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Global Residues for Sparse Polynomial Systems

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1. Introduction

Let $f_1 = \cdots = f_n = 0$ be a system of Laurent polynomial equations in n variables whose Newton polytopes are $\Delta_1, \ldots, \Delta_n$. Suppose the solution set Z_f in the algebraic torus $\mathbb{T}^n = (\mathbb{C} - \{0\})^n$ is finite. This will be true if the coefficients of the f_i are generic.

The *global residue* assigns to every polynomial g the sum over Z_f of Grothendieck residues of the meromorphic form

$$\omega_g = \frac{g}{f_1 \ldots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \cdots \wedge \frac{\mathrm{d}t_n}{t_n}.$$

It is a symmetric function of the solutions and hence depends rationally on the coefficients of the system. Global residues were studied by Tsikh, Gelfond and Khovanskii, and later by Cattani, Dickenstein and Sturmfels [16,9,3,4]. There are numerous applications of global residues ranging from elimination algorithms in commutative algebra [4] to integral representation formulae in complex analysis [16].

There have been several approaches to the problem of computing the global residue explicitly. Gelfond and Khovanskii considered systems with *generically positioned* Newton polytopes. The latter means that for any linear function ξ, among the corresponding extremal faces $\Delta_1^\xi, \ldots, \Delta_n^\xi$ at least one is a vertex. In this case Gelfond and Khovanskii found an explicit formula for the global residue as a Laurent polynomial in the coefficients of the system [9]. In general, the global residue is not a Laurent polynomial and one should not expect a closed formula for it.

Another approach was taken by Cattani and Dickenstein in the *generalized unmixed* case when all Δ_i are dilates of a single n-dimensional polytope (see [3]). Their algorithm requires computing a certain element in the homogeneous coordinate ring of a toric variety whose *toric residue* equals 1. Then the *Codimension Theorem* for toric varieties [6] allows global residue computation using Gröbner bases techniques. This approach was then extended by D'Andrea and Khetan to a slightly more general *ample* case [7]. However, it turned out to be a hard (and still open) problem to find an element of toric residue 1 for an arbitrary collection of polytopes.[1]

In this paper we present a new algorithm for computing the global residue when $\Delta_1, \ldots, \Delta_n$ are arbitrary n-dimensional polytopes (see Section 5). The proof of its correctness is in the spirit of Cattani and Dickenstein's arguments, but avoids the toric residue problem. It relies substantially on the Toric Euler–Jacobi theorem due to Khovanskii [13]. Also in our algorithm we replace Gröbner bases computations with solving a linear system. This gives an expression of the global residue as a quotient of two determinants. The same idea was previously used by D'Andrea and Khetan in [7].

There is an intimate relation between the global residue and polynomial interpolation. In the classical case this was understood already by Kronecker. In Section 3 we show how the Toric Euler–Jacobi theorem gives sparse polynomial interpolation. Another consequence of this theorem is a lower bound on the number of interior lattice points in the Minkowski sum of n full-dimensional lattice polytopes in $\mathbb{R}^n$ (Corollary 3.2). Finally, our results give rise to interesting optimization problems about Minkowski sums which we discuss in Section 6.

2. Global residues in $\mathbb{C}^n$ and Kronecker interpolation

We begin with classical results on polynomial interpolation that show how global residues come into play.

Let $f_1, \ldots, f_n \in \mathbb{C}[t_1, \ldots, t_n]$ be polynomials of degrees $d_1, \ldots, d_n$, respectively. Let $\rho = \sum d_i - n$ denote the *critical degree* for the f_i. We assume that the solution set $Z_f \subset \mathbb{C}^n$ of $f_1 = \cdots = f_n = 0$ consists of a finite number of simple roots. The following is a classical problem on polynomial interpolation.

Problem 1. Given $\phi : Z_f \to \mathbb{C}$ find a polynomial $g \in \mathbb{C}[t_1, \ldots, t_n]$ of degree at most ρ such that $g(a) = \phi(a)$ for all $a \in Z_f$.

Kronecker [14] suggested the following solution to this problem. Since each f_i vanishes at every $a \in Z_f$ we can write (non-uniquely):

$$f_i = \sum_{j=1}^{n} g_{ij}(t_j - a_j), \quad a = (a_1, \ldots, a_n), \quad g_{ij} \in \mathbb{C}[t_1, \ldots, t_n]. \tag{2.1}$$

The determinant g_a of the polynomial matrix $[g_{ij}]$ is a polynomial of degree at most ρ. Notice that the value of g_a at $t = a$ equals the value of the Jacobian $J_f = \det\left(\frac{\partial f_i}{\partial t_j}\right)$ at $t = a$, since $\frac{\partial f_i}{\partial t_j}(a) = g_{ij}(a)$ from (2.1). Also $g_a(a') = 0$ for any $a' \in Z_f$, $a' \neq a$. Indeed, substituting $t = a'$ into (2.1) we get

$$0 = \sum_{j=1}^{n} g_{ij}(a')(a'_j - a_j),$$

which means that the non-zero vector $a' - a$ is in the kernel of the matrix $[g_{ij}(a')]$, and hence $\det[g_{ij}(a')] = 0$. It remains to put

$$g = \sum_{a \in Z_f} \frac{\phi(a)}{J_f(a)} g_a. \tag{2.2}$$

[1] For a combinatorial construction of such an element that provides a solution to the problem for a wide class of polytopes see [10].

Remark 2.1. When choosing g_{ij} in (2.1) one can assume that

$$g_{ij} = \frac{\partial F_i}{\partial t_j} + \text{lower degree terms},$$

where F_i is the main homogeneous part of f_i. Then (2.2) becomes

$$g = \left(\sum_{a \in Z_f} \frac{\phi(a)}{J_f(a)} \right) J_F + \text{lower degree terms}, \tag{2.3}$$

where J_F is the Jacobian of the F_i.

Definition 2.2. Given $g \in \mathbb{C}[t_1, \dots, t_n]$ the sum of the local Grothendieck residues

$$\mathcal{R}_f(g) = \sum_{a \in Z_f} \operatorname{res}_a \left(\frac{g}{f_1 \dots f_n} \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_n \right)$$

is called the *global residue* of g for the system $f_1 = \dots = f_n = 0$. In the case of simple roots of the system we get

$$\mathcal{R}_f(g) = \sum_{a \in Z_f} \frac{g(a)}{J_f(a)}.$$

Theorem 2.3 (*The Euler–Jacobi Theorem*). *Let $f_1 = \dots = f_n = 0$ be a generic polynomial system with* $\deg(f_i) = d_i$. *Then for any h of degree less than $\rho = \sum d_i - n$ the global residue $\mathcal{R}_f(h)$ is zero.*

Proof. Consider the function $h : Z_f \to \mathbb{C}$, $a \mapsto h(a)$. According to the previous discussion there is a polynomial of the form (2.3) which takes the same values on Z_f as h. In other words,

$$h \equiv \left(\sum_{a \in Z_f} \frac{h(a)}{J_f(a)} \right) J_F + \text{lower degree terms} \pmod{I},$$

where we used that the ideal $I = \langle f_1, \dots, f_n \rangle$ is radical and the roots are simple. Comparing the homogeneous parts of degree ρ in this equation we see that either $J_F \in I$, which is equivalent to the F_i having a non-trivial common zero (this does not happen generically), or the coefficient of J_F (the global residue of h) is zero. □

3. Global residues in $\mathbb{T}^n$ and sparse polynomial interpolation

Now we will consider *sparse* polynomial systems and define the global residue in this situation. The word "sparse" indicates that instead of fixing the degrees of the polynomials we fix their *Newton polytopes*. The Newton polytope $\Delta(f)$ of a (Laurent) polynomial f is defined as the convex hull in $\mathbb{Z}^n$ of the exponent vectors of all the monomials appearing in f. Note that the Newton polytope of a generic polynomial of degree d is a d-dilate of the standard n-simplex. Such polynomials are usually called *dense*. In what follows when we say generic sparse polynomial we will mean that its Newton polytope is fixed and the coefficients are generic.

Let $f_1, \dots, f_n \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be Laurent polynomials whose Newton polytopes $\Delta_1, \dots, \Delta_n$ are n-dimensional. We will assume that the solution set $Z_f \subset \mathbb{T}^n$ of the system $f_1 = \dots = f_n = 0$ is finite. Here $\mathbb{T}^n$ denotes the n-dimensional algebraic torus $(\mathbb{C} - \{0\})^n$.

Like in the affine case define the *global residue* of a Laurent polynomial g as the sum of the local Grothendieck residues

$$\mathcal{R}_f^{\mathbb{T}}(g) = \sum_{a \in Z_f} \operatorname{res}_a \left(\frac{g}{f_1 \dots f_n} \frac{\mathrm{d}t_1}{t_1} \wedge \dots \wedge \frac{\mathrm{d}t_n}{t_n} \right).$$

When the roots of the system are simple we obtain

$$\mathcal{R}_f^{\mathbb{T}}(g) = \sum_{a \in Z_f} \frac{g(a)}{J_f^{\mathbb{T}}(a)},$$

where $J_f^{\mathbb{T}} = \det\left(t_j \frac{\partial f_i}{\partial t_j}\right)$ is the *toric Jacobian* of the polynomials $f_1, \ldots, f_n$.

Notice that $\mathcal{R}_f^{\mathbb{T}}$ is a linear function on the space of Laurent polynomials and depends rationally on the coefficients of the f_i, since it is symmetric in the roots Z_f. In Section 5 we will give an algorithm for computing $\mathcal{R}_f^{\mathbb{T}}(g)$ as a rational function of the coefficients of the system.

The following theorem due to A. Khovanskii is a far reaching generalization of the Euler–Jacobi theorem.

Theorem 3.1 *([13]). Let $f_1 = \cdots = f_n = 0$ be a generic system of Laurent polynomials with n-dimensional Newton polytopes $\Delta_1, \ldots, \Delta_n$. Let $\Delta = \Delta_1 + \cdots + \Delta_n$ be the Minkowski sum. Then*

(1) *(Toric Euler–Jacobi) for any Laurent polynomial h whose Newton polytope lies in the interior of Δ the global residue $\mathcal{R}_f^{\mathbb{T}}(h)$ is zero;*
(2) *(Inversion of Toric Euler–Jacobi) for any $\phi : Z_f \to \mathbb{C}$ with $\sum_{a \in Z_f} \phi(a) = 0$ there exists a polynomial h whose Newton polytope lies in the interior of Δ such that $\phi(a) = h(a)/J_f^{\mathbb{T}}(a)$.*

Let us denote by S_{Δ° the vector space of all Laurent polynomials whose Newton polytope lies in the interior Δ° of Δ. We have a linear map

$$\mathcal{A} : S_{\Delta^\circ} \to \mathbb{C}^{|Z_f|}, \qquad h \mapsto \left(\frac{h(a)}{J_f^{\mathbb{T}}(a)}, a \in Z_f\right). \tag{3.1}$$

Then the above theorem says that the image of $\mathcal{A}$ is the hyperplane $\{\sum x_i = 0\}$ in $\mathbb{C}^{|Z_f|}$. By Bernstein's theorem the number of solutions $|Z_f|$ is equal to the normalized mixed volume $n!\, V(\Delta_1, \ldots, \Delta_n)$ of the polytopes [2]. We thus obtain a lower bound on the number of interior lattice points of Minkowski sums.

Corollary 3.2. *Let $\Delta_1, \ldots, \Delta_n$ be n-dimensional lattice polytopes in $\mathbb{R}^n$ and Δ their Minkowski sum. Then the number of lattice points in the interior of Δ is at least $n! V(\Delta_1, \ldots, \Delta_n) - 1$.*

It would be interesting to give a direct geometric proof of this inequality and determine all collections $\Delta_1, \ldots, \Delta_n$ for which the bound is attained. In the unmixed case $\Delta_1 = \cdots = \Delta_n = \Delta$ the inequality becomes

$$(n\Delta)^\circ \cap \mathbb{Z}^n \geq n!\mathrm{Vol}_n(\Delta) - 1 \tag{3.2}$$

and can be deduced from Stanley's positivity theorem for the Ehrhart polynomial [15]. Recently Batyrev and Nill described all possible Δ which give equality in (3.2) (see [1]). Here is a mixed case example which shows that the bound in Corollary 3.2 is sharp.

Example 1. Let $\Gamma(m)$ denote the simplex defined as the convex hull in $\mathbb{R}^n$ of $n+1$ points $\{0, e_1, \ldots, e_{n-1}, me_n\}$, where e_i is the i-th standard basis vector.

Consider a collection of n such simplices $\Gamma(m_1), \ldots, \Gamma(m_n)$ with $m_1 \leq \cdots \leq m_n$. It is not hard to see that their mixed volume equals m_n. For example, one can consider a generic system with these Newton polytopes and eliminate all but the last variable to obtain a univariate polynomial of degree m_n. The number of solutions of such a system is m_n, which is the mixed volume by Bernstein's theorem.

Also one can see that the number of interior lattice points in $\Gamma(m_1) + \cdots + \Gamma(m_n)$ is exactly $m_n - 1$. (In fact, these lattice points are precisely the points $(1, \ldots, 1, k)$ for $1 \leq k < m_n$.)

Corollary 3.3 (*Sparse Polynomial Interpolation*). *Let $f_1 = \cdots = f_n = 0$ be a generic system with n-dimensional Newton polytopes $\Delta_1, \ldots, \Delta_n$ and let Δ be their Minkowski sum. Let $Z_f \subset \mathbb{T}^n$ denote the solution set of the system. Then for any function $\phi : Z_f \to \mathbb{C}$ there is a polynomial g with $\Delta(g) \subseteq \Delta$ such that $g(a) = \phi(a)$. Moreover, g can be chosen to be of the form $g = h + cJ_f^{\mathbb{T}}$ for some h with $\Delta(h) \subset \Delta^\circ$ and a constant c.*

Proof. Consider a new function $\psi : Z_f \to \mathbb{C}$ given by

$$\psi(a) = \frac{\phi(a)}{J_f^{\mathbb{T}}(a)} - \frac{c}{MV}, \quad \text{where } c = \sum_{a \in Z_f} \frac{\phi(a)}{J_f^{\mathbb{T}}(a)}, MV = n!V(\Delta_1, \ldots, \Delta_n).$$

Then the sum of the values of ψ over the points of Z_f equals zero. Therefore there exists $h \in S_{\Delta^\circ}$ such that $h(a) = J_f^{\mathbb{T}}(a)\psi(a) = \phi(a) - \frac{c}{MV} J_f^{\mathbb{T}}(a)$ for all $a \in Z_f$. It remains to put $g = h + \frac{c}{MV} J_f^{\mathbb{T}}$. □

Remark 3.4. Theorem 3.1 is an instance of a more general result. Let $f_1, \ldots, f_k$ be generic Laurent polynomials with n-dimensional Newton polytopes $\Delta_1, \ldots, \Delta_k$, for $k \leq n$. The set of their common zeros defines an algebraic variety Z_k in $\mathbb{T}^n$. There is a way to embed $\mathbb{T}^n$ into a projective toric variety X so that the algebraic closure of $f_i = 0$ defines a Cartier divisor D_i on X and the closure $\overline{Z}_k$ is a complete intersection in X. In [12] Khovanskii described the space of top degree holomorphic forms on $\overline{Z}_k$. The special case $k = n$ corresponds to the space of all functions on the finite set $\overline{Z}_n = Z_f$ whose description we gave in Theorem 3.1. It follows from cohomology computation on complete intersections. In particular, for $k = n$, there is an exact sequence of global sections

$$\cdots \to \bigoplus_{i=1}^{n} H^0(X, \mathcal{O}(D - D_i + K)) \to H^0(X, \mathcal{O}(D + K)) \to \mathbb{C}^{|Z_f|} \to \mathbb{C} \to 0. \tag{3.3}$$

Here $D = D_1 + \cdots + D_n$, K the canonical divisor, and $\mathcal{O}(L)$ the invertible sheaf corresponding to a divisor L. The first non-zero map in (3.3) (from the right) is the trace map, the second map is the residue map we considered in (3.1), and the third one is given by $(f_1, \ldots, f_n)$.

4. Some commutative algebra

As before consider a system of Laurent polynomial equations $f_1 = \cdots = f_n = 0$ whose Newton polytopes $\Delta_1, \ldots, \Delta_n$ are full-dimensional. For generic coefficients the system will have a finite number of simple roots Z_f in the torus $\mathbb{T}^n$. We concentrate on the following problem.

Problem 2. Given a Laurent polynomial g compute the global residue $\mathcal{R}_f^{\mathbb{T}}(g)$ as a rational function of the coefficients of the f_i.

We postpone the algorithm to the next section and now formulate our main tool for solving the problem.

Theorem 4.1. *Let $\Delta_0, \ldots, \Delta_n$ be $n + 1$ full-dimensional lattice polytopes in $\mathbb{R}^n$ and assume that Δ_0 contains the origin in its interior. Put*

$$\tilde{\Delta} = \Delta_0 + \cdots + \Delta_n \quad \text{and} \quad \tilde{\Delta}_{(i)} = \Delta_0 + \cdots + \Delta_{i-1} + \Delta_{i+1} + \cdots + \Delta_n. \tag{4.1}$$

Then for generic polynomials f_i with Newton polytopes Δ_i, for $1 \leq i \leq n$, the linear map

$$\bigoplus_{i=0}^{n} S_{\tilde{\Delta}_{(i)}^\circ} \oplus \mathbb{C} \to S_{\tilde{\Delta}^\circ}, \qquad (h_0, \ldots, h_n, c) \mapsto h_0 + \sum_{i=1}^{n} h_i f_i + c J_f^{\mathbb{T}} \tag{4.2}$$

is surjective.

Proof. Let g be in $S_{\tilde{\Delta}^\circ}$. By Corollary 3.3 there exists a polynomial h_0 supported in $\Delta^\circ = \tilde{\Delta}_{(0)}^\circ$ and a constant c such that $g(a) = h_0(a) + cJ_f^{\mathbb{T}}(a)$ for all $a \in Z_f$, i.e. the polynomial $g - h_0 - cJ_f^{\mathbb{T}} \in S_{\tilde{\Delta}^\circ}$ vanishes on Z_f. Now the statement follows from Theorem 4.2 below. □

The following statement can be considered as the toric version of the classical Noether theorem in $\mathbb{P}^n$ (see [16], Section 20.2).

Theorem 4.2. *Let $f_1, \ldots, f_n$ be generic Laurent polynomials with n-dimensional Newton polytopes $\Delta_1, \ldots, \Delta_n$. Let h be a Laurent polynomial vanishing on Z_f. Assume $\Delta(h)$ lies in the interior of $\tilde{\Delta} = \Delta_0 + \Delta_1 + \cdots + \Delta_n$ for some n-dimensional polytope Δ_0. Then h can be written in the form*

$$h = h_1 f_1 + \cdots + h_n f_n, \quad \textit{with } \Delta(h_i) \subset \tilde{\Delta}^{\circ}_{(i)},$$

where $\tilde{\Delta}_{(i)}$ *as in* (4.1).

Proof. First we note that the statement remains true when $\Delta_0 = \{0\}$, i.e. h is supported in the interior of $\Delta = \Delta_1 + \cdots + \Delta_n$. This follows from the exact sequence (3.3). Indeed, if one considers the toric variety X associated with Δ then each f_i defines a semiample divisor D_i on X with polytope Δ_i. It is well known that for any semiample divisor L

$$H^{n-\dim \Delta_L}(X, \mathcal{O}(L+K)) \cong S_{\Delta_L^{\circ}}, \tag{4.3}$$

where Δ_L is the polytope of L (see, for example [11,8]). Thus the first term in (3.3) is isomorphic to $S_{\Delta^{\circ}_{(1)}} \oplus \cdots \oplus S_{\Delta^{\circ}_{(n)}}$, where $\Delta_{(i)} = \sum_{j \neq i} \Delta_j$. Since the sequence is exact and h lies in the kernel of the second map we get the required representation.

Now assume Δ_0 is n-dimensional. Let X be the toric variety associated with $\tilde{\Delta}$. Let f_0 be any monomial supported in Δ_0. Then $f_0, \ldots, f_n$ define $n+1$ semiample divisors $D_0, \ldots, D_n$ on X whose polytopes are $\Delta_0, \ldots, \Delta_n$. Since $f_1, \ldots, f_n$ are generic (and so all their common zeros lie in $\mathbb{T}^n$) and f_0 is a monomial, the divisors $D_0, \ldots, D_n$ have empty intersection in X. Then the following twisted Koszul complex of sheaves on X is exact (see [6,7]):

$$0 \to \mathcal{O}(K) \to \bigoplus_{i=0}^{n} \mathcal{O}(D_i + K) \to \cdots \to \bigoplus_{i=0}^{n} \mathcal{O}(\tilde{D} - D_i + K) \to \mathcal{O}(\tilde{D} + K) \to 0,$$

where $\tilde{D} = D_0 + \cdots + D_n$ and K the canonical divisor on X. The first few terms of the cohomology sequence are

$$\cdots \to \bigoplus_{i=0}^{n} H^0(X, \mathcal{O}(\tilde{D} - D_i + K)) \to H^0(X, \mathcal{O}(\tilde{D} + K)) \to 0,$$

where the middle map is given by $(f_0, \ldots, f_n)$. This, with the help of (4.3), provides

$$h = h_0 f_0 + h_1 f_1 + \cdots + h_n f_n, \quad \text{where } \Delta(h_i) \subset \tilde{\Delta}^{\circ}_{(i)}.$$

Notice that h_0 vanishes on Z_f and is supported in the interior of $\tilde{\Delta}_{(0)} = \Delta$. Therefore, there exist h'_i such that

$$h_0 = h'_1 f_1 + \cdots + h'_n f_n, \quad \text{with } \Delta(h'_i) \subset \Delta^{\circ}_{(i)}$$

by the previous case. It remains to note that $\Delta(h'_i f_0) \subset \Delta^{\circ}_{(i)} + \Delta_0 = \tilde{\Delta}^{\circ}_{(i)}$. □

Remark 4.3. Theorem 4.1 has interpretation in terms of the homogeneous coordinate ring S_X of the toric variety X associated with $\tilde{\Delta}$ (see [5]). One can homogenize $f_0, \ldots, f_n$ to get elements $F_0, \ldots, F_n \in S_X$ of *big* and *nef* degrees. According to the Codimension Theorem of Cox and Dickenstein (see [6]) the codimension of $I = \langle F_0, \ldots, F_n \rangle$ in *critical* degree (corresponding to the interior of $\tilde{\Delta}$) equals 1. Then Theorem 4.1 says that the homogenization of the Jacobian $J_f^{\mathbb{T}}$ to the critical degree generates the critical degree part of the quotient S_X/I.

5. Algorithm for computing the global residue in $\mathbb{T}^n$

Now we will present an algorithm for computing the global residue $\mathcal{R}_f^{\mathbb{T}}(g)$ for any Laurent polynomial g assuming that the Newton polytopes of the system are full-dimensional.

Algorithm 1. Let $f_1 = \cdots = f_n = 0$ be a system of Laurent polynomial equations with n-dimensional Newton polytopes $\Delta_1, \ldots, \Delta_n$. As before we let Δ denote the Minkowski sum of the polytopes.

Input: A Laurent polynomial g with Newton polytope $\Delta(g)$.

Step 1: Choose an n-dimensional polytope Δ_0 with $0 \in \Delta_0^{\circ}$ such that the Minkowski sum $\tilde{\Delta} = \Delta_0 + \Delta$ contains $\Delta(g)$ in its interior.

Step 2: Solve the system of linear equations

$$g = h_0 + \sum_{i=1}^{n} h_i f_i + c J_f^{\mathbb{T}} \tag{5.1}$$

for c, where h_i are polynomials with unknown coefficients supported in the interior of $\tilde{\Delta}_{(i)}$ (see (4.1)).

Output: The global residue $\mathcal{R}_f^{\mathbb{T}}(g) = c\,n!V(\Delta_1, \ldots, \Delta_n)$.

Proof. According to Theorem 4.1, given g with $\Delta(g) \subset \tilde{\Delta}^{\circ}$ there exist h_i supported in $\tilde{\Delta}^{\circ}_{(i)}$ and $c \in \mathbb{C}$ such that $g = h_0 + \sum_{i=1}^{n} h_i f_i + c J_f^{\mathbb{T}}$. Taking the global residue we have

$$\mathcal{R}_f^{\mathbb{T}}(g) = \mathcal{R}_f^{\mathbb{T}}(h_0) + \mathcal{R}_f^{\mathbb{T}}\left(\sum_{i=1}^{n} h_i f_i\right) + c\mathcal{R}_f^{\mathbb{T}}(J_f^{\mathbb{T}}) = cn!V(\Delta_1, \ldots, \Delta_n),$$

where the first two terms vanish by Theorem 3.1(1) and the definition of the global residue. □

Remark 5.1. Notice that we can ignore those terms of $J_f^{\mathbb{T}}$ whose exponents lie in the interior of Δ since their residue is zero by Theorem 3.1(1), and work with the "restriction" of $J_f^{\mathbb{T}}$ to the boundary of Δ.

We illustrate the algorithm with a small 2-dimensional example.

Example 2. Consider a system of two equations in two unknowns.

$$\begin{cases} f_1 = a_1 x + a_2 y + a_3 x^2 y^2, \\ f_2 = b_1 x + b_2 x y^2 + b_3 x^2 y^2. \end{cases}$$

The Newton polytopes Δ_1, Δ_2 and their Minkowski sum Δ are depicted in Fig. 1.

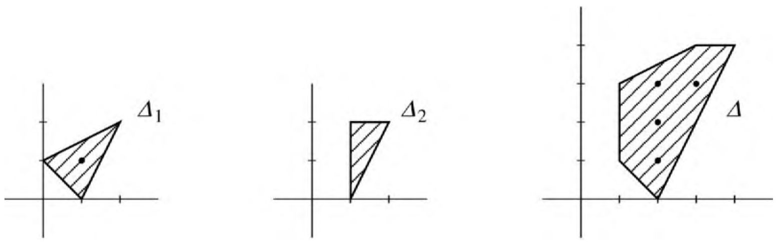

Fig. 1. The Newton polygons and their Minkowski sum.

We compute the global residue of $g = x^5 y^4$. Let Δ_0 be the triangle with vertices $(-1, 0)$, $(0, -1)$ and $(2, 1)$. Then the Minkowski sum $\tilde{\Delta} = \Delta + \Delta_0$ contains $\Delta(g) = (5, 4)$ in the interior (see Fig. 2).

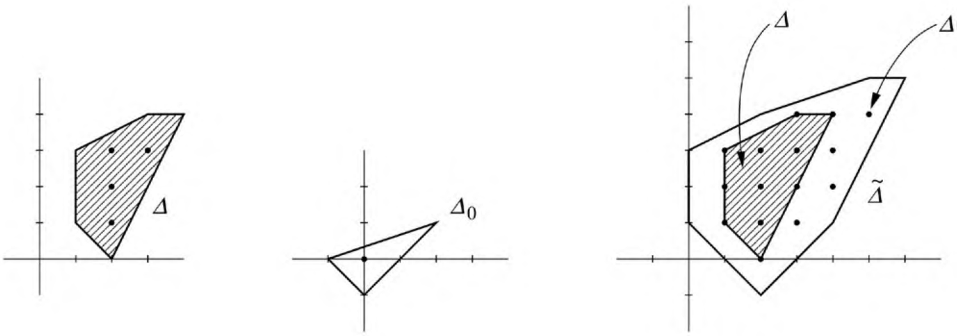

Fig. 2. The Minkowski sum of Δ and Δ_0 contains $\Delta(g)$ in the interior.

The vector space $S_{\tilde{\Delta}^\circ}$ has dimension 15 and a monomial basis

$$S_{\tilde{\Delta}^\circ} = \langle xy, xy^2, xy^3, x^2, x^2y, x^2y^2, x^2y^3, x^3y, x^3y^2, x^3y^3, x^3y^4, x^4y^2, x^4y^3, x^4y^4, x^5y^4 \rangle.$$

Now $\tilde{\Delta}_{(0)} = \Delta_1 + \Delta_2 = \Delta$, $\tilde{\Delta}_{(1)} = \Delta_0 + \Delta_2$ and $\tilde{\Delta}_{(2)} = \Delta_0 + \Delta_1$ and the corresponding vectors spaces are of dimension 4, 6 and 6, respectively. Here are their monomial bases:

$$S_{\tilde{\Delta}^\circ_{(0)}} = \langle x^2y, x^2y^2, x^2y^3, x^3y^3 \rangle, \qquad S_{\tilde{\Delta}^\circ_{(1)}} = \langle x, xy, xy^2, x^2y, x^2y^2, x^3y^2 \rangle,$$
$$S_{\tilde{\Delta}^\circ_{(2)}} = \langle y, x, xy, x^2y, x^2y^2, x^3y^2 \rangle,$$

as Fig. 3 shows.

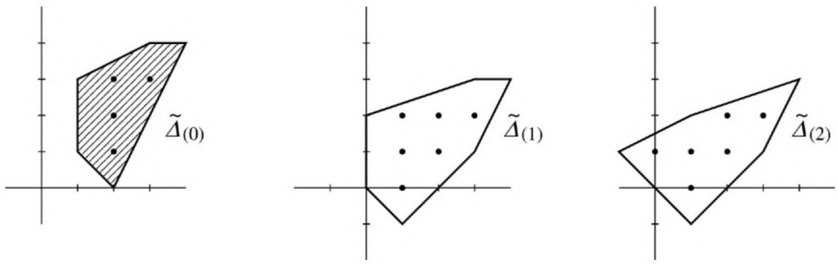

Fig. 3. The polygons $\tilde{\Delta}_{(i)}$.

We have

$$J_f^{\mathbb{T}} = -a_2b_1xy - a_2b_2xy^3 + 2a_1b_2x^2y^2 - 2a_2b_3x^2y^3 + 2(a_1b_3 - a_3b_1)x^3y^2 + 2a_3b_2x^3y^4,$$

where we can ignore the third and the fourth terms (see Remark 5.1). Now the map (4.2) written in the above bases has the following 15×17 matrix, which we denote by $\mathbf{A}$.

$$\begin{bmatrix}
0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 & -a_2b_1 \\
0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 & -a_2b_2 \\
0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & b_3 & 0 & b_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_1 & 0 & 0 & b_3 & 0 & 0 & b_1 & 0 & 2(a_1b_3 - a_3b_1) \\
0 & 0 & 0 & 1 & 0 & a_3 & 0 & 0 & 0 & a_2 & 0 & 0 & b_3 & b_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & 2a_3b_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & b_3 & b_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 & b_3 & 0
\end{bmatrix}.$$

It remains to solve the system $\mathbf{Ac} = \mathbf{b}$, where $\mathbf{c}$ is the vector of unknowns (the coefficients of the h_i and c) and $\mathbf{b}$ is the monomial $g = x^5y^4$ written in the basis for $S_{\tilde{\Delta}^\circ}$, i.e. $\mathbf{b} = (0, \ldots, 0, 1)^t$. With the help of `Maple` we obtain

$$c = \frac{1}{4}\frac{{a_1}^2b_2}{a_3(a_1b_3 - a_3b_1)^2}.$$

Since the mixed volume (area) of Δ_1 and Δ_2 equals 4, we conclude that

$$\mathcal{R}_f^{\mathbb{T}}(x^5y^4) = \frac{{a_1}^2b_2}{a_3(a_1b_3 - a_3b_1)^2}.$$

6. Some geometry

The first step of our algorithm is constructing a lattice n-dimensional polytope Δ_0 such that the Minkowski sum $\tilde{\Delta} = \Delta_0 + \Delta$ contains both Δ and $\Delta(g)$ in its interior. There are many ways of doing that. For example, one can take Δ_0 to be a sufficiently large dilate of Δ (translated so it contains the origin in the interior). In general, this could result in an unnecessarily large dimension of $S_{\tilde{\Delta}^\circ}$, which determines the size of the linear system (5.1). Therefore, to minimize the size of the linear system one would want to solve the following problem.

Problem 3. Given two lattice polytopes Δ and Δ' in $\mathbb{R}^n$, $\dim \Delta = n$, find an n-dimensional lattice polytope Δ_0 such that $\Delta + \Delta_0$ contains both Δ and Δ' in its interior and has the smallest possible number of interior lattice points.

This appears to be a hard optimization problem. Instead we will consider a less challenging one. First, since the global residue is linear we can assume that g is a monomial, i.e. $\Delta(g)$ is a point.

Problem 4. Given a convex polytope Δ and a point u in $\mathbb{R}^n$ find a segment I starting at the origin such that u is contained in the Minkowski sum $\Delta + I$ and the volume of $\Delta + I$ is minimal.

If $I = [0, m]$, for $m \in \mathbb{Z}^n$, is such a segment then we can take Δ_0 to be a "narrow" polytope with 0 in the interior and m one of the vertices, as in Fig. 2. Then the volume (and presumably the number of interior lattice points) of $\tilde{\Delta} = \Delta + \Delta_0$ will be relatively small.

We will now show how Problem 4 can be reduced to a linear programming problem. Let $\Delta \subset \mathbb{R}^n$ be an n-dimensional polytope, $I = [0, v]$ a segment, $v \in \mathbb{R}^n$. First, notice that the volume of $\Delta + I$ equals

$$\mathrm{Vol}_n(\Delta + I) = \mathrm{Vol}_n(\Delta) + |I| \cdot \mathrm{Vol}_{n-1}(\mathrm{pr}_I \Delta),$$

where $\mathrm{pr}_I \Delta$ is the projection of Δ onto the hyperplane orthogonal to I, Vol_k the k-dimensional volume, and $|I|$ the length of I. For each facet $\Gamma \subset \Delta$ let n_Γ denote the outer normal vector whose length equals the $(n-1)$-dimensional volume of Γ. Then we can write

$$|I| \cdot \mathrm{Vol}_{n-1}(\mathrm{pr}_I(\Delta)) = \frac{1}{2} \sum_{\Gamma \subset \Delta} |\langle n_\Gamma, v\rangle|.$$

But the latter is the support function h_Z of a convex polytope (zonotope) Z, which is the Minkowski sum of segments:

$$h_Z(v) = \sum_{\Gamma \subset \Delta} |\langle n_\Gamma, v\rangle|, \qquad Z = \sum_{\Gamma \subset \Delta} [-n_\Gamma, n_\Gamma].$$

Indeed, h_Z is the sum of the support functions of the segments. Also it is clear that

$$h_{[-n_\Gamma, n_\Gamma]}(v) = \max_{-1 \le t \le 1} \langle t n_\Gamma, v\rangle = |\langle n_\Gamma, v\rangle|.$$

An example of the polytopes Δ and Z, and the normal fan Σ_Z of Z is given in Fig. 4.

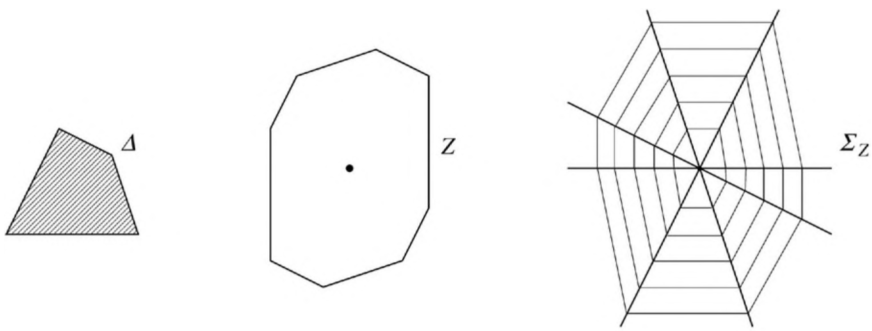

Fig. 4. The zonotope Z associated with Δ and its normal fan Σ_Z.

Now we get back to Problem 4. After translating everything by $-u$ we may assume that u is at the origin. Then Problem 4 is equivalent to finding $x \in \Delta$ such that the volume of $\Delta + [0, -x]$ is minimal, which by the previous discussion means minimizing the support function $h_Z(-x) = h_Z(x)$ on Δ.

We can interpret this geometrically. The function h_Z is a non-negative continuous function, linear on every cone of the normal fan Σ_Z. Its graph above the polytope Δ is a "convex down" polyhedral set in $\mathbb{R}^{n+1}$ (see Fig. 5). The set of points with the smallest last coordinate is a face of this polyhedral set. The projection of this face to Δ gives the solution to our minimization problem.

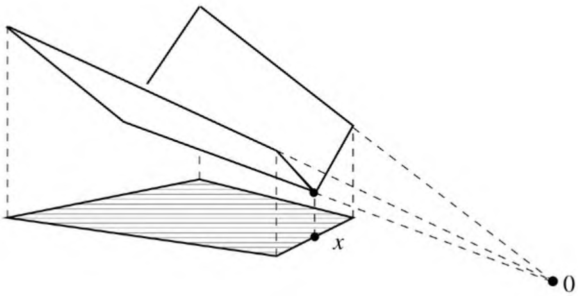

Fig. 5. The graph of h_Z above the polytope Δ.

Finally, note that the normal fan Σ_Z has a simple description. It is obtained by translating all the facet hyperplanes H_Γ, for $\Gamma \subset \Delta$, to the origin.

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