = 1, 2, 3, \ldots$. Here τ is the length of the period. In this case we can define

$$\mathcal{P}(m) = P(m, m + \tau - 1) = P(j\tau + m, j\tau + m + \tau - 1), \quad \forall j \geq 1. \tag{2.10}$$

Let us assume that there exists a stationary distribution $\mathbf{y}(m)$ given by

$$\mathbf{y}(m) = \boldsymbol{x}(m + k\tau) = \boldsymbol{x}(m + k\tau)\mathcal{P}(m), \quad k = 0, 1, 2, \ldots, \text{ with } \mathbf{y}(m)\mathbf{1} = 1.$$

Next we state the conditions under which such a stationary distribution exists and show how to obtain it.

Consider the matrix $\mathcal{A}^{(m)} = \sum_{v=0}^{\infty} A_v^{(m,m+\tau-1)}$. This matrix is stochastic and let us assume that it is irreducible. If so, then there exists a vector $\boldsymbol{\pi}^{(m)}$ such that $\boldsymbol{\pi}^{(m)} = \boldsymbol{\pi}^{(m)}\mathcal{A}^{(m)}$ with $\boldsymbol{\pi}^{(m)}\mathbf{1} = 1$. Further let $\boldsymbol{\psi}^{(m)} = \sum_{v=0}^{\infty} v\mathcal{A}_v^{(m)}\mathbf{1}$. Using the results in Neuts (1981), the stability conditions for the Markov chain represented by the matrix $\mathcal{P}(m)$ is

$$\boldsymbol{\pi}^{(m)}\boldsymbol{\psi}^{(m)} > 1.$$

Matrix $\mathcal{P}(m)$ can be re-blocked into the form:

$$\mathcal{P}(m) = \begin{bmatrix} \mathbf{B}_{0,0}^{(m)} & \mathbf{C}_0^{(m)} & & & \\ \mathbf{B}_{1,0}^{(m)} & \mathbf{A}_1^{(m)} & \mathbf{A}_0^{(m)} & & \\ \mathbf{B}_{2,0}^{(m)} & \mathbf{A}_2^{(m)} & \mathbf{A}_1^{(m)} & \mathbf{A}_0^{(m)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.11}$$

We see that this is of the skip-free GI/M/1-type. For this Markov chain, provided the stability conditions are satisfied, it is known $\mathbf{y}(m)$ exists. Most important is that there is a matrix $R(m)$ which is the minimal non-negative solution to the matrix polynomial equation

$$R(m) = \sum_{v=0}^{\infty} R(m)^v \mathbf{A}_v^{(m)}.$$

If we partition $\mathbf{y}(m)$ as

$$\mathbf{y}(m) = [\mathbf{y}(m)_0, \mathbf{y}(m)_1, \mathbf{y}(m)_2, \mathbf{y}(m)_3, \ldots],$$

then

$$\mathbf{y}(m)_{j+1} = \mathbf{y}(m)_j R(m).$$

The key requirement is to determine $R(m)$ efficiently.

Let us partition $R(m)$ as

$$R(m) = [\mathbf{r}(m)_1, \mathbf{r}(m)_2, \ldots, \mathbf{r}(m)_\tau],$$

where each $\mathbf{r}(m)_v,\ \forall v$, is a matrix of order $M\tau \times M$. Let $\mathbf{r}(m) = \mathbf{r}(m)_1$. Let

$$\mathbf{r}(m) = \begin{bmatrix} r_1(m) \\ r_2(m) \\ \vdots \\ r_\tau(m) \end{bmatrix}.$$

From the results of Gail et al. (1997) we have

$$R(m) = [\mathbf{r}(m), C(m)\mathbf{r}(m), C(m)^2\mathbf{r}(m), \ldots, C(m)^{\tau-1}\mathbf{r}(m)],$$

where

$$C(m) = \begin{bmatrix} 0 & 0 & \dots & 0 & r(m)_1 \\ I & 0 & \dots & 0 & r(m)_2 \\ 0 & I & \dots & 0 & r(m)_3 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \dots & I & r(m)_\tau \end{bmatrix} \quad \text{and} \quad \mathbf{r}(m) = \begin{bmatrix} \mathbf{A}_{\tau-1}^{(m)} \\ \mathbf{A}_{\tau-2}^{(m)} \\ \vdots \\ \mathbf{A}_0^{(m)} \end{bmatrix} + \sum_{v=\tau}^{\infty} C(m)^{v-\tau}\mathbf{r}(m)\mathbf{A}_v^{(m)}.$$

Consider the $\text{Geo}_n/\text{Geo}_n^X/1$ system in which we define $\beta_{i,j}$ as the probability that the batch size served at time n is i of the j waiting customers. For this system we study the number of waiting customers and not the number in the system. It is easy to show that the Markov chain of this $\text{Geo}_n/\text{Geo}_n^X/1$ type is a small example of the GI/M/1 type Markov chain discussed. In this case we have $M_0 = M = 1$. If however we bound the batch size that can be served at one time interval to be $c_n < \infty$ then we have the $\text{Geo}_n/\text{Geo}_n^{1,c_n}/1$ system which has the same structure as the unbounded one except that the associated matrix now has one super-diagonal, a diagonal and c_n sub-diagonals. The multiserver system $\text{Geo}_n/\text{Geo}_n/c_n$ has the same structure as the $\text{Geo}_n/\text{Geo}_n^{1,c_n}/1$ system. Since both the $\text{Geo}_n/\text{Geo}_n^{1,c_n}/1$ and the $\text{Geo}_n/\text{Geo}_n/c_n$ systems have similar structures we use the multiserver system to illustrate our technique.

2.2 Queue length of the $\text{Geo}_n/\text{Geo}_n/c_n$ system

We consider the discrete time-varying multi-server queue for which service rates and arrival rates may vary with time. The $\text{Geo}_n/\text{Geo}_n/c_n$ system is the discrete analogue of the $M_t/M_t/c_t$ system. This system is a special case of the time-inhomogeneous GI/M/1 type Markov chain.

First, we define $\beta_{v,w}^{(m)}$ to be the probability that of the ongoing w services, v get completed in the current time-interval, m. We also define $\alpha^{(m)}$ and $\bar{\alpha}^{(m)}$ respectively, as the probability of an arrival and the probability of no arrival during interval m. In order to avoid unnecessary details in explaining the associated Markov chain we focus on the case where $c_n = c < \infty$, $\forall n$. The value c may be the maximum number of servers possible at any time epoch. We can therefore drop subscript n for the number of servers.

The matrix $P(m)$ associated with this Markov chain can be written as

$$P(m) = \begin{bmatrix} d_{0,0}^{(m)} & d_{0,1}^{(m)} & & & & & & \\ d_{1,0}^{(m)} & d_{1,1}^{(m)} & d_{1,2}^{(m)} & & & & & \\ d_{2,0}^{(m)} & d_{2,1}^{(m)} & d_{2,2}^{(m)} & d_{2,3}^{(m)} & & & & \\ \vdots & & \ddots & \ddots & \ddots & & & \\ d_{c-1,0}^{(m)} & d_{c-1,1}^{(m)} & \dots & \dots & d_{c-1,c-1}^{(m)} & d_{c-1,c}^{(m)} & & \\ u_{-c}^{(m)} & u_{-c+1}^{(m)} & \dots & \dots & u_{-1}^{(m)} & u_0^{(m)} & u_1^{(m)} & \\ & u_{-c}^{(m)} & u_{-c+1}^{(m)} & \dots & \dots & u_{-1}^{(m)} & u_0^{(m)} & u_1^{(m)} \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where $\bar{\alpha}^{(m)} = d_{0,0}^{(m)}$, $\alpha^{(m)} = d_{0,1}^{(m)}$, $d_{v-1,v}^{(m)} = \alpha^{(m)}\beta_{0,v}^{(m)}$, $v \geq 1$, also, $d_{c-1,0}^{(m)} = \bar{\alpha}^{(m)}\beta_{c-1,c-1}^{(m)}$, $d_{i,j}^{(m)} = \alpha^{(m)}\beta_{i-j+1,i}^{(m)} + \bar{\alpha}^{(m)}\beta_{i-j,i}^{(m)}$, $i \leq i \leq j < c$, $u_{-c}^{(m)} = \bar{\alpha}^{(m)}\beta_{c,c}^{(m)}$, $u_1^{(m)} = \alpha^{(m)}\beta_{0,c}^{(m)}$, $u_j^{(m)} =$

$\bar{\alpha}^{(m)}\beta^{(m)}_{-j,c}+\alpha^{(m)}\beta^{(m)}_{-j+1,c}$, $0 \leq j \leq c$. No modification is required in this analysis to accommodate the time-varying server case versus constant server, except to include the rules needed to deal with what happens to the customers in service when the number of servers changes, i.e. whether it is preemptive or non-preemptive. The matrix $P(m)$ will reflect the number of servers working at time m.

By properly blocking this matrix we have

$$P(m) = \begin{bmatrix} B^{(m)}_{0,0} & C^{(m)}_0 & & & & & & & \\ B^{(m)}_{1,0} & A^{(m)}_1 & A^{(m)}_0 & & & & & & \\ B^{(m)}_{2,0} & A^{(m)}_2 & A^{(m)}_1 & A^{(m)}_0 & & & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & \\ B^{(m)}_{c,0} & A^{(m)}_c & A^{(m)}_{c-1} & \dots & \dots & A^{(m)}_0 & & & \\ & A^{(m)}_{c+1} & A^{(m)}_c & A^{(m)}_{c-1} & \dots & \dots & A^{(m)}_0 & & \\ & & A^{(m)}_{c+1} & A^{(m)}_c & A^{(m)}_{c-1} & \dots & \dots & A^{(m)}_0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where $B_{j,0} = 0, \forall j \geq c+1$ and $A_j = 0, \forall j \geq c+1$, with

$$B^{(m)}_{0,0} = \begin{bmatrix} d^{(m)}_{0,0} & d^{(m)}_{0,1} & & & & \\ d^{(m)}_{1,0} & d^{(m)}_{1,1} & d^{(m)}_{1,2} & & & \\ d^{(m)}_{2,0} & d^{(m)}_{2,1} & d^{(m)}_{2,2} & d^{(m)}_{2,3} & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ d^{(m)}_{c-2,0} & d^{(m)}_{c-2,1} & \dots & \dots & \dots & d^{(m)}_{c-2,c-1} \\ d^{(m)}_{c-1,0} & d^{(m)}_{c-1,1} & \dots & \dots & \dots & d^{(m)}_{c-1,c-1} \end{bmatrix},$$

$$C^{(m)}_0 = \begin{bmatrix} 0 & 0 & \dots & d^{(m)}_{c-1,c} \end{bmatrix}^T, \qquad B^{(m)}_{1,0} = \begin{bmatrix} u^{(m)}_{-c} & u^{(m)}_{-c+1} & \dots & u^{(m)}_{-1} \end{bmatrix},$$
$$A^{(m)}_j = u^{(m)}_{-j+1}.$$

Consider a $c \times c$ matrix

$$B_j = \begin{bmatrix} \mathbf{0}_{j\times(c-j)} & I_j \\ \mathbf{0}_{(c-j)\times(c-j)} & \mathbf{0}_{(c-j)\times j} \end{bmatrix},$$

then $B^{(m)}_{j,0} = B^{(m)}_{1,0} B_{c-j+1}$, $j = 1, 2, \dots, c$. In this example, $M_0 = c$ and $M = 1$.

If we now define $P(m,n)_{l,k;i,j} = \Pr\{X_{n+1} = i, J_{n+1} = j | X_m = l, J_m = k\}$, it is immediately clear that $P(m,n)$ is a special case of $P(m,n)$ given in (2.1) and it is given as

$$P(m,n) = P(m) \times P(m+1) \times \cdots \times P(n-1) \times P(n)$$

for $n > m$ and $P(m,m) = P(m)$.

Defining the matrix $\mathcal{P}(\cdot)$ in a manner analogous to the definition in (2.10), the matrix $\mathcal{P}(m) = P(m)P(m+1)\ldots P(\tau)P(1)P(2)\ldots P(m-1)$. The block form representation of this matrix obtained by applying (2.11) to this QBD is

$$\mathcal{P}(m) = \begin{bmatrix} \mathbf{B}_{0,0}^{(m)} & \mathbf{C}_0^{(m)} & & & \\ \mathbf{B}_{1,0}^{(m)} & \mathbf{A}_1^{(m)} & \mathbf{A}_0^{(m)} & & \\ & \mathbf{A}_2^{(m)} & \mathbf{A}_1^{(m)} & \mathbf{A}_0^{(m)} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

We simply apply the results of Sect. 3.1 to solve this system.

We examine the block matrices $\mathbf{A}_i^{(m)}$ in greater detail. The first row of the three blocks: $[\mathbf{A}_2^{(m)} \,|\, \mathbf{A}_1^{(m)} \,|\, \mathbf{A}_0^{(m)}]$ is given by $[a_{-c\tau}^{(m)}, a_{-c\tau+1}^{(m)}, \ldots, a_{-1}^{(m)}, a_0^{(m)}, a_1^{(m)}, \ldots, a_\tau^{(m)}, 0, \ldots, 0]$ with $2c\tau - \tau - 1$ zeros. Each subsequent row is equal to the preceding row with the nonzero elements $(a_{-c\tau}^{(m)}, \ldots, a_\tau^{(m)})$ shifted one column to the right. $a_i^{(m)}$ is given by

$$a_i^{(m)} = \sum_{\{r_1,\ldots,r_\tau\}\in S_\tau} \prod_{\sum_{j=1}^{\tau} k_j = i} u_{k_j}^{(r_j)},$$

where S_τ is the set of all permutations of $\{1, 2, \ldots, \tau\}$. Note that $a_i^{(m)}$ does not depend on m, so we may drop the superscript and write

$$\mathcal{P}(m) = \begin{bmatrix} \mathbf{B}_{0,0}^{(m)} & \mathbf{C}_0^{(m)} & & & \\ \mathbf{B}_{1,0}^{(m)} & \mathbf{A}_1 & \mathbf{A}_0 & & \\ & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The time-dependence is still present in the block matrices $\mathbf{B}_{0,0}^{(m)}, \mathbf{B}_{1,0}^{(m)}$, and $C_0^{(m)}$ and in the blocks of the $P(m)$ matrices, but because of the underlying diagonal structure of the $P(m)$ matrices at the scalar level and because scalar multiplication is commutative, the relation $\mathbf{A}_i^{(m)} = \mathbf{A}_i$ holds for all m, $m = 1, \ldots, \tau$ and $i = 0, 1, 2$.

This is computationally important in that for the $\mathrm{Geo}_n/\mathrm{Geo}_n/c_n$ queue, it is not necessary to compute a separate R matrix for each time $m = 1, 2, \ldots, \tau$ within the period since R depends only on the $\mathbf{A}_i^{(m)} = \mathbf{A}_i$ matrices which do not depend on time. The periodicity is captured in the time-dependent matrices, which come into play at the boundary. The stationary distribution, $\mathbf{y}(m)$ will still be governed by the formula:

$$\mathbf{y}(m)_{j+1} = \mathbf{y}(m)_j R,$$

for all times m within the period, and in general, $\mathbf{y}(m)$ will not be equal to $\mathbf{y}(k)$ if m and k represent different times within the period.

2.3 Some examples of $\mathrm{Geo}_n/\mathrm{Geo}_n/c_n$ queues

This example is based on data from the Cleveland Police 4th district. The graph in Fig. 1 shows the average number of calls for service per hour (hourly rate), the number of servers

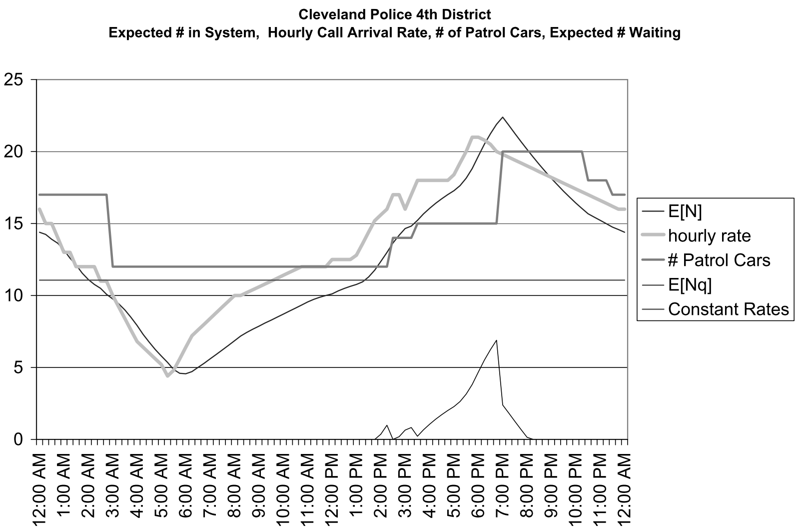

Fig. 1 Cleveland police example

(# of patrol cars), and the expected number of callers either waiting or in service (E[N]), and the expected number of callers waiting for a patrol car (E[Nq]). The computation was done by assuming a constant arrival rate for each of the 96 quarter hours in the day. Each 15 minute interval was then further subdivided so that the probability of a single arrival (the probability of receiving a single call for service) was roughly 4% and the probability of more than one arrival was very small. For this example, $\tau = 8{,}200$, but only 96 of the $P(m)$ matrices are unique. The rows of $[\mathbf{A}_2 \,|\, \mathbf{A}_1 \,|\, \mathbf{A}_0]$ can be approximated by

$$a_i \approx \frac{e^{-(i-(\bar{\lambda}-\bar{\mu}))^2/(2(\bar{\lambda}+\bar{\mu}))}}{\sqrt{2\pi(\bar{\lambda}+\bar{\mu})}},$$

where $\bar{\lambda}$ is the average number of customers arriving during the period (calls for service received during the day), and $\bar{\mu}$ is the average effective service rate (service rate times number of servers). In other words, the components of the rows of the $\mathbf{A}_i$ matrices will be approximately discretized normal with mean $\bar{\lambda}-\bar{\mu}$, and variance $\bar{\lambda}+\bar{\mu}$. For the example, $\bar{\mu} = 422.4$, and $\bar{\lambda} = 327.925$. This convergence to a discretized normal distribution was used to reduce the size of the matrices in the computations. A naïve application of the method described here would require matrices with 164,000 rows ($8{,}200 \times 20$, where 20 is the maximum number of servers). The largest matrices used in the computation were 600×600. This size was chosen because the boundary conditions had minimal effect on the rows of the $\mathbf{A}_i$ matrices when blocks of this sized are used.

2.4 Queue length of the $\text{Geo}_n/\text{Geo}_n/1$ system

We consider the $\text{Geo}_n/\text{Geo}_n/1$ system which is a special case of the $\text{Geo}_n/\text{Geo}_n/c_n$. For this special case we let $\alpha^{(m)}$ be the probability of an arrival at time m, with $\bar{\alpha}^{(m)} = 1 - \alpha^{(m)}$. We also define $\beta^{(m)}$ be the probability of a service completion at time m, with $\bar{\beta}^{(m)} = 1 - \beta^{(m)}$. We focus our attention on how to compute the stationary distribution of the discrete periodic queue for each time $m = 1, 2, \ldots, \tau$ within the period. That is, for each time m with $\mathcal{P}(m)$ as given below, we obtain the stationary distribution $\lim_{k\to\infty} \mathbf{x}(m+k\tau) = \mathbf{y}(m) = \mathbf{y}(m)\mathcal{P}(m)$.

Let the matrix $P(m)$ be written as

$$P(m) = \begin{bmatrix} B_{0,0}^{(m)} & C_0^{(m)} & & & \\ A_2^{(m)} & A_1^{(m)} & A_0^{(m)} & & \\ & A_2^{(m)} & A_1^{(m)} & A_0^{(m)} & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} c^{(m)} & f^{(m)} & & & \\ d^{(m)} & g^{(m)} & u^{(m)} & & \\ & d^{(m)} & g^{(m)} & u^{(m)} & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

where

$$c^{(m)} = \bar{\alpha}^{(m)}, \quad f^{(m)} = \alpha^{(m)}, \quad d^{(m)} = \bar{\alpha}^{(m)}\beta^{(m)},$$
$$u^{(m)} = \alpha^{(m)}\bar{\beta}^{(m)}, \quad \text{and} \quad g^{(m)} = \alpha^{(m)}\beta^{(m)} + \bar{\alpha}^{(m)}\bar{\beta}^{(m)}.$$

Here we note that matrix $B_{1,0}^{(m)}$ is the same as $A_2^{(m)}$ and $M_0 = M = 1$. Hence this has the same general structure as the $\mathrm{Geo}_n/\mathrm{Geo}_n/c_n$ with an additional special property, i.e. $B_{1,0}^{(m)} = A_2^{(m)}$.

Let us now consider the periodic case. We may write the matrix $\mathcal{P}(m)$ in a form which looks very similar to the matrix $P(m)$, but in the matrix $P(m)$, the blocks are 1×1 and each non-zero entry is time-dependent. For the matrix $\mathcal{P}(m)$, the blocks are $\tau \times \tau$ and as in the case of the $\mathrm{Geo}_n/\mathrm{Geo}_n/c_n$ queue, there is no time-dependence in the matrices of the interior, i.e. $\mathbf{A}_j,\ j = 0, 1, 2$. However, $\mathbf{B}_{0,0}^{(m)}$ and $\mathbf{C}_0^{(m)}$ are time-dependent. We will therefore drop the (m) superscript when it is not necessary. Hence we may write matrix $\mathcal{P}(m)$ as

$$\mathcal{P}(m) = \begin{bmatrix} \mathbf{B}_{0,0}^{(m)} & \mathbf{C}_0^{(m)} & & & & \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & & & \\ & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & & \\ & & \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where we can write the block matrices as

$$\mathbf{B}_{0,0}^{(m)} = \begin{bmatrix} b_{0,0}^{(m)} & b_{0,1}^{(m)} & \ldots & b_{0,\tau-1}^{(m)} \\ b_{1,0}^{(m)} & b_{1,1}^{(m)} & \ldots & b_{1,\tau-1}^{(m)} \\ b_{2,0}^{(m)} & b_{2,1}^{(m)} & \ldots & b_{2,\tau-1}^{(m)} \\ \vdots & \vdots & \vdots & \vdots \\ b_{\tau-1,0}^{(m)} & b_{\tau-1,1}^{(m)} & \ldots & b_{\tau-1,\tau-1}^{(m)} \end{bmatrix}, \qquad \mathbf{C}_0^{(m)} = \begin{bmatrix} b_{0,\tau}^{(m)} & & & \\ b_{1,\tau}^{(m)} & a_\tau & & \\ \vdots & \vdots & \ddots & \\ b_{\tau-1,\tau}^{(m)} & a_2 & \ldots & a_\tau \end{bmatrix},$$

$$\mathbf{A}_0 = \begin{bmatrix} a_\tau & & & \\ a_{\tau-1} & a_\tau & & \\ \vdots & \vdots & \ddots & \\ a_1 & a_2 & \ldots & a_\tau \end{bmatrix}, \qquad \mathbf{A}_1 = \begin{bmatrix} a_0 & a_1 & \ldots & a_{\tau-1} \\ a_{-1} & a_0 & \ldots & a_{\tau-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{-\tau+1} & a_{-\tau+2} & \ldots & a_0 \end{bmatrix},$$

and

$$\mathbf{A}_2 = \begin{bmatrix} a_{-\tau} & a_{-\tau+1} & \cdots & a_{-1} \\ & a_{-\tau} & \cdots & a_{-2} \\ & & \ddots & \vdots \\ & & & a_{-\tau} \end{bmatrix}.$$

2.4.1 Stability

Let $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2$. It is straightforward to see that $\mathbf{A}$ is an irreducible Markov chain, and since it is a finite Markov chain we know that it has a unique stationary distribution. Let this distribution be $\boldsymbol{\pi} = [\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots, \boldsymbol{\pi}_\tau]$, with

$$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{A}, \qquad \boldsymbol{\pi}\mathbf{1} = 1.$$

The matrix $\mathbf{A}$ is a circulant matrix given by

$$\mathbf{A} = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{\tau-1} & f_\tau \\ f_\tau & f_1 & f_2 & \cdots & f_{\tau-2} & f_{\tau-1} \\ f_{\tau-1} & f_k & f_1 & \cdots & f_{\tau-3} & f_{\tau-2} \\ f_{\tau-2} & f_{\tau-1} & f_\tau & \cdots & f_{\tau-4} & f_{\tau-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_2 & f_3 & f_4 & \cdots & f_\tau & f_1 \end{bmatrix},$$

where

$$f_1 = a_\tau + a_0 + a_{-\tau} \quad \text{and} \quad f_v = a_{v-1} + a_{-\tau+v-1}, \quad 2 \le v \le \tau.$$

Hence

$$\boldsymbol{\pi} = \tau^{-1}\mathbf{1}^T.$$

By applying the mean drift result in Neuts (1981), i.e. that the stability conditions required are that

$$\boldsymbol{\pi}\boldsymbol{\psi} > 1 \quad \Rightarrow \quad \boldsymbol{\pi}\mathbf{A}_2\mathbf{1} > \boldsymbol{\pi}\mathbf{A}_0\mathbf{1},$$

we end up with the stability conditions for this system to be

$$\sum_{v=1}^{\tau} \beta^{(v)}\bar{\alpha}^{(v)} > \sum_{v=1}^{\tau} \alpha^{(v)}\bar{\beta}^{(v)} \quad \Rightarrow \quad \sum_{v=1}^{\tau} \beta^{(v)} > \sum_{v=1}^{\tau} \alpha^{(v)}.$$

The proof is straightforward. All we need to observe is that the term $\tau^{-1}\mathbf{1}'\mathbf{A}_2\mathbf{1}$ is the total number of effective service completions during a cycle and $\tau^{-1}\mathbf{1}'\mathbf{A}_0\mathbf{1}$ is simply the total number of effective arrivals during a cycle. In the Appendix we give a detail example of this for the cases of $\tau = 1, 2$. A detailed general case will involve unnecessary laborious algebra, and therefore is not pursued.

Let the matrix R be the minimal non-negative solution to the quadratic matrix equation

$$R = \mathbf{A}_0 + R\mathbf{A}_1 + (R)^2\mathbf{A}_2.$$

Further partition R as

$$R = [\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_\tau],$$

where each $\mathbf{r}_v$, $\forall v$ is a column vector of order τ. Let $\mathbf{r} = \mathbf{r}_1$. Let

$$\mathbf{r} = [r_1, r_2, \ldots, r_\tau]^T.$$

From the results of Gail et al. (1997) we have

$$R = [\mathbf{r}, C_j(r)\mathbf{r}, C_j(r)^2\mathbf{r}, \ldots, C_j(r)^{\tau-1}\mathbf{r}],$$

where

$$C_j(r) = \begin{bmatrix} 0 & 0 & \ldots & 0 & r_1 \\ 1 & 0 & \ldots & 0 & r_2 \\ 0 & 1 & \ldots & 0 & r_3 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \ldots & 1 & r_\tau \end{bmatrix}.$$

The column vector $\mathbf{r}$ is obtained iteratively from

$$\mathbf{r} = [a_\tau, a_{\tau-1}, \ldots, a_1]^T + \sum_{v=0}^{\tau} C_j(r)^v \mathbf{r} a_{-v}.$$

Alternatively we may first compute the matrix G which is the minimal non-negative solution to the quadratic matrix equation

$$G = \mathbf{A}_2 + \mathbf{A}_1 G + \mathbf{A}_2(G)^2,$$

by using the efficient quadratically convergent cyclic reduction method of Bini and Meini (1995). The matrix R can then be obtained from

$$R = \mathbf{A}_0(I - \mathbf{A}_1 - \mathbf{A}_0 G)^{-1}.$$

3 Time-inhomogeneous M/G/1 type Markov chain

Consider the discrete-time Markov chain $\{X_n, J_n\}, n \geq 0$, with the state space $\{(\{0\} \times \{1, 2, \ldots, M_0\}) \cup (\{1, 2, \ldots\} \times \{1, 2, \ldots, M\})\}$ and the following transition matrix

$$P(n) = \begin{bmatrix} C_0^{(n)} & C_1^{(n)} & C_2^{(n)} & \ldots & \\ E_0^{(n)} & A_1^{(n)} & A_2^{(n)} & \ldots & \\ & A_0^{(n)} & A_1^{(n)} & \ldots & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The time-varying state probability vector for this system can be written as $\boldsymbol{x}(n) = [\boldsymbol{x}(n)_0, \boldsymbol{x}(n)_2, \ldots, \boldsymbol{x}(n)_i, \ldots]$, with

$$\boldsymbol{x}(n)_i = [\boldsymbol{x}(n)_{i,1}, \boldsymbol{x}(n)_{i,2}, \ldots, \boldsymbol{x}(n)_{i,M}].$$

The element $\boldsymbol{x}(n)_{i,j} = \Pr\{X_n = i, J_n = j\}$. It is easy to see that

$$\boldsymbol{x}(n+1) = \boldsymbol{x}(n)P(n), \quad n \geq 0.$$

Let us now define $P(m,n)_{l,k;i,j} = \Pr\{X_{n+1} = i, J_{n+1} = j | X_m = l, J_m = k\}$. It is immediately clear that

$$P(m,m) = P(m)$$

and

$$\begin{aligned} P(m,n) &= P(m) \times P(m+1) \times \cdots \times P(n-1) \times P(n) \\ &= P(m,n-1)P(n) \end{aligned}$$

for $n > m$. This matrix is of the form

$$P(m,n) = \begin{bmatrix} C_{0,0}^{(m,n)} & C_{0,1}^{(m,n)} & C_{0,2}^{(m,n)} & C_{0,3}^{(m,n)} & C_{0,4}^{(m,n)} & \cdots \\ C_{1,0}^{(m,n)} & C_{1,1}^{(m,n)} & C_{1,2}^{(m,n)} & C_{1,3}^{(m,n)} & C_{1,4}^{(m,n)} & \cdots \\ C_{2,0}^{(m,n)} & C_{2,1}^{(m,n)} & C_{2,2}^{(m,n)} & C_{2,3}^{(m,n)} & C_{2,4}^{(m,n)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{n-m,0}^{(m,n)} & C_{n-m,1}^{(m,n)} & C_{n-m,2}^{(m,n)} & C_{n-m,3}^{(m,n)} & C_{n-m,4}^{(m,n)} & \cdots \\ E_0^{(m,n)} & A_1^{(m,n)} & A_2^{(m,n)} & A_3^{(m,n)} & A_4^{(m,n)} & \cdots \\ & A_0^{(m,n)} & A_1^{(m,n)} & A_2^{(m,n)} & A_3^{(m,n)} & \cdots \\ & & A_0^{(m,n)} & A_1^{(m,n)} & A_2^{(m,n)} & \cdots \\ & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where the block matrices are obtained using the same concept as in equations (2.2–2.9) of Sect. 2. The resulting Markov chain has the non-skip-free properties as described by Gail et al. (1997) in that a transition from level $\mathbf{i}$ to level $\mathbf{i-k}$, $1 \leq k \leq n-m$, does not have to go through levels $\mathbf{i-1}, \mathbf{i-2}, \ldots, \mathbf{i-k+1}$ before reaching level $\mathbf{i-k}$, if $k < n-m$. However, by reblocking the block matrices into $M_0(n-m+1) \times M_0(n-m+1)$ for $C_{0,0}^{(m)}$, $M_0(n-m+1) \times M(n-m+1)$ for $C_{0,i}^{(m)}$, $i \geq 1$, $M(n-m+1) \times M_0(n-m+1)$ for $E^{(m)}$, and $M(n-m+1) \times M(n-m+1)$ for A_j, $j \geq 0$ blocks we reduce this Markov chain to the M/G/1 type with skip-free properties, and then analyze it using the same techniques as for the M/G/1 type. Further we have

$$\boldsymbol{x}(n+1) = \boldsymbol{x}(m)P(m,n), \quad m \leq n.$$

3.1 The case of periodic transition probabilities

Now we consider the case when there exists an integer $\tau < \infty$, such that $P(m,n) = P(m+j\tau, n+j\tau)$, $j = 1, 2, 3, \ldots$. In this case we can define

$$\mathcal{P}(m) = P(m, m+\tau-1) = P(m+j\tau, m+j\tau+\tau-1), \quad \forall j \geq 1.$$

Let us assume that there exists a stationary distribution $\mathbf{y}(m)$ given by

$$\mathbf{y}(m) = \boldsymbol{x}(m + k\tau) = \boldsymbol{x}(m + k\tau)\mathcal{P}(m), \quad k = 0, 1, 2, \ldots, \text{ with } \mathbf{y}(m)\mathbf{1} = 1.$$

Next we state the conditions under which such a stationary distribution exists and show how to obtain it.

Consider the matrix $\mathcal{A}^{(m)} = \sum_{v=0}^{\infty} A_v^{(m,m+\tau-1)}$. This matrix is stochastic and let us assume that it is irreducible. If so, then there exists a vector $\boldsymbol{\pi}^{(m)}$ such that $\boldsymbol{\pi}^{(m)} = \boldsymbol{\pi}^{(m)}\mathcal{A}^{(m)}$ with $\boldsymbol{\pi}^{(m)}\mathbf{1} = 1$. Further let $\boldsymbol{\psi}^{(m)} = \sum_{v=0}^{\infty} v\mathcal{A}_v^{(m)}\mathbf{1}$. Using the results in Neuts (1989), the stability condition for the Markov represented by the matrix $\mathcal{P}(m)$ is

$$\boldsymbol{\pi}^{(m)}\boldsymbol{\psi}^{(m)} < 1.$$

The matrix $\mathcal{P}(m)$ is of the M/G/1-type which is non-skip-free as discussed by Gail et al. (1997). Let us reblock the matrix $\mathcal{P}(m)$ to the form

$$\mathcal{P}(m) = \begin{bmatrix} \mathbf{C}_0^{(m)} & \mathbf{C}_1^{(m)} & \mathbf{C}_2^{(m)} & \mathbf{C}_3^{(m)} & \mathbf{C}_4^{(m)} & \cdots \\ \mathbf{E}^{(m)} & \mathbf{A}_1^{(m)} & \mathbf{A}_2^{(m)} & \mathbf{A}_3^{(m)} & \mathbf{A}_4^{(m)} & \cdots \\ & \mathbf{A}_0^{(m)} & \mathbf{A}_1^{(m)} & \mathbf{A}_2^{(m)} & \mathbf{A}_3^{(m)} & \cdots \\ & & \mathbf{A}_0^{(m)} & \mathbf{A}_1^{(m)} & \mathbf{A}_2^{(m)} & \cdots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is known from Neuts (1989) that there exists a matrix $G(m)$ which is the non-negative solution to the matrix polynomial equation

$$G(m) = \mathbf{A}(G(m)) = \sum_{v=0}^{\infty} \mathbf{A}_v^{(m)} G(m)^v.$$

This matrix $G(m)$ is stochastic if the stability conditions hold. Once $G(m)$ is known, then using methods due to Ramaswami (1988) we can easily compute the vector $\mathbf{y}(m)$. The key issue is how to compute $G(m)$ efficiently. We use an approach developed by Gail et al. (1997).

3.1.1 Computing $G(m)$

Let the matrix $G(m)$ be written as

$$G(m) = \begin{bmatrix} \mathbf{g}(m)_1 \\ \mathbf{g}(m)_2 \\ \vdots \\ \mathbf{g}(m)_\tau \end{bmatrix},$$

where $\mathbf{g}(m)_i$ is an $M \times M\tau$ matrix. For brevity, let us write $\mathbf{g}(m) = \mathbf{g}(m)_1$. Using the Corollary 3 of Gail et al. (1997) we can write

$$G(m) = \mathcal{C}(\mathbf{g})^\tau,$$

where

$$\mathcal{C}(\mathbf{g})^{\tau} = \begin{bmatrix} \mathbf{g} \\ \mathbf{g}\mathcal{C}(\mathbf{g}) \\ \mathbf{g}\mathcal{C}(\mathbf{g})^2 \\ \vdots \\ \mathbf{g}\mathcal{C}(\mathbf{g})^{\tau-1} \end{bmatrix},$$

and

$$\mathcal{C}(\mathbf{g}) = \begin{bmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & I \\ \mathbf{g}(m)_0^* & \mathbf{g}(m)_1^* & \mathbf{g}(m)_2^* & \mathbf{g}(m)_3^* & \dots & \mathbf{g}(m)_{\tau-1}^* \end{bmatrix},$$

where $\mathbf{g}(m) = [\mathbf{g}(m)_0^*, \mathbf{g}(m)_1^*, \mathbf{g}(m)_2^*, \mathbf{g}(m)_3^*, \dots, \mathbf{g}(m)_{\tau-1}^*]$.

Define an $M \times M\tau$ matrix $\mathbf{f}_j = [\mathbf{0}, \dots, \mathbf{0}, I, \mathbf{0}, \dots, \mathbf{0}]$ with I in the jth block column and $\mathbf{0}$ elsewhere. Also for an $M \times M\tau$ matrix $\mathbf{p}$ let $\delta(\mathbf{p})$ be given as

$$\delta(\mathbf{p}) = \mathbf{f}_0 a(\mathcal{C}(\mathbf{p})^{\tau}).$$

Let $\mathbf{p}^*$ be a unique solution to

$$\mathbf{p}^* = \delta(\mathbf{p}^*),$$

then $\mathbf{p}^* = \mathbf{g}$.

3.2 The $\mathrm{Geo}_n^X/\mathrm{Geo}_n/1$ system

We consider the $\mathrm{Geo}_n^X/\mathrm{Geo}_n/1$ system, a batch arrival system, which is a special case of the time-varying M/G/1 type Markov chain.

At time n, let $\alpha_i^{(n)}$, $i \geq 0$ be the probability that i arrivals occur in an interval and $\beta^{(n)}$ be the probability of a service completion, with $\bar{\beta}^{(n)} = 1 - \beta^{(n)}$. Then the transition matrix of the Markov chain of this system is

$$P(n) = \begin{bmatrix} \alpha_0^{(n)} & \alpha_1^{(n)} & \alpha_2^{(n)} & \alpha_3^{(n)} & \dots \\ \alpha_0^{(n)}\beta^{(n)} & \alpha_0^{(n)}\bar{\beta}^{(n)} + a_1^{(n)}\beta^{(n)} & \alpha_1^{(n)}\bar{\beta}^{(n)} + \alpha_2^{(n)}\beta^{(n)} & \alpha_2^{(n)}\bar{\beta}^{(n)} + \alpha_3^{(n)}\beta^{(n)} & \dots \\ & \alpha_0^{(n)}\beta^{(n)} & \alpha_0^{(n)}\bar{\beta}^{(n)} + \alpha_1^{(n)}\beta^{(n)} & \alpha_1^{(n)}\bar{\beta}^{(n)} + \alpha_2^{(n)}\beta^{(n)} & \dots \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is straightforward to develop the transition matrix of the period case from this and then apply the non-skip free M/G/1 results of Gail et al. (1997). We will not pursue the details in this paper.

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Appendix: Stability condition for $\tau = 2$

For $\tau = 1$: We have $\pi = 1$, and $\mathbf{A}_2 = A_2 = d^{(1)} = \bar{\alpha}^{(1)}\beta^{(1)}$, $\mathbf{A}_0 = A_0 = u^{(1)} = \alpha^{(1)}\bar{\beta}^{(1)}$. So it is clear that

$$\begin{aligned}\pi A_2\mathbf{1} > \pi A_0\mathbf{1} \quad &\Rightarrow \quad \bar{\alpha}^{(1)}\beta^{(1)} > \alpha^{(1)}\bar{\beta}^{(1)} \quad \Rightarrow \quad (1-\alpha^{(1)})\beta^{(1)} > (1-\beta^{(1)})\alpha^{(1)} \\ &\Rightarrow \quad \beta^{(1)} > \alpha^{(1)}.\end{aligned}$$

For $\tau = 2$: We have $\pi = [0.5\ 0.5]$ and

$$\mathbf{A}_2 = \begin{bmatrix} d^{(1)}d^{(2)} & d^{(1)}g^{(2)} + g^{(1)}d^{(2)} \\ 0 & d^{(1)}d^{(2)} \end{bmatrix}, \qquad \mathbf{A}_0 = \begin{bmatrix} u^{(1)}u^{(2)} & 0 \\ g^{(1)}u^{(2)} + u^{(1)}g^{(2)} & u^{(1)}u^{(2)} \end{bmatrix}.$$

We have

$$\begin{aligned}\pi\mathbf{A}_2\mathbf{1} &= 0.5[d^{(1)}(d^{(2)} + g^{(2)}) + d^{(2)}(g^{(1)} + d^{(1)})] \\ &= 0.5[d^{(1)}(1-u^{(2)}) + d^{(2)}(1-u^{(1)})] = 0.5[d^{(1)} - d^{(1)}u^{(2)} + d^{(2)} - d^{(2)}u^{(1)}].\end{aligned}$$

We also have

$$\begin{aligned}\pi\mathbf{A}_0\mathbf{1} &= 0.5[u^{(2)}(u^{(1)} + g^{(1)}) + u^{(1)}(g^{(2)} + u^{(2)})] \\ &= 0.5[u^{(2)}(1-d^{(1)}) + u^{(1)}(1-d^{(2)})] = 0.5[u^{(2)} - u^{(2)}d^{(1)} + u^{(1)} - u^{(1)}d^{(2)}].\end{aligned}$$

Hence we have

$$\begin{aligned}d^{(1)} + d^{(2)} > u^{(1)} + u^{(2)} \quad &\Rightarrow \quad \beta^{(1)}\bar{\alpha}^{(1)} + \beta^{(2)}\bar{\alpha}^{(2)} > \alpha^{(1)}\bar{\beta}^{(1)} + \alpha^{(2)}\bar{\beta}^{(2)} \\ &\Rightarrow \quad \beta^{(1)} + \beta^{(2)} > \alpha^{(1)} + \alpha^{(2)}.\end{aligned}$$

This argument can also be used for the cases of $\tau \geq 3$, but is not necessary since it is easy to see that what the condition is telling us is that the total average number of arrivals during a cycle τ should be less than the total average number that can be served during the cycle. Hence for any τ, all we need is that $\sum_{v=1}^{\tau}\beta^{(v)} > \sum_{v=1}^{\tau}\alpha^{(v)}$.

References

Alfa, A. S. (1982). Time-inhomogeneous bulk server queue in discrete time: A transportation type problem. *Operations Research*, *30*, 650–658.

Bini, D., & Meini, B. (1995). On cyclic reduction applied to a class of Toeplitz-like matrices arising in queueing problems. In W. J. Stewart (Ed.), *Computations with Markov chains* (pp. 21–38). Amsterdam: Kluwer Academic.

Breuer, L. (2001). The periodic BMAP/PH/*c* queue. *Queueing Systems*, *38*, 67–76.

Gail, H. R., Hantler, S. L., & Taylor, B. A. (1997). Non-skip-free M/G/1 and GI/M/1 types Markov chains. *Advances in Applied Probability*, *29*(3), 733–758.

Hall, R. W. (1991). *Queueing methods for services and manufacturing*. Englewood Cliffs: Prentice Hall.

Ingolfsson, A. (2005). *Modeling the* M(t)/M/$s(t)$ *queue with an exhaustive discipline* (Working paper). Available online at http://www.business.ualberta.ca/aingolfsson/documents/PDF/MMs_note.pdf.

Margolius, B. (2005). Transient solution to the time-dependent multi-server Poisson queue. *Journal of Applied Probability*, *42*, 766–777.

Massey, W. A., & Whitt, W. (1998). Uniform acceleration expansions for Markov chains with time-varying rates. *Annals of Applied Probability*, *8*(4), 1130–1155.

Neuts, M. F. (1981). *Matrix-geometric solutions in stochastic models: An algorithmic approach*. Baltimore: Hopkins University Press.

Neuts, M. F. (1989). *Structured stochastic matrices of the M/G/1 type and their applications*. New York: Marcel Dekker.

Ramaswami, V. (1988). Stable recursion for the steady state vector for Markov chains of M/G/1 type. *Communications in Statistics-Stochastic Models*, *4*, 183–188.

Yin, G., & Zhang, H. (2005). Two-time-scale Markov chains and applications to quasi-birth-death queues. *SIAM Journal on Applied Mathematics*, *65*(2), 567–568.