6-10-2016

Characterization of Simplices via the Bezout Inequality for Mixed Volumes

Christos Saroglou
Kent State University, csaroglo@math.kent.edu

Ivan Soprunov
Cleveland State University, i.soprunov@csuohio.edu

Arten Zvavitch
Kent State University, zvavitch@math.kent.edu

CHARACTERIZATION OF SIMPLICES VIA THE BEZOUT INEQUALITY FOR MIXED VOLUMES

CHRISTOS SAROGLOU, IVAN SOPRUNOV, AND ARTEM ZVAVITCH

Abstract. We consider the following Bezout inequality for mixed volumes:

$$V(K_1,\dots,K_r,\Delta[n-r])V_n(\Delta)^{r-1}\le\prod_{i=1}^{r}V(K_i,\Delta[n-1])\ \text{ for } 2\le r\le n.$$

It was shown previously that the inequality is true for any n-dimensional simplex Δ and any convex bodies $K_1,\dots,K_r$ in $\mathbb{R}^n$. It was conjectured that simplices are the only convex bodies for which the inequality holds for arbitrary bodies $K_1,\dots,K_r$ in $\mathbb{R}^n$. In this paper we prove that this is indeed the case if we assume that Δ is a convex polytope. Thus the Bezout inequality characterizes simplices in the class of convex n-polytopes. In addition, we show that if a body Δ satisfies the Bezout inequality for all bodies $K_1,\dots,K_r$ then the boundary of Δ cannot have *strict* points. In particular, it cannot have points with positive Gaussian curvature.

1. Introduction

It was noticed in [SZ] that the classical Bezout inequality in algebraic geometry [F, Sec. 8.4] together with the Bernstein–Kushnirenko–Khovanskii bound [B, Ku, Kh] produces a new inequality involving mixed volumes of convex bodies:

$$V(K_1,\dots,K_r,\Delta[n-r])V_n(\Delta)^{r-1}\le\prod_{i=1}^{r}V(K_i,\Delta[n-1])\ \text{ for } 2\le r\le n. \tag{1.1}$$

Here Δ is an n-dimensional simplex and $K_1,\dots,K_r$ are arbitrary convex bodies in $\mathbb{R}^n$. Throughout the paper $V_n(K)$ denotes the n-dimensional Euclidean volume of a body K and $V(K_1,\dots,K_n)$ denotes the n-dimensional mixed volume of bodies $K_1,\dots,K_n$. Furthermore, $K[m]$ indicates that the body K is repeated m times in the expression for the mixed volume.

In [SZ] it was conjectured that the Bezout inequality characterizes simplices, that is if Δ is a convex body such that (1.1) holds for all convex bodies $K_1,\dots,K_r$ then Δ is necessarily a simplex (see [SZ, Conjecture 1.2]). It was proved that Δ has to be indecomposable (see [SZ, Theorem 3.3]) which, in particular, confirms the conjecture in dimension $n=2$. In the present paper we prove this conjecture for the class of convex polytopes.

2010 *Mathematics Subject Classification.* Primary 52A39, 52A40, 52B11.

Key words and phrases. Convex Bodies, Mixed Volume, Convex Polytopes, Bezout Inequality, Aleksandrov–Fenchel Inequality.

The third author is supported in part by U.S. National Science Foundation Grant DMS-1101636 and by the Simons Foundation.

Theorem 1.1. *Fix $2 \leq r \leq n$. Let Δ be a convex n-dimensional polytope in $\mathbb{R}^n$ satisfying (1.1) for all convex bodies $K_1, \dots, K_r$ in $\mathbb{R}^n$. Then Δ is a simplex.*

Although the above theorem covers a most natural class of convex bodies, in full generality the conjecture remains open. Going outside of the class of polytopes we show that if a convex body Δ satisfies (1.1) for all convex bodies $K_1, \dots, K_r$ in $\mathbb{R}^n$ then Δ cannot have strict points. We say a boundary point $x \in K$ is a *strict point* if x does not belong to any segment contained in the boundary of K.

Theorem 1.2. *Fix $2 \leq r \leq n$. Let Δ be an n-dimensional convex body in $\mathbb{R}^n$ satisfying (1.1) for all convex bodies $K_1, \dots, K_r$ in $\mathbb{R}^n$. Then Δ does not contain any strict points.*

In particular, we see that Δ cannot have points with positive Gaussian curvature.

Let us say a few words about the idea behind the proofs of Theorems 1.1 and 1.2. First, note that it is enough to prove the theorems in the case of $r = 2$ as this implies the general statement. Thus we are going to restate (1.1) for $r = 2$ as follows

$$V(L, M, K[n-2])V_n(K) \leq V(L, K[n-1])V(M, K[n-1]), \tag{1.2}$$

where L and M are convex bodies and K is a polytope. The fact that there is equality in (1.2) when $L = K$ allows us to see this as a variational problem, by fixing an appropriate body M and using an appropriate deformation $L = K_t$ of K. In the case of Theorem 1.1, K_t is obtained from K by moving one of its facets along the direction of its normal unit vector. In the case of Theorem 1.2, K_t is obtained from K by cutting out a small cup in a neighborhood of a strict point.

2. Preliminaries

In this section we collect basic definitions and set up notation. As a general reference on the theory of convex sets and mixed volumes we use R. Schneider's book "Convex bodies: the Brunn-Minkowski theory" [Sch].

A *convex body* is a non-empty convex compact set. A *(convex) polytope* is the convex hull of a finite set of points. An n-dimensional polytope is called an *n-polytope* for short. For $x, y \in \mathbb{R}^n$ we write $\langle x, y\rangle$ for the inner product of x and y. We use $\mathbb{S}^{n-1}$ to denote the $(n-1)$-dimensional unit sphere and $B(x, \delta)$ to denote the closed Euclidean ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$.

For a convex body K the function $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, $h_K(u) = \max\{\langle x, u\rangle \mid x \in K\}$ is the *support function* of K. For every $u \in \mathbb{S}^{n-1}$ we write $H_K(u)$ to denote the supporting hyperplane for K with outer normal u

$$H_K(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle = h_K(u)\}.$$

Furthermore, we use K^u to denote the face $K \cap H_K(u)$ of K.

Let β be a subset of the boundary ∂K of a convex body K. The *spherical image* $\sigma(K, \beta)$ of β with respect to K is defined by

$$\sigma(K, \beta) = \{u \in \mathbb{S}^{n-1} : \exists x \in \beta, \text{ such that } \langle x, u\rangle = h_K(u)\}.$$

If Ω is a subset of $\mathbb{S}^{n-1}$ define the *inverse spherical image* $\tau(K, \Omega)$ of Ω with respect to K by

$$\tau(K, \Omega) = \big\{x \in \partial K : \exists u \in \Omega, \text{ such that } \langle x, u\rangle = h_K(u)\big\}.$$

The *surface area measure* $S(K,\cdot)$ of K (viewed as a measure on $\mathbb{S}^{n-1}$) is defined as

$$S(K,\Omega) = \mathcal{H}^{n-1}\big(\tau(K,\Omega)\big), \quad \text{for } \Omega \text{ a Borel subset of } \mathbb{S}^{n-1}.$$

Here $\mathcal{H}^{n-1}(\cdot)$ stands for the $(n-1)$-dimensional Hausdorff measure.

Let $V(K_1,\dots,K_n)$ denote the n-dimensional mixed volume of n convex bodies $K_1,\dots,K_n$ in $\mathbb{R}^n$. We write $V(K_1[m_1],\dots,K_r[m_r])$ for the mixed volume of the bodies $K_1,\dots,K_r$ where each K_i is repeated m_i times and $m_1+\cdots+m_r=n$. In particular, $V(K[n]) = V_n(K)$, the n-dimensional Euclidean volume of K.

Let $S(K_1,\dots,K_{n-1},\cdot)$ be the *mixed area measure* for bodies $K_1,\dots,K_{n-1}$ defined by

$$V(L,K_1,\dots,K_{n-1}) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L dS(K_1,\dots,K_{n-1},\cdot)$$

for any compact convex set L. In particular, when the K_i are polytopes the mixed area measure $S(K_1,\dots,K_{n-1},\cdot)$ has finite support and for every $u\in\mathbb{S}^{n-1}$ we have

$$S(K_1,\dots,K_{n-1},u) = V(K_1^u,\dots,K_{n-1}^u), \tag{2.1}$$

where $V(K_1^u,\dots,K_{n-1}^u)$ is the $(n-1)$-dimensional mixed volume of the faces K_i^u translated the the subspace orthogonal to u, see [Sch, Sec 5.1].

Finally, for $u\in\mathbb{S}^{n-1}$ the orthogonal projection of a set $A\subset\mathbb{R}^n$ onto the subspace $u^\perp$ orthogonal to u is denoted by $A|u^\perp$.

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. As mentioned in the introduction, it is enough to prove it for $r=2$ in which case we write the Bezout inequality as

$$V(L,M,K[n-2])V_n(K) \le V(L,K[n-1])V(M,K[n-1]). \tag{3.1}$$

We assume that L, M are arbitrary convex bodies and K is a polytope in $\mathbb{R}^n$.

We need to set up additional notation. Let K be defined by inequalities

$$K = \bigcap_{j=1}^{N} \{x\in\mathbb{R}^n : \langle x,u_j\rangle \le h_K(u_j)\},$$

where u_j are the outer normals to the facets of K (in some fixed order) and N is the number of facets of K. Denote by $K_{t,i}$ the polytope obtained by moving the i-th facet of K by t, that is

$$K_{t,i} = \bigcap_{\substack{j=1\\ j\neq i}}^{N} \{x\in\mathbb{R}^n : \langle x,u_j\rangle \le h_K(u_j)\} \bigcap \{x\in\mathbb{R}^n : \langle x,u_i\rangle \le h_K(u_i)+t\}.$$

By abuse of notation we let K_t denote $K_{t,N}$.

Lemma 3.1. *Let K and K_t be as above. Then there exists $\delta=\delta(K)$ such that the following supports are equal*

$$\operatorname{supp} S(K_t[r],K[n-1-r],\cdot) = \operatorname{supp} S(K,\cdot)$$

for any $0\le r\le n-1$ and any $t\in(-\delta,\delta)$.

Proof. By (2.1) it is enough to show that $V(K_t^u[r], K^u[n-1-r]) = 0$ if and only if $V_{n-1}(K^u) = 0$, that is K^u is not a facet of K. Indeed, by choosing δ small enough we can ensure that K_t has the same facet normals as K and so $\dim K_t^u = n-1$ whenever K^u is a facet of K. In this case $V(K_t^u[r], K^u[n-1-r]) > 0$.

Conversely, assume K^u is a face of K of dimension less than $n-1$. As before, for small enough t the face K_t^u also has dimension less than $n-1$. First, suppose K^u is not contained in the moving facet $F = K \cap H_K(u_N)$. Then $h_K(u) = h_{K_t}(u)$ and so $K^u \subseteq K_t^u$ for $t \geq 0$ and $K^u \supseteq K_t^u$ for $t < 0$. Then, by the monotonicity of the mixed volume, if $t \geq 0$ then

$$0 \leq V(K_t^u[r], K^u[n-1-r]) \leq V_{n-1}(K_t^u) = 0,$$

and so $V(K_t^u[r], K^u[n-1-r]) = 0$. The case $t < 0$ is similar.

Now suppose K^u is contained in the moving facet F. Then $K^u \subseteq H_K(u) \cap H_K(u_N)$ and $K_t^u \subseteq H_{K_t}(u) \cap H_{K_t}(u_N)$. This shows that K^u and K_t^u are contained in two affine $(n-2)$-dimensional subspaces which are translates of the same linear subspace of dimension $n-2$. Therefore, for any collection of line segments $(L_1, \ldots, L_{n-1})$, where $L_i \subset K_t^u$ for $1 \leq i \leq r$ and $L_i \subset K^u$ for $r+1 \leq i \leq n-1$, the L_i have linearly dependent directions. The latter implies that $V(K_t^u[r], K^u[n-1-r]) = 0$ by [Sch, Theorem 5.1.7].

□

Proposition 3.2. *Let K, P be n-polytopes with the following properties:*

(1) $\operatorname{supp} S(P, \cdot) = \operatorname{supp} S(K, \cdot)$,
(2) *there exists a constant $\lambda > 0$ such that $V(L, P[n-1]) \leq \lambda V(L, K[n-1])$ for all convex bodies L,*
(3) $V(K, P[n-1]) = \lambda V_n(K)$.

Then,

$$S(P, \cdot) = \lambda S(K, \cdot) \ .$$

Proof. As before, let $\{u_1, \ldots, u_N\}$ be the outer normals to the facets of K. By assumption (1) they are the outer normals to the facets of P as well. Fix $1 \leq i \leq N$ and let $L = K_{s,i}$ be the polytope obtained from K by moving its i-th facet by a small number $s \in (-\delta_i, \delta_i)$ as in Lemma 3.1.

By assumption (2), for any $s \in (-\delta_i, \delta_i)$ we have

$$V(K_{s,i}, P[n-1]) \leq \lambda V(K_{s,i}, K[n-1]).$$

Consider the function

$$F(s) = \lambda V(K_{s,i}, K[n-1]) - V(K_{s,i}, P[n-1]).$$

Then $F(s) \geq 0$ and $F(0) = 0$. Below we show that $F(s)$ is, in fact, linear on $(-\delta_i, \delta_i)$. But then $F(s)$ is identically zero on $(-\delta_i, \delta_i)$, which implies that

$$V(K_{s,i}, P[n-1]) = \lambda V(K_{s,i}, K[n-1]) \tag{3.2}$$

for all $s \in (-\delta_i, \delta_i)$. We claim that this also implies that

$$S(P, u_i) = \lambda S(K, u_i), \tag{3.3}$$

and since i is chosen arbitrarily and the supports of the two measures are equal, the statement of the proposition follows.

Now we show that $F(s)$ is linear and then prove that (3.2) implies (3.3). Since the polytopes P and K have the same set of facet normals $\{u_1, \dots, u_N\}$, we obtain:

$$\begin{aligned} nV(K_{s,i}, P[n-1]) &= \sum_{j=1}^{N} h_{K_{s,i}}(u_j) V_{n-1}(P^{u_j}) \\ &= \sum_{j=1}^{N} h_K(u_j) V_{n-1}(P^{u_j}) + (h_K(u_i) + s) V_{n-1}(P^{u_i}) \\ &= nV(K, P[n-1]) + sV_{n-1}(P^{u_i}) \\ &= n\lambda V_n(K) + sV_{n-1}(P^{u_i}). \end{aligned} \tag{3.4}$$

Similarly,

$$nV(K_{s,i}, K[n-1]) = nV_n(K) + sV_{n-1}(K^{u_i}). \tag{3.5}$$

Substituting (3.4) and (3.5) into the definition of $F(s)$ and using assumption (3), we see that $F(s) = \lambda s$ for some λ, that is $F(s)$ is linear.

It remains to show that (3.2) implies (3.3). Since $F(s)$ is identically zero we have $\lambda = 0$, which translates to

$$V_{n-1}(P^{u_i}) = \lambda V_{n-1}(K^{u_i}).$$

But that is precisely what (3.3) is stating, which completes the proof of the proposition.

□

Lemma 3.3. *Let K be an n-polytope satisfying (3.1) for all bodies L and for all $M = K_t$ where $t \in (-\delta, \delta)$ as in Lemma 3.1. Then*

$$S(K_t[r], K[n-1-r], \cdot) = \frac{V(K_t, K[n-1])^r}{V_n(K)^r} S(K, \cdot)$$

for all $0 \le r \le n-1$ and all $t \in (-\delta, \delta)$.

Proof. For $0 \le r \le n-1$, set P_r to be the polytope whose surface area measure equals $S(K_t[r], K[n-r-1], \cdot)$ and let $\lambda := V(K_t, K[n-1])/V_n(K)$. For each r the existence and uniqueness of P_r is ensured by the Minkowski Existence and Uniqueness Theorem (see [Sch, Sections 7.1, 7.2]). We need to prove that

$$S(P_r, \cdot) = \lambda^r S(K, \cdot), \qquad r = 0, 1, \dots, n-1. \tag{3.6}$$

Note that by Lemma 3.1, we have:

$$\operatorname{supp} S(P_r, \cdot) = \operatorname{supp} S(K, \cdot), \qquad r = 1, \dots, n-1. \tag{3.7}$$

We prove (3.6) by induction on r. The case $r = 0$ is trivial. For the case $r = 1$ we apply Proposition 3.2 with $P = P_1$. Indeed, by our assumption, (3.1) is satisfied for $M = K_t$ and becomes equality when $L = K$. Thus the conditions (1)–(3) of Proposition 3.2 hold and so $S(P_1, \cdot) = \lambda S(K, \cdot)$, as required.

Now assume (3.6) holds for $1 \le m \le r-1$. This is equivalent to the following:

$$V(L, P_m[n-1]) = \lambda^m V(L, K[n-1]), \tag{3.8}$$

for all convex bodies L and $1 \leq m \leq r-1$. Next fix a convex body $L \subset \mathbb{R}^n$ and apply the Aleksandrov-Fenchel inequality

$$\begin{aligned}
& V(L, P_{r-1}[n-1])^2 = V(L, K_t[r-1], K[n-r])^2 \\
= \; & V(K, K_t, K_t[r-2], K[n-r-1], L)^2 \\
\geq \; & V(K, K, K_t[r-2], K[n-r-1], L) V(K_t, K_t, K_t[r-2], K[n-r-1], L) \\
= \; & V(L, K_t[r-2], K[n-r+1]) V(L, K_t[r], K[n-r-1]) \\
= \; & V(L, P_{r-2}[n-1]) V(L, P_r[n-1]) \;,
\end{aligned}$$

which, by (3.8) with $m = r-2$ and $m = r-1$, gives

$$\lambda^{2(r-1)} V(L, K[n-1])^2 \geq \lambda^{r-2} V(L, K[n-1]) V(L, P_r[n-1]).$$

Thus

$$V(L, P_r[n-1]) \leq \lambda^r V(K, P_r[n-1]). \tag{3.9}$$

Furthermore, using (3.8) for $m = r-1$, we get:

$$\begin{aligned}
V(K, P_r[n-1]) &= V(K, K_t[r], K[n-1-r]) \\
&= V(K_t, K_t[r-1], K[n-r]) \\
&= V(K_t, P_{r-1}[n-1]) \\
&= \lambda^{r-1} V(K_t, K[n-1]) \\
&= \frac{V(K_t, K[n-1])^{r-1}}{V_n(K)^{r-1}} V(K_t, K[n-1]) = \lambda^r V_n(K). \qquad (3.10)
\end{aligned}$$

Now, as in the case of $r = 1$, (3.7), (3.9), (3.10) together with Proposition 3.2, show that $S(P_r, \cdot) = \lambda^r S(K, \cdot)$, which completes the proof of the lemma. □

Now we are ready to prove the main theorem which implies Theorem 1.1.

Theorem 3.4. *Let K be an n-polytope in $\mathbb{R}^n$. Suppose that*

$$V(L, M, K[n-2]) V_n(K) \leq V(L, K[n-1]) V(M, K[n-1]) \tag{3.11}$$

holds for all convex bodies L and M in $\mathbb{R}^n$. Then K is a simplex.

Proof. Let K_t be the polytope obtained by moving one of the facets of K for t small enough. Then Lemma 3.3 with $r = n-1$ implies that the surface area measures of K_t and K are proportional, and hence, K_t is homothetic to K.

We may assume that one of the vertices of K not lying on the moving facet is at the origin, so $K_t = \lambda K$ for some $\lambda \neq 1$. For every vertex v in K, λv must be a vertex of λK. Therefore, the origin is the only vertex of K not lying on the moving facet. In other words, K is the cone over the moving facet. But since the facet was chosen arbitrarily, for every vertex v the polytope K is the convex hull of v and the facet not containing v. This implies that K is a simplex. □

4. Proof of Theorem 1.2

Recall that a boundary point $y \in \partial K$ is *strict* if it does not belong to any segment contained in ∂K. Note that points with positive Gaussian curvature and, more generally, regular exposed points are strict points (see [Sch] for the definitions). Clearly the boundary of a polytope does not contain any strict points, but there are other convex bodies having this property (for example, a cylinder).

As before it is enough to prove Theorem 1.2 in the case of $r = 2$. It follows from the theorem below.

Theorem 4.1. *Let K be a convex body whose boundary contains at least one strict point. Then there exist convex bodies L and M such that*

$$V(L, M, K[n-2])V_n(K) > V(L, K[n-1])V(M, K[n-1]). \tag{4.1}$$

Proof. First let us fix some notation. For $a > 0$ and $u \in \mathbb{S}^{n-1}$, define the closed half-spaces:

$$H_a^+(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle \geq a\} \quad \text{and} \quad H_a^-(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq a\}.$$

Also set $H_a(u) := H_a^+(u) \cap H_a^-(u)$. With this notation, the supporting hyperplane of K whose unit normal vector is u, can be written as $H_{h_K(u)}(u)$.

Let y be a strict point of ∂K and u be a normal vector of K at y. Choose $v \in \mathbb{S}^{n-1}$, such that $y|v^\perp \in \operatorname{relint}(K|v^\perp)$, where $\operatorname{relint}(K|v^\perp)$ denotes the relative interior of the body $K|v^\perp$ in $v^\perp$. We claim that there exists $\varepsilon > 0$, such that

$$\Big(K \cap H^-_{h_K(u)-\varepsilon}(u)\Big)|v^\perp = K|v^\perp. \tag{4.2}$$

To see this, assume that (4.2) is not true for all $\varepsilon > 0$. This means that for any $\varepsilon > 0$, there exists a point $x_\varepsilon \in \partial K$, such that $x_\varepsilon|v^\perp \in \partial(K|v^\perp)$ and $x_\varepsilon \in H^+_{h_K(u)-\varepsilon}(u)$. Let x_0 be an accumulation point of the set $\{x_\varepsilon : \varepsilon > 0\}$. Then, by compactness, $x_0 \in \partial K$, $x_0|v^\perp \in \partial(K|v^\perp)$, and $x_0 \in H_{h_K(u)}(u)$ (because $x_0 \in H^+_{h_K(u)}(u)$ and $x_0 \in K$). Note that, since $x_0|v^\perp \in \partial(K|v^\perp)$ and $y|v^\perp \in \operatorname{relint}(K|v^\perp)$, we have $x_0 \neq y$. It follows that the segment $[x_0, y]$ is contained in a supporting hyperplane of K, thus $[x_0, y] \subseteq \partial K$, which contradicts the assumption that y is strict. Hence, (4.2) holds for some $\varepsilon > 0$.

Next, set $K_\varepsilon := K \cap H^-_{h_K(u)-\varepsilon}(u)$. Clearly, $h_{K_\varepsilon} \leq h_K$. We claim that there exists an open subset $\beta \subset \partial K \setminus \partial K_\varepsilon$, such that $y \in \beta$ and

$$h_{K_\varepsilon}(u) < h_K(u), \qquad \text{for all } u \in \sigma(K, \beta). \tag{4.3}$$

Suppose not. Then for any δ-neighborhood $\beta_\delta = (\partial K \setminus \partial K_\varepsilon) \cap B(y, \delta)$ of y there exists a unit vector $u_\delta \in \sigma(K, \beta_\delta)$ such that $h_K(u_\delta) = h_{K_\varepsilon}(u_\delta)$. In other words, there exist points $y_\delta \in \beta_\delta$ and $x_\delta \in \partial K_\varepsilon$ lying in the same hyperplane $H_K(u_\delta)$. But then, by compactness, there exist a point $x \in \partial K_\varepsilon$ and a unit vector u, which is normal for K at y and at x. This shows again that the points y and x of K lie in the same supporting hyperplane $H_K(u)$, thus $[y, x]$ is a boundary segment of K, which contradicts our assumption. Therefore, (4.3) holds for some open set $\beta \subseteq \partial K \setminus \partial K_\varepsilon$.

Note, furthermore, that $\tau(K, \sigma(K, \beta)) \supseteq \beta$, thus $\mathcal{H}^{n-1}\big(\tau(K, \sigma(K, \beta))\big) > 0$, which shows that

$$S(K, \sigma(K, \beta)) > 0. \tag{4.4}$$

Now we are ready to exhibit examples of compact convex sets L and M satisfying (4.1). Set $L = [-v, v]$ and $M = K_\varepsilon$. Then, by (5.3.23) in [Sch, p. 294] and applying (4.2) we obtain

$$V(L, M, K[n-2]) = V(K_\varepsilon|v^\perp, K|v^\perp[n-2]) = V_{n-1}(K|v^\perp) = V(L, K[n-1]).$$

On the other hand, by (4.3) and (4.4), we have:

$$\begin{aligned} V(M, K[n-1]) = V(K_\varepsilon, K[n-1]) &= \frac{1}{n}\int_{S^{n-1}} h_{K_\varepsilon} dS(K, \cdot) \\ &< \frac{1}{n}\int_{S^{n-1}} h_K dS(K, \cdot) = V_n(K). \end{aligned}$$

This shows that

$$V(L, M, K[n-2])V_n(K) > V(L, K[n-1])V(M, K[n-1]),$$

as asserted. □

Remark 4.2. One might ask the following: If K is a convex body whose boundary contains at least one strict point x, is it true that ∂K has an open neighborhood that does not contain any line segments, i.e. K is strictly convex in a neighborhood of x? If yes, this would simplify the proof of Theorem 4.1 considerably. The following simple 3-dimensional example shows, however, that this is not the case. Take K equal to

$$\{x \in \mathbb{R}^3 : x_3 \le 1\} \bigcap \mathrm{conv}\Big(\{(0, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_2^2\} \cup \{(x_1, 0, x_3) \in \mathbb{R}^3 : x_3 = x_1^2\}\Big).$$

Then the origin is a strict point of the boundary of K, but no neighborhood of the origin is strictly convex.

References

[B] D. N. Bernstein, *The number of roots of a system of equations*. Funct. Anal. and Appl. 9 (2), (1975) 183–185.

[F] W. Fulton, *Intersection Theory*, 2nd edition, Springer, 1998.

[Kh] A. G. Khovanskii, *Newton polyhedra and the genus of complete intersections*, Funct. Anal. Appl., 12 (1978), pp. 38-46.

[Ku] A. G. Kushnirenko, *Newton polyhedra and Bezout's theorem* (Russian) Funkcional. Anal. i Prilozhen. 10, no. 3, (1976) 82–83.

[Sch] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.

[SZ] I. Soprunov, A. Zvavitch, *Bezout Inequality for Mixed volumes*, arXiv:1507.00765 [math.MG], International Mathematics Research Notices, to appear.

(Christos Saroglou) Department of Mathematical Sciences, Kent State University, Kent, OH USA

E-mail address: `csaroglo@math.kent.edu`

(Ivan Soprunov) Department of Mathematics, Cleveland State University, Cleveland, OH USA

E-mail address: `i.soprunov@csuohio.edu`

(Artem Zvavitch) Department of Mathematical Sciences, Kent State University, Kent, OH USA

E-mail address: `zvavitch@math.kent.edu`