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Hereditary Properties of Co-Kähler Manifolds

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1. Introduction

Co-Kähler manifolds may be thought of as odd-dimensional versions of Kähler manifolds and various structure theorems explicitly display how the former are constructed from the latter (see [3,24]).

In this paper, we take the point of view that topological and geometric properties of co-Kähler manifolds are inherited from those of the Kähler manifolds that construct them. We shall see this in both topological and geometric contexts. First, let us recall some basic definitions (see [5] for a detailed introduction).

Definition 1.1. An **almost contact metric structure** (J, ξ, η, g) on a manifold M^{2n+1} consists of a tensor J of type $(1, 1)$, a vector field ξ, a 1-form η and a Riemannian metric g such that

$$J^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(JX, JY) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

for vector fields X and Y, I the identity transformation on TM.

A local J-basis for TM, $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n, \xi\}$, may be found with $\eta(X_i) = 0$ for $i = 1, \ldots, n$. The *fundamental* 2-*form* on M is given by

$$\omega(X, Y) = g(JX, Y),$$

and if $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \eta\}$ is a local 1-form basis dual to the local J-basis, then

$$\omega = \sum_{i=1}^{n} \alpha_i \wedge \beta_i.$$

Note that $\imath_\xi \omega = 0$.

Definition 1.2. The geometric structure $(M^{2n+1}, J, \xi, \eta, g)$ is

- **co-symplectic** if $d\omega = 0 = d\eta$;
- **normal** if $[J, J] + 2\, d\eta \otimes \xi = 0$;
- **co-Kähler** if it is co-symplectic and normal; equivalently, if J is parallel with respect to the metric g.

Recently, co-symplectic geometry has attracted a great deal of interest, especially in the context of Poisson geometry, where co-symplectic structures are interpreted as corank 1 Poisson structures (see for instance [8,13,19,20,26]). Sasakian structures also belong to this family; more precisely, they are normal structures such that $d\eta = \omega$ (see [7,9,10]).

Two crucial facts about co-Kähler manifolds are contained in the following lemma. For a direct proof of these facts, see [3].

Lemma 1.3. *On a co-Kähler manifold, the vector field ξ is Killing and parallel. Furthermore, the 1-form η is parallel and harmonic.*

Lemma 1.3 is a key point in Theorem 1.5 below. In fact, in [24] it is shown that we can replace η by a harmonic integral form η_θ with dual parallel vector field ξ_θ and associated metric g_θ, $(1, 1)$-tensor J_θ and closed 2-form ω_θ with $i_{\xi_\theta} \omega_\theta = 0$. Then we have the following result of H. Li.

Theorem 1.4 *([24]). If M^{2n+1} is compact, with the structure $(J_\theta, \xi_\theta, \eta_\theta, g_\theta)$, there are a compact Kähler manifold (K, h) and a Hermitian isometry $\psi\colon K \to K$ such that M is diffeomorphic to the mapping torus*

$$K_\psi = \frac{K \times [0, 1]}{(x, 0) \sim (\psi(x), 1)}$$

with associated fibre bundle $K \to M = K_\psi \to S^1$.

In [3], the following refinement of Li's result is proved:

Theorem 1.5 *([3], Theorem 3.3). Let $(M^{2n+1}, J, \xi, \eta, g)$ be a compact co-Kähler manifold with integral structure and mapping torus bundle $K \to M \to S^1$. Then M splits as $M \cong S^1 \times_{\mathbb{Z}_m} K$, where $S^1 \times K \to M$ is a finite cover with structure group $\mathbb{Z}_m$ acting diagonally and by translations on the first factor. Moreover, M fibres over the circle $S^1/(\mathbb{Z}_m)$ with finite structure group.*

The first true study of the topological properties of co-Kähler manifolds was made in [11] where the focus was on things such as Betti numbers and a modified Lefschetz property. The two results above allow us to say something about the fundamental group and, moreover, to display the higher homotopy groups as those of the constituent Kähler manifold K (groups which, of course, are generally unknown as well). Nevertheless, the principle (which gives rise to the paper's title) remains that topological qualities of a co-Kähler manifold are intimately tied up with those of the Kähler manifold that constructs it. In this paper, we shall explore this principle in several ways. We begin by examining the cohomology algebra of a co-Kähler manifold and its effect on the manifold's rational homotopy structure. We then will consider the structure of the minimal models (in the sense of Sullivan) of co-Kähler manifolds in terms of the decompositions given in Theorem 1.4 and Theorem 1.5. In Section 3, we go beyond algebraic considerations in showing that co-Kähler manifolds satisfy the so-called Toral Rank Conjecture. This theorem strongly connects the geometry of the co-Kähler manifold to the size of its cohomology.

In a previous version of this paper, a fourth section was included, which has now been taken out and constitutes the new paper [4].

We have written this paper for an audience of geometers who may not be experts in rational homotopy. Therefore, we have included a substantial review of basic facts in the subject. The main references for this material are [15–17].

2. The Lefschetz property and associated algebraic models

2.1. Cohomology algebra structure

Using Theorem 1.5, the following description of the cohomology of a compact co-Kähler manifold was obtained in [3].

Theorem 2.1 *([3], Theorem 4.3). If $(M^{2n+1}, J, \xi, \eta, g)$ is a compact co-Kähler manifold with integral structure and splitting $M \cong K \times_{\mathbb{Z}_m} S^1$, then*

$$H^*(M;\mathbb{R}) \cong H^*(K;\mathbb{R})^G \otimes H^*(S^1;\mathbb{R}), \tag{2}$$

as commutative graded algebras, where $G = \mathbb{Z}_m$. Hence, the Betti numbers of M satisfy:

(i) *$b_s(M) = \overline{b}_s(K) + \overline{b}_{s-1}(K)$, where $\overline{b}_s(K)$ denotes the dimension of G-invariant cohomology $H^s(K;\mathbb{R})^G$;*
(ii) *$b_1(M) \leq b_2(M) \leq \ldots \leq b_n(M) = b_{n+1}(M)$.*

In order to study cohomological properties of co-Kähler manifolds, we recall the notion of cohomologically Kählerian differential graded algebra.

Definition 2.2. Let (A, d) be a commutative differential graded algebra (cdga) of cohomological dimension $2n$, whose cohomology algebra satisfies Poincaré duality. The cdga (A, d) is called *cohomologically Kählerian* if there exists a closed element $\omega \in A^2$ such that the map

$$\mathcal{L}^{n-p} \colon H^p(A) \to H^{2n-p}(A), \quad [\sigma] \mapsto [\omega]^{n-p} \cdot [\sigma]$$

is an isomorphism for every $0 \leq p \leq n$. Note that we include the case where (A, d) has $d = 0$ and we then refer to A as a commutative graded algebra (cga).

Clearly, if K is a Kähler manifold, the cohomology algebra $H^*(K;\mathbf{k})$ is cohomologically Kähler, where $\mathbf{k}$ can be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Note that there exist examples of non-Kähler manifolds whose de Rham algebra is cohomologically Kähler (see for instance [18]).

Let (M, J, ξ, η, g) be a compact co-Kähler manifold with integral structure and mapping torus bundle $K \to M \to S^1$ and consider the cohomology algebra $H^*(K;\mathbb{R})$ of the Kähler manifold K. The finite group $G \cong \mathbb{Z}_m$ acts on $H^*(K;\mathbb{R})$ and, according to Theorem 2.1, the cohomology algebra of M is the product of the invariant part of the cohomology of K and the cohomology of S^1. We now show that the invariant part of $H^*(K;\mathbb{R})$ is a cohomologically Kählerian algebra.

Proposition 2.3. *The cga $H^*(K;\mathbb{R})^G$ is cohomologically Kählerian.*

Proof. In [3, Lemma 4.2], it is proved that $H^*(K;\mathbb{R})^G$ contains a G-invariant element ω of degree 2 which behaves like a symplectic form. Such an element is the pullback of the Kähler form in $K \times S^1$ under the inclusion $K \hookrightarrow K \times S^1$. In order to see that $H^*(K;\mathbb{R})^G$ is cohomologically Kähler, we must show, further, that the Lefschetz map $H^p(K;\mathbb{R})^G \to H^{2n-p}(K;\mathbb{R})^G$ is an isomorphism for every $0 \leq p \leq n$. We check injectivity first. Since $H^*(K;\mathbb{R})$ is cohomologically Kählerian, the multiplication by ω^{n-p} is injective on the whole space $H^p(K;\mathbb{R})$ and remains injective when restricted to $H^p(K;\mathbb{R})^G$. For surjectivity, take $\tau \in H^{2n-p}(K;\mathbb{R})^G$. Again, since $H^*(K;\mathbb{R})$ is cohomologically Kählerian, there exists $\tau' \in H^p(K;\mathbb{R})$ such that $\tau = \tau' \wedge \omega^{n-p}$. We must show that $\tau' \in H^p(K;\mathbb{R})^G$. We have

$$\tau' \wedge \omega^{n-p} = \tau = g(\tau) = g(\tau' \wedge \omega^{n-p}) = g(\tau') \wedge g(\omega^{n-p}) = g(\tau') \wedge \omega^{n-p},$$

so $(\tau' - g(\tau')) \wedge \omega^{n-p} = 0$. But $H^*(K;\mathbb{R})$ is cohomologically Kählerian, so the multiplication by ω^{n-p} is injective, and therefore $g(\tau') = \tau'$ and $\tau' \in H^p(K;\mathbb{R})^G$. $\square$

This immediately gives another old result about the Betti numbers of co-Kähler manifolds [11, Theorem 11].

Corollary 2.4. *Let $(M^{2n+1}, J, \xi, \eta, g)$ be a compact co-Kähler manifold. Then, for $0 \leq i \leq n$, the differences $b_{2i+1}(M) - b_{2i}(M)$ are even integers and non-negative if $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. By Theorem 2.1, once we replace the co-Kähler structure on M with an integral one, we obtain $b_i(M) = \bar{b}_i(K) + \bar{b}_{i-1}(K)$, where M sits in the mapping torus fibration $K \to M \to S^1$, with K a $2n$-dimensional Kähler manifold and $\bar{b}_i(K) = \dim H^i(K;\mathbb{R})^G$. We proved in Proposition 2.3 that $H^*(K;\mathbb{R})^G$ is cohomologically Kähler, and this implies that $\bar{b}_{2i+1}(K)$ is even for $0 \leq i \leq n-1$. Therefore,

$$\begin{aligned} b_{2i+1}(M) - b_{2i}(M) &= \bar{b}_{2i+1}(K) + \bar{b}_{2i}(K) - \bar{b}_{2i}(K) - \bar{b}_{2i-1}(K) = \\ &= \bar{b}_{2i+1}(K) - \bar{b}_{2i-1}(K), \end{aligned}$$

which is an even number and non-negative if $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. $\square$

Now, according to Theorem 1.4, the choice of an integral co-Kähler structure $(J_\theta, \xi_\theta, \eta_\theta, g_\theta)$ on M produces a mapping torus bundle $K \to M \to S^1$. By Theorem 1.5, this gives in turn a finite cover $K \times S^1 \to M$ with deck group $G \cong \mathbb{Z}_m$. We then have the following *fundamental data* for a co-Kähler manifold.

Definition 2.5. Let (M, J, ξ, η, g) be a compact co-Kähler manifold and choose an integral structure $(J_\theta, \xi_\theta, \eta_\theta, g_\theta)$ on M. The data (K, G) of the mapping torus bundle $K \to M \to S^1$ and the finite G-cover $K \times S^1 \to M$ form a **presentation** of (M, J, ξ, η, g).

Clearly the presentation of (M, J, ξ, η, g) depends on the choice of an integral co-Kähler structure on M. Nevertheless, according to Theorem 2.1, $H^*(M;\mathbb{Q}) \cong H^*(K;\mathbb{Q})^G \otimes H^*(S^1;\mathbb{Q})$ and therefore, if we are given two different presentations

$$K_1 \to M \to S^1, \quad K_1 \times S^1 \stackrel{/G_1}{\to} M$$

and

$$K_2 \to M \to S^1, \quad K_2 \times S^1 \stackrel{/G_2}{\to} M,$$

then we must have

$$H^*(K_1;\mathbb{Q})^{G_1} \cong H^*(K_2;\mathbb{Q})^{G_2}\,.$$

Let (M, J, ξ, η, g) be a compact co-Kähler manifold and fix a presentation (K, G). The finite cyclic group G acts on the product $K \times S^1$ and, according to Theorem 1.5, G acts by translations on S^1, so the G action is free. But the action need not remain free when we restrict it to K. Therefore, the quotient space K/G need not be a manifold. However, since G is a finite, cyclic group, K/G is a Kähler orbifold which is canonically associated to the presentation of the co-Kähler manifold M. Indeed, G acts by Hermitian isometries on K, so the Kähler structure is preserved under the G-action, and passes to the quotient. Since G is finite, we have $H^*(K/G;\mathbb{Q}) \cong H^*(K;\mathbb{Q})^G$, so the rational cohomology of the quotient K/G is computed by the invariant rational cohomology of K. In view of (2), the cohomology of M contains information about the cohomology of the Kähler orbifold K/G. Such a Kähler orbifold K/G is associated to the chosen presentation $K \to M \to S^1$ of the co-Kähler manifold (M, J, ξ, η, g), but since all possible presentations yield diffeomorphic M, the corresponding orbifolds have the same rational cohomology.

2.2. Rational homotopy structure

In fact, the algebra splitting of Theorem 2.1 tells us much more about the structure of the co-Kähler manifold M. For this we need to recall some notions from Rational Homotopy Theory. The reader is referred to [15], [17, Chapters 2 and 3] and [2] for details and proofs of the statements that follow.

A commutative graded algebra (cga) over a field of characteristic zero $\mathbf{k}$, A, is called *free graded commutative* if A is the quotient of TV, the tensor algebra on the graded vector space V, by the bilateral ideal generated by the elements $a \otimes b - (-1)^{|a|\cdot|b|} b \otimes a$, where a and b are homogeneous elements of A. As an algebra, A is the tensor product of the symmetric algebra on V^{even} with the exterior algebra on V^{odd}:

$$A = \mathrm{Symmetric}(V^{\mathrm{even}}) \otimes \mathrm{Exterior}(V^{\mathrm{odd}})\,.$$

We denote the free commutative graded algebra on the graded vector space V by $\wedge V$. Note that this notation refers to a free commutative graded algebra and not necessarily to an exterior algebra alone. We usually write $\wedge V = \wedge(x_i)$, where x_i is a homogeneous basis of V. Clearly the cohomology of a cdga is a commutative graded algebra. A morphism of cdga's inducing an isomorphism in cohomology will be called a *quasi-isomorphism*. A *Sullivan cdga* is a cdga $(\wedge V, d)$ whose underlying algebra is free commutative, with $V = \{V^n\}$, $n \geq 1$, and such that V admits a basis x_α indexed by a well-ordered set such that $d(x_\alpha) \in \wedge(x_\beta)_{\beta<\alpha}$. A *(Sullivan) minimal cdga* is a Sullivan cdga $(\wedge V, d)$ satisfying the additional property that $d(V) \subset \wedge^{\geq 2} V$. Minimal cdga's play an important role because they are tractable models for "all" other cdga's. (For the path-connected non-simply-connected case of the following result, see [22, Chapter 6] or, from a functorial viewpoint, [2], especially Chapters 7 and 12.)

Theorem 2.6 *(Existence and uniqueness of the minimal model). Let (A, d) be a cdga over $\mathbf{k}$ satisfying $H^0(A, d) = \mathbf{k}$, where $\mathbf{k}$ is $\mathbb{R}$ or $\mathbb{Q}$ and $\dim(H^p(A, d)) < \infty$ for all p. Then,*

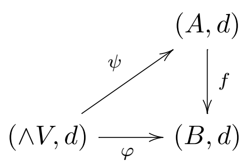
(1) *There is a quasi-isomorphism* $\varphi\colon (\wedge V, d) \to (A, d)$*, where* $(\wedge V, d)$ *is a minimal cdga.*
(2) *The minimal cdga* $(\wedge V, d)$ *is unique in the following sense: If* $(\wedge W, d)$ *is a minimal cdga and* $\psi\colon (\wedge W, d) \to (A, d)$ *is a quasi-isomorphism, then there is an isomorphism* $f\colon (\wedge V, d) \to (\wedge W, d)$ *such that* $\psi \circ f$ *is homotopic (see [15]) to* φ*.*

The cdga $(\wedge V, d)$ *is then called the* minimal model *of* (A, d)*.*

The connection between this type of algebra and topology is via the de Rham cdga of differential forms on the manifold M, $(\Omega(M), d)$, when $\mathbf{k}$ is $\mathbb{R}$ and Sullivan's rational polynomial forms on M (thought of as a simplicial complex, say), $(A_{PL}(M), d)$, when $\mathbf{k}$ is $\mathbb{Q}$. Applying Theorem 2.6 to these cdga's produces a *minimal model of the space* M denoted by $\varphi\colon \mathcal{M}_M = (\wedge V, d) \to A$, where we let A stand for either the de Rham or Sullivan algebras. We shall not distinguish the minimal models depending on the field because the context will always be clear. The minimal model thus provides a special type of cdga associated to a space. Note that the condition $H^0(A, d) = \mathbf{k}$ in Theorem 2.6 means that any path-connected space has a minimal model. There are two key facts that make minimal cdga's an important tool.

Lemma 2.7.

(1) *If* $f\colon (\wedge V, d) \to (\wedge Z, d)$ *is a quasi-isomorphism between minimal cdga's, then* f *is an isomorphism.*
(2) *For a Sullivan cdga* $(\wedge V, d)$*, a cdga quasi-isomorphism* $f\colon (A, d) \to (B, d)$ *and a cdga morphism* $\varphi\colon (\wedge V, d) \to (B, d)$*, there is a cdga morphism* $\psi\colon (\wedge V, d) \to (A, d)$ *such that* $f \circ \psi$ *is homotopic (see [15]) to* φ*.*



Here is one application. Say that the spaces X and Y have *the same rational homotopy type* if there is a finite chain of maps $X \to Y_1 \leftarrow Y_2 \to \cdots \to Y$ such that each induced map in rational cohomology is an isomorphism. If we consider the cdga morphisms

$$\mathcal{M}_{Y_1} \to A_{PL}(Y_1) \to A_{PL}(X) \leftarrow \mathcal{M}_X$$

and apply (2) of Lemma 2.7, we obtain a cdga morphism $\mathcal{M}_{Y_1} \to \mathcal{M}_X$ which is a quasi-isomorphism (since the other morphisms are). By (1) of Lemma 2.7, we then have $\mathcal{M}_{Y_1} \cong \mathcal{M}_X$. We carry on this process through the chain of maps to get $\mathcal{M}_Y \cong \mathcal{M}_X$.

Proposition 2.8. *If* X *and* Y *have the same rational homotopy type, then their minimal models are isomorphic. Moreover, if* X *and* Y *are nilpotent spaces (e.g. simply connected), then the converse is true.*

The second statement follows from the existence of spatial rationalizations coming from homotopical localization theory. In general, these do not exist for non-nilpotent spaces. This is important to note because compact co-Kähler manifolds are rarely nilpotent spaces (they are never simply connected of course). So, in the case of non-nilpotent spaces such as typical co-Kähler manifolds, it is the isomorphism class of the minimal model that really represents some sort of rational type. Of course, everything we have said applies to models over $\mathbb{R}$ as well.

Some minimal models are even more special; they are isomorphic to the minimal models associated to the cohomology algebra (considered as a cdga with zero differential). Spaces with this property are called *formal*. Lemma 2.7 implies that there is the following equivalent definition.

Definition 2.9. A space X, with minimal model $(\wedge V, d)$, is called **formal** if there is a quasi-isomorphism

$$\theta\colon (\wedge V, d) \to (H^*(X;\mathbb{Q}), 0)\,.$$

Remark 2.10. We can also define a cdga (A, d) to be *formal* if there is a chain of quasi-isomorphisms

$$(A, d) \leftarrow (B_1, d_1) \to \cdots (B_k, d_k) \to (H^*(A), 0)\,.$$

We can take the minimal models of (A, d), the minimal models of the (B_i, d_i) and the minimal models of the morphisms and apply Lemma 2.7 to see that this is equivalent to Definition 2.9.

The last piece of Rational Homotopy Theory that we shall need is the notion of an equivariant minimal model. Let Γ be a finite group. A Γ-cdga is a cdga on which the group Γ acts by a homomorphism $\Gamma \to \mathrm{aut}_{cdga}(A, d_A)$.

Definition 2.11. A Γ-cdga (A, d_A) is called Γ-**minimal** if $(A, d_A) = (\wedge V, d)$ with

(1) $d(V) \subset \wedge^{\geq 2}(V)$;
(2) Each V^n is a Γ-module (i.e. this gives a Γ-structure to $\wedge V$);
(3) d is Γ-equivariant: $d(ga) = gd(a)$;
(4) V admits a filtration by sub Γ-spaces

$$0 \subset V(0) \subset V(1) \subset \cdots \subset V(n) \subset \cdots \subset V = \cup_n V(n)\,,$$

with $d(V(n)) \subset (\wedge V(n-1))$.

Generalizing the non-equivariant case, we have the following. Note that, while all proofs (e.g. [17, Theorem 3.26]) of this result assume $H^1(A, d_A) = 0$, this is for convenience only. In the same way that the ordinary minimal model can be constructed for general path-connected spaces (see Theorem 2.6) by a limiting process, we can also construct an equivariant model.

Theorem 2.12. *Let (A, d_A) be a Γ-cdga. Suppose that $H^0(A, d_A) = \mathbf{k}$, where $\mathbf{k} = \mathbb{R}$, or $\mathbf{k} = \mathbb{Q}$. Then there exists a Γ-minimal algebra $(\wedge V, d)$ and a Γ-equivariant quasi-isomorphism $\varphi\colon (\wedge V, d) \to (A, d_A)$. The Γ-minimal algebra $(\wedge V, d)$ is called the Γ-minimal model of the Γ-cdga (A, d_A), and it is unique up to Γ-isomorphism.*

Suppose a finite group Γ acts on the space X. If the Γ-equivariant minimal model of X, $(\wedge V, d)$, is equivariantly isomorphic to the Γ-equivariant minimal model of $H^*(X;\mathbf{k})$, then we say that (X, Γ) is Γ-*formal*. It can be shown that a formal Γ-space is Γ-formal [28]. That is, if X is a formal space with an action of a finite group Γ, then the equivariant minimal model can be constructed from the action of Γ on $H^*(X;\mathbf{k})$. Moreover, in this situation, we can show that the minimal model of X/Γ is the minimal model of $H^*(X/\Gamma;\mathbf{k})$, so that X/Γ is formal. To see this, let $\phi\colon (\wedge W, d) \to (\wedge V, d)^\Gamma$ be the minimal model of $(\wedge V, d)^\Gamma$. By computing the invariant part of cohomology, we know that $(\wedge W, d)$ is the minimal model of X/Γ (see [17, Corollary 3.29]). Now consider the commutative diagram below, where the right square comes from the inclusion of invariant elements and the equivariant formality quasi-isomorphism θ, and the left square comes from lifting the composition $(\wedge W, d) \xrightarrow{\phi} (\wedge V, d)^\Gamma \xrightarrow{\theta^\Gamma} H^*(X; \mathbf{k})^\Gamma$ through the isomorphism $H^*(X/\Gamma; \mathbf{k}) \cong H^*(X; \mathbf{k})^\Gamma$.

$$\begin{array}{ccccc}
(\wedge W, d) & \xrightarrow{\phi} & (\wedge V, d)^\Gamma & \longrightarrow & (\wedge V, d) \\
\downarrow & & \downarrow{\scriptstyle \theta^\Gamma} & & \downarrow{\scriptstyle \theta} \\
H^*(X/\Gamma; \mathbf{k}) & \xrightarrow{\cong} & H^*(X; \mathbf{k})^\Gamma & \longrightarrow & H^*(X; \mathbf{k})
\end{array}$$

Then, since θ is a quasi-isomorphism, so is θ^Γ. But then the lift $(\wedge W, d) \to H^*(X/\Gamma; \mathbf{k})$ is also a quasi-isomorphism. Hence, X/Γ is formal if X is.

Let (M, J, ξ, η, g) be a compact co-Kähler manifold and let (K, G) be a presentation. Let $(\mathcal{M}_K, d)$ denote the minimal model of K. Then its invariant part is a rational model for the space K/G. A main result of [14] states that compact Kähler manifolds are formal. This means, among other things, that the minimal model of a Kähler manifold K is determined by its rational cohomology algebra $H^*(K; \mathbb{Q})$. (Also, note that formality does not depend on the field $\mathbf{k}$.) Since K is a formal space, so is K/G, and hence its minimal model can be computed from the cohomology algebra $H^*(K/G; \mathbb{Q}) \cong H^*(K; \mathbb{Q})^G$. Furthermore, co-Kähler manifolds are also formal (see [11]), so the rational minimal model of M can be constructed directly from its rational cohomology, which in view of Theorem 2.1, is isomorphic to $H^*(K; \mathbb{Q})^G \otimes H^*(S^1; \mathbb{Q})$, for any presentation (K, G) of M. Putting this together, we obtain the following result.

Theorem 2.13. *Let M be a compact co-Kähler manifold and let (K, G) be a presentation. Then the minimal model of M has the following cdga splitting:*

$$\mathcal{M}_M \cong \mathcal{M}_{K/G} \otimes \mathcal{M}_{S^1}.$$

Proof. Because all spaces are formal, we have the following diagram:

$$\begin{array}{ccc}
\mathcal{M}_M & \overset{\phi}{\dashrightarrow} & \mathcal{M}_{K/G} \otimes \mathcal{M}_{S^1} \\
{\scriptstyle \theta_M}\downarrow & & \downarrow{\scriptstyle \theta_{K/G} \otimes \theta_{S^1}} \\
H^*(M; \mathbb{Q}) & \xrightarrow{\cong} & H^*(K; \mathbb{Q})^G \otimes H^*(S^1; \mathbb{Q}),
\end{array}$$

where the top arrow comes from Lemma 2.7. Also by Lemma 2.7, we see that ϕ is an isomorphism. □

Theorem 2.13 is quite interesting, in the following sense. Since a compact co-Kähler manifold (M, J, ξ, η, g) is never simply connected, when its fundamental group $\pi_1(M)$ is not nilpotent, or acts non-nilpotently on higher homotopy groups (see [17]), the minimal model of M does not give, in general, information about the usual rational homotopy structure of M. For instance, we don't see things such as rational homotopy groups and Whitehead products. However, the isomorphism class of a minimal model is always an invariant attached to any space, so Theorem 2.13 says that inside the minimal model of M we can see the "auxiliary" Kähler orbifold K/G (i.e. its minimal model). So the minimal model provides a new type of (geometric) information that is non-classical.

Example 2.14. Here is another description of the Chinea–de León–Marrero example contained in [11]. Consider the torus T^2 with its standard Kähler structure and let $\phi\colon T^2 \to T^2$ be the holomorphic isometry covered by the linear transformation $A\colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Betti numbers of the mapping torus T^2_ϕ are easily computed to be the following:

- $b_0(T^2_\phi) = b_3(T^2_\phi) = 1$;
- $b_1(T^2_\phi) = 1$, generated by the volume form of the circle S^1;
- $b_2(T^2_\phi) = 1$, generated by the Kähler class of the torus T^2.

The minimal model of T^2_ϕ is

$$(\wedge(t, u, v), |t| = 1, |u| = 2, |v| = 3, dv = u^2),$$

which is isomorphic to the minimal model of $S^2 \times S^1$. The automorphism ϕ of T^2 has order 4 and M can be seen as the quotient of $T^2 \times S^1$ by the $\mathbb{Z}_4$-action given by

$$(x, y, z) \mapsto (y, -x, z + 1/4).$$

Now consider the quotient T^2/G. When we think of T^2 as the square $[0, 1] \times [0, 1]$ with the sides identified, the action of G on T^2 is a rotation of $\pi/2$ around the center of the square. There are therefore 2 fixed points, $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$. Using the Riemann–Hurwitz formula, one sees that the quotient T^2/G is a compact surface of genus 0, hence topologically a sphere S^2.

Example 2.15. Take two copies $(\mathbb{C}P^1_i, \omega_i)$, $i = 1, 2$, of $\mathbb{C}P^1$ with its standard Kähler structure, and consider the manifold $K = \mathbb{C}P^1 \times \mathbb{C}P^1$ endowed with Kähler structure $\omega = \omega_1 + \omega_2$. Let $\phi\colon K \to K$ denote the map $\phi(p, q) = (q, p)$. Then ϕ is a holomorphic isometry of K. The rational cohomology of the co-Kähler manifold $N = K_\phi$ is:

- $H^1(K_\phi; \mathbb{Q}) = \langle [u] \rangle$, generated by the class of the circle S^1;
- $H^2(K_\phi; \mathbb{Q}) = \langle [\omega] \rangle$, generated by the Kähler class of K;
- $H^3(K_\phi; \mathbb{Q}) = \langle [\omega \wedge u] \rangle$;
- $H^4(K_\phi; \mathbb{Q}) = \langle [\omega^2] \rangle$;
- $H^5(K_\phi; \mathbb{Q}) = \langle [\omega^2 \wedge u] \rangle$.

The minimal model of K_ϕ is

$$(\wedge(t, u, v), |t| = 1, |u| = 2, |v| = 5, dv = u^3),$$

which is isomorphic to the minimal model of $\mathbb{C}P^2 \times S^1$. The automorphism ϕ of K has order 2 and M can be seen as the quotient of $K \times S^1$ by the $\mathbb{Z}_2$-action given by

$$(p, q, t) \mapsto (q, p, t + 1/2).$$

It is not hard to see that the quotient $K/\mathbb{Z}_2$ is smooth, and isomorphic (as algebraic varieties) to $\mathbb{C}P^2$. Indeed, let $D = (p, p) \subset K$ be the diagonal; the Segre map gives an embedding $\imath\colon K \to \mathbb{C}P^3$ which realizes K as a smooth quadric $\mathcal{Q}$. The projection from $\mathcal{Q}$ to a plane $\pi \subset \mathbb{C}P^3$ is a $2:1$ cover, branched over the conic $\mathcal{C} \subset \pi$ which is the image under the projection of $\imath(D)$. Therefore, the quotient $\mathcal{Q}/\mathbb{Z}^2$ is precisely $\pi \cong \mathbb{C}P^2$.

Remark 2.16. It is worth pointing out here that in neither of the two examples does the minimal model compute typical rational homotopy information (beyond cohomology) about the corresponding Kähler mapping torus. Indeed, in the first case, T^2_ϕ is an aspherical manifold, as can be seen directly from the long exact sequence of homotopy groups of the fibration $T^2 \to T^2_\phi \to S^1$, but the minimal model of T^2_ϕ has generators in degree 2 and 3. In the second case, by the same method one sees that $\pi_2(K_\phi) = \mathbb{Z} \oplus \mathbb{Z}$, but the minimal model of K_ϕ has only one generator in degree 2. In both cases, the reason for this apparent mis-match is that neither T^2_ϕ nor K_ϕ are nilpotent spaces.

3. Toral rank of co-Kähler manifolds

The *Toral Rank Conjecture* (TRC), due to Halperin [21], has been a very influential and motivating problem in the development of Rational Homotopy Theory. In this section we show that a co-Kähler manifold satisfies the conjecture. This is again an instance of our principle that co-Kähler manifolds inherit properties from their constituent Kähler manifolds since the TRC has long been known in the Kähler case (see [1,25]). Before we state the conjecture, recall that a compact Lie group G (continuously) acts *almost freely* on a space X if all isotropy groups are finite. The *toral rank* of a space X, $\mathrm{rk}(X)$, is the dimension of the largest torus that can act almost freely on X.

Conjecture 3.1 *(Toral rank conjecture). If the toral rank of a space X is r, then*

$$\dim\, H^*(X;\mathbb{Q}) \geq 2^r.$$

The notation $\dim V$ means (total) dimension of V as a rational graded vector space. Our methods allow us to establish this conjecture for a large class of spaces, which (strictly) contains co-Kählerian manifolds. Furthermore we obtain a strong form of the conjecture; namely, we will show that, for our class of spaces, the rational cohomology algebra actually contains a "cohomological r-torus". Note that toral rank is a homeomorphism invariant, but is not a homotopy invariant. This suggests that we are getting at deeper topological qualities of co-Kähler manifolds than Betti numbers or even the full algebra structure of cohomology. We begin with some terminology.

Definition 3.2 *(Property B).* Say that a graded algebra H has **Property B** if, for any negative-degree derivation θ of H, we have

$$\theta\left(H^1\right) = 0 \Longrightarrow \theta(H) = 0.$$

We say that a space X has Property B if its (rational) cohomology algebra has Property B.

For example, any simply connected space whose rational cohomology algebra does not admit a non-zero, negative-degree derivation has Property B. Also, it is known that any cohomologically Kählerian space has Property B. This fact is due to Blanchard [6, Th. II.1.2], and this accounts for our choice of the letter B here. Of course, since the property is intrinsic to the cohomology algebra, any space with the same cohomology algebra has Property B. A main result about Property B spaces is the following (see for instance [17, Proposition 4.40, Theorem 4.36]).

Proposition 3.3. *Suppose $F \to E \to X$ is a fibration such that F satisfies Property B and X is simply connected. In the Leray–Serre spectral sequence, if $d_2(H^1(F;\mathbb{Q})) = 0$, then the spectral sequence collapses and $H^*(E;\mathbb{Q}) \cong H^*(X;\mathbb{Q}) \otimes H^*(F;\mathbb{Q})$ as $H^*(X;\mathbb{Q})$-modules. In particular, if F is cohomologically Kählerian and $d_2(H^1(F;\mathbb{Q})) = 0$, then the spectral sequence collapses.*

Now suppose that we have an action $T^r \times X \to X$, of an r-torus on a space X. Recall that we say the action is *homologically injective* if the orbit map $T^r \to X$ of the action induces an injection $H_1(T^r;\mathbb{Q}) \to H_1(X;\mathbb{Q})$ on first rational homology groups. This property of actions has been extensively studied (see, e.g. [12]). In [1], Allday and Puppe show that a cohomologically Kählerian space satisfies Conjecture 3.1. (In [25], this can be extended to spaces of Lefschetz type.) Distilling their argument a little reveals that it is really Property B that is the key, and not the cohomologically Kählerian structure, as such. In the following result, we extend the Allday–Puppe result by relaxing their hypothesis. Nonetheless, the basic argument, which we repeat here for the convenience of the reader, remains that of [1, Th. 2.2].

Theorem 3.4. *Let X be a space that satisfies Property B above. If an r-torus T^r acts almost freely on X, then the action is homologically injective, and we have*

$$H^*(X;\mathbb{Q}) \cong H^*(X_{T^r};\mathbb{Q}) \otimes H^*(T^r;\mathbb{Q})$$

as graded algebras, where $X_{T^r} = ET^r \times_{T^r} X$ is the Borel construction. In particular, we have

$$\dim\, H^*(X;\mathbb{Q}) \geq \dim\, H^*(T^r;\mathbb{Q}) = 2^r\,,$$

and thus X satisfies Conjecture 3.1.

Proof. Suppose that a torus $\mathbb{T} = T^r$ acts almost freely on the space X for some r. Let $E\mathbb{T} \to B\mathbb{T}$ be a universal principal $\mathbb{T}$-bundle and let $X_{\mathbb{T}} = (X \times E\mathbb{T})/\mathbb{T}$ be the Borel construction. Let $\{E_k^{p,q}\}$ be the rational cohomology Leray–Serre spectral sequence of $X \to X_{\mathbb{T}} \to B\mathbb{T}$ and let s be the rank of the linear map

$$d_2\colon E_2^{0,1} = H^1(X;\mathbb{Q}) \to E_2^{2,0} = H^2(B\mathbb{T};\mathbb{Q})\,.$$

Now, we can choose a basis for $H^*(B\mathbb{T};\mathbb{Q})$ so that $H^*(B\mathbb{T};\mathbb{Q}) \cong \mathbb{Q}[a_1,\ldots,a_r]$ with $|a_i| = 2$ for $i = 1,\ldots,r$ and $d_2(y_i) = a_i$, $i = 1,\ldots,s$ for $y_1,\ldots,y_s \in H^1(X;\mathbb{Q})$. Since d_2 is a derivation, we obtain

$$d_2(y_{i_1}\cdots y_{i_{j+1}}) = \sum_{\ell=1}^{j+1} \pm a_{i_\ell} \otimes y_{i_1}\cdots \hat{y}_{i_\ell}\cdots y_{i_{j+1}}.$$

By induction, using the algebraic independence of the a_j, we see that $y := y_1\cdots y_s$ must also be non-zero.

Suppose that $s < r$. By duality, the Hurewicz theorem and the fact that elements of $\pi_1(\mathbb{T})$ are realizable by homomorphisms from S^1, we can obtain a sub-torus $\mathbb{S} \subseteq \mathbb{T}$ which realizes the subalgebra $\langle a_1,\ldots,a_s\rangle$. Now, every sub-torus of a torus has a complement, so let $\mathbb{K} \subseteq \mathbb{T}$ be such that $\mathbb{T} = \mathbb{S} \times \mathbb{K}$. In particular, $\dim(\mathbb{K}) = r - s$. We then see that $\mathbb{K}$ is the sub-torus such that the ideal generated by the a_i, $i = 1,\ldots,s$ is the kernel of the projection in cohomology:

$$(a_1,\ldots,a_s) = \ker(H^*(B\mathbb{T};\mathbb{Q}) \to H^*(B\mathbb{K};\mathbb{Q}))\,.$$

We now restrict the action of $\mathbb{T}$ on X to $\mathbb{K}$ and note that it is also almost free. If we form the Borel fibration for the $\mathbb{K}$ action, then the Leray–Serre spectral sequence for $X_{\mathbb{T}}$ pulls back to that for $X_{\mathbb{K}}$. Then, because $\mathrm{Im}((d_2)_{\mathbb{T}}) \subseteq \ker(H^*(B\mathbb{T};\mathbb{Q}) \to H^*(B\mathbb{K};\mathbb{Q}))$, we have $(d_2)_{\mathbb{K}} = 0$ on $H^1(X;\mathbb{Q})$. But because X satisfies Property B, Proposition 3.3 guarantees that the spectral sequence collapses. However, this implies that $H^*(B\mathbb{K};\mathbb{Q}) \to H^*(X_{\mathbb{K}};\mathbb{Q})$ is injective and the Borel fixed point theorem then says that the fixed point set $X^{\mathbb{K}}$ is non-empty, contradicting the fact that $\mathbb{K}$ acts almost freely. Hence, $\mathbb{K}$ is trivial and $r = s$.

Thus we have $y_1,\ldots,y_r \in H^*(X;\mathbb{Q})$ which generate an exterior algebra. In fact, stepping back in the Barratt–Puppe sequence to the fibration $\mathbb{T} \to X \to X_{\mathbb{T}}$, we see that $\langle y_1,\ldots,y_r\rangle$ maps onto $H^*(\mathbb{T};\mathbb{Q})$.

Therefore this spectral sequence collapses and $H^*(X;\mathbb{Q}) \cong H^*(X_{\mathbb{T}};\mathbb{Q}) \otimes H^*(\mathbb{T};\mathbb{Q})$. Thus, $\dim(H^*(X;\mathbb{Q})) \geq \dim(H^*(\mathbb{T};\mathbb{Q})) = 2^r$. □

Next, we show that the class of graded algebras that satisfy Property B is closed under tensor products.

Proposition 3.5. *If H and G are graded algebras that satisfy Property B, then so too $H \otimes G$ satisfies Property B.*

Proof. Suppose H and G have Property B, and that $\theta\colon H \otimes G \to H \otimes G$ is a negative degree derivation that vanishes on $(H \otimes G)^1 = H^1 \otimes 1 + 1 \otimes G^1$. We wish to show that θ must be zero.

First, we show that θ vanishes on H. For suppose that $\theta(H) \neq= 0$, and let $k \geq 0$ be the smallest integer for which $\theta(H) \cap H \otimes G^k \neq= 0$. Take any $\chi \in H$, and write $\theta(\chi) = \theta_k(\chi) + \theta_{k+1}(\chi)$, with $\theta_k(\chi) \in H \otimes G^k$ and $\theta_{k+1}(\chi) \in I(G^{\geq k+1})$, the ideal of $H \otimes G$ generated by elements of G of degree $k+1$ or greater. Further, suppose that we have a basis $\{g^i\}$ of G^k. Then we may write $\theta_k(\chi) = \sum_i \theta_k^i(\chi) \otimes g^i$. This defines linear maps $\theta_k^i\colon H \to H$, of negative degree — in fact of degree equal to $|\theta| - k$. It is straightforward to check that each θ_k^i is a derivation of H, so $\theta_k^i(\chi) = 0$, for each $\chi \in H^1$, by the assumption that H has Property B. But this implies that we have $\theta(H) \subseteq H \otimes G^{\geq k+1}$, which contradicts our assumption on k. Therefore, we must have $\theta(H) = 0$. The same argument, with H and G interchanged, gives that θ must vanish on G. Hence, $\theta = 0$ and $H \otimes G$ has Property B. □

Corollary 3.6. *If M is a compact co-Kähler manifold, then it satisfies the Toral Rank Conjecture.*

Proof. By Theorem 2.1, $H^*(M;\mathbb{R}) = H^*(K;\mathbb{R})^G \otimes H^*(S^1;\mathbb{R})$ and by Proposition 2.3, $H^*(K;\mathbb{R})^G$ is Kählerian and has Property B. Clearly $H^*(S^1;\mathbb{R})$ has Property B for degree reasons. Hence, by Proposition 3.5, $H^*(M;\mathbb{R})$ has Property B. Now apply Theorem 3.4. □

This result points out again that properties of co-Kähler manifolds often derive from properties of the constituent Kähler manifold. Also note that, by [3], a co-Kähler manifold always has toral rank at least equal to one. Note that we also have the following result, where the 1 is added to account for the S^1 factor in cohomology.

Corollary 3.7. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold with presentation (K, G) so that $M = (K \times S^1)/G$. Then*

$$\mathrm{rk}(M) \leq \tilde{\alpha}_1(K) + 1\,,$$

where $\tilde{\alpha}_1(K)$ is the maximal number of algebraically independent elements in $H^1(K;\mathbb{Q})$ which are fixed by the induced G-action on $H^1(K;\mathbb{Q})$.

By the Myers–Steenrod theorem [27], the isometry group $\mathrm{Isom}(M, g)$ of a compact Riemannian manifold is a compact Lie group. As a consequence, it is observed in [3] that, when (M, J, ξ, η, g) is compact co-Kähler, the closure of the Reeb flow in $\mathrm{Isom}(M, g)$ is a compact torus T, which acts almost freely on M. Therefore M is endowed with an almost free torus action.

Corollary 3.8. *Let (M, J, ξ, η, g) be a compact co-Kähler manifold and assume that $M = (K \times S^1)/G$ for a Kähler manifold K. Let $T \subset \mathrm{Isom}(M, g)$ be the closure of the Reeb flow in the isometry group of M. Then*

$$\dim(T) \leq \tilde{\alpha}_1(K) + 1\,.$$

When $b_1(M) = 1$, then $\dim(T) = 1$ and the Reeb flow generates a homologically injective circle action on M.

Proof. The torus T acts almost freely on M, hence $\dim(T) \le \mathrm{rk}(M)$. By Corollary 3.7, we get $\dim(T) \le \mathrm{rk}(M) \le \tilde{\alpha}_1(K) + 1$. When $b_1(M) = 1$, we have $H^1(K;\mathbb{Q})^G = 0$ by Theorem 2.1, so the group G fixes no element on $H^1(K;\mathbb{Q})$. Therefore $\tilde{\alpha}_1(K) = 0$ and $\mathrm{rk}(M) \le 1$. Notice that $\dim(T) \ge 1$, since the flow of the Reeb vector field ξ generates at least a circle in $\mathrm{Isom}(M, g)$. Therefore we get $1 \le \dim(T) \le \mathrm{rk}(M) \le 1$, and $T = S^1$, hence ξ generates a circle action, which is homologically injective by the argument given in [3, Section 2]. □

Examples 3.9.

(1) As already observed, any cohomologically Kählerian space satisfies Property B.
(2) Any algebra generated in degree 1 (tautologically) satisfies Property B. In particular, this remark applies to $H^*(T^r;\mathbb{Q})$ for each $r \ge 1$. Note that these algebras are cohomologically Kählerian only for even r.
(3) Therefore, if H is cohomologically Kählerian and G is generated in degree 1, then the tensor product $H \otimes G$ satisfies Property B.
(4) Suppose that H is a finite-dimensional complete intersection, i.e., generated by even-degree generators with ideal of relations generated by a (maximal length) regular sequence. Another long-standing, open conjecture due to Halperin is that any such algebra does not admit any non-zero negative-degree derivation. This conjecture has been established in many cases. If H is any such algebra for which this conjecture is true, then tensor products of the form $H \otimes G$, with G generated in degree 1 (or, more generally, any algebra with Property B) satisfy Property B.

Note that these examples include many that are neither cohomologically Kählerian, nor finite-dimensional complete intersections.

Remark 3.10. If X and Y are spaces that satisfy Property B, then by Theorem 3.4 each satisfies Conjecture 3.1, and by Proposition 3.5 and once again Theorem 3.4, their product $X \times Y$ also satisfies Conjecture 3.1. In this way, we are able to generate spaces that are products, and that satisfy Conjecture 3.1. It is worth emphasizing that, in general, the toral rank—the maximum rank of a torus that may act almost freely — may not behave well with respect to products. In [23], an example is given of a product $X \times Y$ that admits a free circle action, and yet neither X nor Y admit an almost-free circle action. Generally, therefore, the toral rank does not behave in a “sub additive” way with respect to products. This means, in particular, that as yet there is no *a priori* reason to conclude $X \times Y$ satisfies Conjecture 3.1, simply because X and Y do.

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