1-1-1989

Asymptotic Analysis of Model Problems for a Coupled System

F. A. Howes
University of California, Davis

S. Shao
Cleveland State University, s.shao@csuohio.edu

ASYMPTOTIC ANALYSIS OF MODEL PROBLEMS FOR A COUPLED SYSTEM

F. A. Howes and S. Shao

Key words and phrases: Boundary layer, shock layer, Dirichlet problem.

1. INTRODUCTION

We present in this paper an analysis of the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the coupled system of equations on (0, 1)

$$\begin{aligned} u'' &= v \\ \varepsilon v'' + f(u, u')v' &= 0, \end{aligned} \tag{1.1}$$

for $f(u, u') = u$ or u', satisfying the Dirichlet boundary conditions $u(0, \varepsilon) = u(1, \varepsilon) = 0$ and $v(0, \varepsilon) = v_0$, $v(1, \varepsilon) = v_1$. Such a system arises naturally in seeking simple models whose solutions have the qualitative properties of solutions of the more complicated system of partial differential equations

$$\left.\begin{aligned} \nabla^2\psi := \psi_{xx} + \psi_{yy} &= -\omega \\ (1/Re)\nabla^2\omega - \psi_y\omega_x + \psi_x\omega_y &= 0, \end{aligned}\right\} \quad (x, y) \text{ in } \Omega, \tag{1.2}$$

that govern the steady-state distribution of the vorticity ω in the limit of infinite Reynolds number Re; cf. for example [1, Chapter 5; 10, Chapter 7]. Here ψ is the streamfunction associated with the two-dimensional velocity field $u := (u_1, u_2, 0)$, that is, $u_1 = \psi_y$ and $u_2 = -\psi_x$, and Ω is a bounded, open subset of the plane. It was this motivation that led Parter and his co-workers (cf. [5, 6]) to study the system (1.1) for certain choices of the boundary values v_0 and v_1 by a judicious use of the maximum principle and *a priori* estimates. Unfortunately their elegant and useful treatment has been overlooked by most workers in asymptotic analysis who are now attempting to study even more challenging systems of singularly perturbed differential equations and by most, if not all, fluid dynamicists who study viscous flow at large Reynolds number. In this paper we approach (1.1) more from the point of view of asymptotic analysis than did Parter *et al.*, in the hope that the techniques displayed may prove useful in the study of more complicated systems of ordinary and partial differential equations, such as (1.2).

The outline of the paper is as follows. In Section 2 we describe the results from singular perturbation theory and the maximum principle that are needed in the discussion of (1.1). The next section contains a discussion of the problem $u'' = v$, $\varepsilon v'' + uv' = 0$, while the last section, Section 4, contains a discussion of the problem $u'' = v$, $\varepsilon v'' + u'v' = 0$.

2. ELEMENTARY ESTIMATES

For the convenience of the reader we collect in this section some elementary results from the theory of singularly perturbed boundary value problems and the theory of *a priori* estimates.

Let us consider first the linear problem on (0, 1)

$$\varepsilon w'' + g(x)w' = 0, \qquad w(0, \varepsilon) = A, \qquad w(1, \varepsilon) = B, \tag{2.1}$$

where g is a continuous function on [0, 1]. On any closed subinterval of (0, 1) the limit of $w(x, \varepsilon)$ is a constant whose value depends upon the function g and the boundary values A, B, since the reduced equation $w' = 0$ has only constant solutions. If there is a constant k such that $g(x) \geq k > 0$ in [0, 1], then as $\varepsilon \to 0^+$ the solution of (2.1) satisfies

$$w(x, \varepsilon) = B + \mathcal{O}(|A - B| \exp[-kx/\varepsilon]) \qquad \text{in } [0, 1], \tag{2.2}$$

that is, there is a boundary layer of width $\mathcal{O}(\varepsilon)$ at $x = 0$. While if $g(x) \leq -k < 0$ in [0, 1], then there is a boundary layer of width $\mathcal{O}(\varepsilon)$ at $x = 1$; to wit,

$$w(x, \varepsilon) = A + \mathcal{O}(|B - A| \exp[-k(1 - x)/\varepsilon]) \qquad \text{in } [0, 1]. \tag{2.3}$$

These results are classical; cf. for example [8, Chapter 3; 3, Chapter 4] or simply integrate the constant coefficient equations $\varepsilon w'' \pm kw' = 0$. In other words, if $g(x) < 0$ near $x = 0$ then there can be no boundary layer at $x = 0$, and if $g(x) > 0$ near $x = 1$ there is likewise no boundary layer at $x = 1$. What happens (or does not happen!) if g has a zero in [0, 1]? The answer here is not as simple. If $g(x) > 0$ in (0, 1] and $g(0) = 0$, then a relation like (2.2) still obtains, only with the exponential term replaced by a more complicated function (cf. (3.5) below), that is, there is a boundary layer at $x = 0$ but its structure is more involved and it may be thicker than $\mathcal{O}(\varepsilon)$. Similarly, if $g(x) < 0$ in [0, 1) and $g(1) = 0$, then there is a boundary layer at $x = 1$ more complicated than the one described by (2.3). These results can be found in [8, Chapter 8; 3, Chapter 4], or simply integrate an equation like $\varepsilon w' + x^n w' = 0$, n a natural number.

Suppose now that g has a single interior zero, say $g(x_0) = 0$, $0 < x_0 < 1$, with $g'(x_0) \neq 0$. Then there are two cases, depending upon whether $g'(x_0) < 0$ or $g'(x_0) > 0$. The first case is illustrated by setting $g(x) = x_0 - x$, that is, $g(x) > 0$ for $0 \leq x < x_0$, $g(x_0) = 0$ and $g(x) < 0$ for $x_0 < x \leq 1$, and taking $A < 0 < B$. Then $w'(x, \varepsilon) > 0$ in [0, 1] ($w'(x, \varepsilon) = \text{const} \cdot \exp[(x^2 - 2xx_0)/(2\varepsilon)]$), and so we can rewrite the equation in (2.1) as $\varepsilon w''/w' = x - x_0$, that is, $\varepsilon\,(\ln w'(x, \varepsilon))' = x - x_0$ or

$$\varepsilon \ln(w'(1, \varepsilon)/w'(0, \varepsilon)) = 1/2 - x_0. \tag{2.4}$$

There are then three subcases. If $0 < x_0 < 1/2$ then (2.4) tells us that $w'(1, \varepsilon) > w'(0, \varepsilon)$, and it follows that the solution of (2.1) has a boundary layer at $x = 1$ as $\varepsilon \to 0^+$ (cf. (2.3) with $k := 1 - x_0$). If $1/2 < x_0 < 1$ then (2.4) implies that $w'(1, \varepsilon) < w'(0, \varepsilon)$, and so the solution of (2.1) has a boundary layer at $x = 0$ as $\varepsilon \to 0^+$ (cf. (2.2) with $k := x_0$). Finally, if $x_0 = 1/2$ then (2.4) implies that $w'(0, \varepsilon) = w'(1, \varepsilon)$. In order to see how w behaves in this case, let us tentatively set

$$w(x, \varepsilon) \sim c + (A - c) \exp[-x/(2\varepsilon)] + (B - c) \exp[-(1 - x)/(2\varepsilon)], \tag{2.5}$$

where the limiting value c is to be determined. It follows that $w'(0, \varepsilon) = w'(1, \varepsilon)$ as $\varepsilon \to 0^+$ if and only if $-(A - c)/(2\varepsilon) = (B - c)/(2\varepsilon)$, that is, c must be equal to $(A + B)/2$, the *average* of the boundary values. These results for $g(x) = x_0 - x$ are likewise classical (cf. [4]).

In order to illustrate the case $g'(x_0) > 0$ we set $g(x) = x - x_0$, that is, $g(x) < 0$ for $0 \leq x < x_0$, $g(x_0) = 0$ and $g(x) > 0$ for $x_0 < x \leq 1$. With such a g we can rule out immediately the occurrence of boundary layer behavior at either endpoint, since g has the "wrong" sign near $x = 0$ and $x = 1$. The only asymptotic behavior available to the solution of (2.1) as $\varepsilon \to 0^+$ is then interior (shock) layer behavior of the form

$$\lim_{\varepsilon \to 0^+} w(x, \varepsilon) = \begin{cases} A, 0 \leq x \leq x_0 - \delta, \\ B, x_0 + \delta \leq x \leq 1, \end{cases}$$

for δ a small positive constant. The interior layer at x_0 can be viewed as a two-sided "boundary layer" centered at x_0 across which there is a change in the convexity of w. Note that the sign of g is compatible with such a layer since $g(x) < 0$ for $x < x_0$ (cf. (2.3)) and $g(x) > 0$ for $x > x_0$ (cf. (2.2)).

Other forms of asymptotic behavior are displayed by solutions of (2.1), depending upon the nature of g; however, the ones just described are sufficient for our purposes here. We turn finally to a description of *a priori* estimates in the study of the problem

$$\begin{aligned} u'' &= v,\ u(0, \varepsilon) = 0,\ u(1, \varepsilon) = 0, \\ \varepsilon v'' + f(u, u')v' &= 0,\ v(0, \varepsilon) = v_0,\ v(1, \varepsilon) = v_1. \end{aligned} \tag{1.1}$$

The basic result is the following. If there exist smooth functions $\alpha_i, \beta_i (i = 1, 2)$ such that $\alpha_i \leq \beta_i$, $\alpha_1(0, \varepsilon) \leq 0 \leq \beta_1(0, \varepsilon)$, $\alpha_1(1, \varepsilon) \leq 0 \leq \beta_1(1, \varepsilon)$, $\alpha_2(0, \varepsilon) \leq v_0 \leq \beta_2(0, \varepsilon)$, $\alpha_2(1, \varepsilon) \leq v_1 \leq \beta_2(1, \varepsilon)$, and for x in $(0, 1)$

$$\begin{aligned} \alpha_1'' &\geq \beta_2, & \beta_1'' &\leq \alpha_2 \\ \varepsilon\alpha_2'' + f(u, z)\alpha_2' &\geq 0, & \varepsilon\beta_2'' + f(u, z)\beta_2' &\leq 0, \end{aligned}$$

for all u in $[\alpha_1, \beta_1]$ and z in $\mathbb{R}$, then the problem (1.1) has a (unique) solution $(u, v) = (u, v)(x, \varepsilon)$ such that for x in $[0, 1]$

$$\begin{aligned} \alpha_1(x, \varepsilon) &\leq u(x, \varepsilon) \leq \beta_1(x, \varepsilon) \\ \alpha_2(x, \varepsilon) &\leq v(x, \varepsilon) \leq \beta_2(x, \varepsilon). \end{aligned} \tag{2.6}$$

The existence and uniqueness of the solution were proved by Dorr and Parter [5], while the estimates (2.6) follow as in [2, Chapter 1]. An easy consequence of (2.6) is contained in the next result, which is obtained by setting $\alpha_1(x, \varepsilon) := M(x^2 - x)/2$, $\beta_1(x, \varepsilon) := m(x^2 - x)/2$, $\alpha_2(x, \varepsilon) := m$ and $\beta_2(x, \varepsilon) := M$, for $m := \min\{v_0, v_1\}$ and $M := \max\{v_0, v_1\}$.

LEMMA 2.1. The solution $(u(x, \varepsilon), v(x, \varepsilon))$ of problem (1.1) satisfies the estimates

$$\begin{aligned} M(x^2 - x)/2 \leq u(x, \varepsilon) &\leq m(x^2 - x)/2 \\ m \leq v(x, \varepsilon) &\leq M, \end{aligned} \tag{2.7}$$

for x in $[0, 1]$ and all $\varepsilon > 0$.

Armed with these estimates and the asymptotic theory, we proceed to study the asymptotic behavior of $(u(x, \varepsilon), v(x, \varepsilon))$ as $\varepsilon \to 0^+$ in the two representative cases $f := u$ and $f := u'$.

3. THE FIRST CASE

The first problem is

$$\text{(E1)}\qquad \begin{aligned} u'' &= v,\ u(0, \varepsilon) = u(1, \varepsilon) = 0, \\ \varepsilon v'' + uv' &= 0,\ v(0, \varepsilon) = v_0,\ v(1, \varepsilon) = v_1, \end{aligned}$$

which illustrates the effect of a driving term like $u(x, \varepsilon)$ itself on the behavior of solutions of the perturbed equation for v. It was first considered by Dorr and Parter in [5] in the case when the boundary values v_0 and v_1 have the same sign. Our discussion of (E1) divides naturally into four cases, depending upon the signs of v_0 and v_1.

Case 1. $v_0 \geq 0$, $v_1 \geq 0$. Lemma 2.1 tells us that for x in $[0, 1]$

$$M(x^2 - x)/2 \leq u(x, \varepsilon) \leq m(x^2 - x)/2, \tag{3.1}$$

where $M := \max\{v_0, v_1\}$ and $m := \min\{v_0, v_1\}$. Thus, if $v_0 = v_1 = 0$ then $u \equiv 0$, $v \equiv 0$ is the solution of (E1). If $v_0 > 0$ and $v_1 > 0$ then (3.1) implies that $u(x, \varepsilon) < 0$ in $(0, 1)$, and we anticipate the occurrence of a boundary layer in v at $x = 1$, with $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = v_0$ for $0 \leq x \leq 1 - \delta$, in view of the discussion in Section 2. Finally, if $v_0 = 0$ or $v_1 = 0$ then $u \leq 0$ in $[0, 1]$ but $u \not\equiv 0$ since $v \not\equiv$ constant, and so we again look for a boundary layer in v at $x = 1$, with $v(x, \varepsilon) \to v_0$ in $[0, 1 - \delta]$. Now in a neighborhood of $x = 1$, $u(x, \varepsilon) \sim l(x^2 - x)/2 = (l/2)x(x - 1) \sim (l/2)(x - 1)$ [where $l := v_0$ if $v_0 > 0$ and $l := v_1$ if $v_0 = 0$], that is, to lowest order in ε, the equation for v in the boundary layer near $x = 1$ reduces to $\varepsilon v'' + (l/2)(x - 1)v' = 0$. Introducing the stretched variable $\xi := (l/2\varepsilon)^{1/2}(x - 1)$ into this equation and setting $\tilde{v}(\xi, \varepsilon) := v(1 + \xi(l/2\varepsilon)^{-1/2}, \varepsilon)$ give us finally the scaled boundary layer equation

$$\ddot{\tilde{v}} + \xi\dot{\tilde{v}} = 0 \quad [\cdot := \mathrm{d}/\mathrm{d}\xi] \tag{3.2}$$

together with the boundary condition

$$\tilde{v}(0, \varepsilon) = v_1 \tag{3.3}$$

and the matching condition

$$\lim_{\xi \to -\infty} \tilde{v}(\xi, \varepsilon) = 0 \text{ (with } \varepsilon > 0 \text{ fixed).} \tag{3.4}$$

The solution of (3.2)-(3.4) is found to be

$$\tilde{v}(\xi, \varepsilon) = v_1(1 + \operatorname{erf}[\xi/2^{1/2}]),$$

where $\operatorname{erf}[z] := (2/\pi^{1/2}) \int_0^z \exp[-s^2]\, \mathrm{d}s$ is the usual error function satisfying $\lim_{z \to -\infty} \operatorname{erf}[z] = -1$. Thus, to lowest order in ε, the solution of the v-problem is

$$v(x, \varepsilon) \sim v_0 + v_1(1 + \operatorname{erf}[(l/4)^{1/2}(x - 1)/\varepsilon^{1/2}]), \qquad 0 \leq x \leq 1, \tag{3.5}$$

revealing clearly that the thickness of the boundary layer at $x = 1$ is of order $\varepsilon^{1/2}$.

Case 2. $v_0 \le 0$, $v_1 \le 0$. Fortunately we can reduce this case to the previous one by making the changes of variable $r := 1 - x$, $\hat{u}(r, \varepsilon) := -u(1 - r, \varepsilon)$ and $\hat{v}(r, \varepsilon) := -v(1 - r, \varepsilon)$, resulting in the new system

$$\hat{u}'' = \hat{v}, \qquad \hat{u}(0, \varepsilon) = \hat{u}(1, \varepsilon) = 0,$$

$$\varepsilon\hat{v}'' + \hat{u}\hat{v}' = 0, \qquad \hat{v}(0, \varepsilon) = \hat{v}_0 := -v_1, \qquad \hat{v}(1, \varepsilon) = \hat{v}_1 := -v_0,$$

where now $' := \mathrm{d}/\mathrm{d}r$. Since $\hat{v}_0 \ge 0$ and $\hat{v}_1 \ge 0$ we know that the solution of the $\hat{v}$-problem satisfies (cf. (3.5))

$$\hat{v}(r, \varepsilon) \sim \hat{v}_0 + \hat{v}_1(1 + \operatorname{erf}[(\hat{l}/4)^{1/2}(r - 1)/\varepsilon^{1/2}],$$

for $l := -v_1$ if $v_1 < 0$ and $\hat{l} := -v_0$ if $v_1 = 0$, that is, the solution of the original v-problem satisfies

$$v(x, \varepsilon) \sim v_1 + v_0(1 + \operatorname{erf}[-(\hat{l}/4)^{1/2}x/\varepsilon^{1/2}]) \qquad \text{in } [0, 1].$$

In other words, for $v_0 < 0$ and/or $v_1 < 0$ there is a boundary layer at $x = 0$ whose thickness is of order $\varepsilon^{1/2}$ and $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = v_1$ for $0 < \delta \le x \le 1$.

Case 3. $v_0 > 0$, $v_1 < 0$. In order to study the solution of (E1) in this case we note that $v'(x, \varepsilon) < 0$ in $[0, 1]$ and $\varepsilon v''/v' = -u$, that is,

$$\varepsilon \ln(|v'(1, \varepsilon)|/|v'(0, \varepsilon)|) = -\int_0^1 u(s, \varepsilon)\, \mathrm{d}s. \tag{3.6}$$

We proceed with the aid of (3.6) to show what types of asymptotic behavior the solution $v(x, \varepsilon)$ cannot display as $\varepsilon \to 0^+$. First of all, we see that

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) \ne c \qquad \text{in } (0, 1)$$

for c a nonzero constant in the interval (v_1, v_0). This follows because if $v(x, \varepsilon) \to c$ then $|v'(0, \varepsilon)| = O(\varepsilon^{-1/2})$ and $|v'(1, \varepsilon)| = O(\varepsilon^{-1/2})$ (cf. (3.5)), that is, $\lim_{\varepsilon \to 0^+} \varepsilon \ln(|v'(1, \varepsilon)|/|v'(0, \varepsilon)|) = 0$; however, $\lim_{\varepsilon \to 0^+} (-\int_0^1 u(s, \varepsilon)\, \mathrm{d}s) = c/12 \ne 0$, in contradiction of (3.6). The latter relation follows because $u'' = v \sim c$ in $(0, 1)$ and $u(0, \varepsilon) = u(1, \varepsilon) = 0$, that is, $u(x, \varepsilon) \sim c(x^2 - x)/2$. It is also not possible that $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = 0$ in $(0, 1)$. The reason is that in this case $v(x, \varepsilon) > 0$ for x near 0 and $v(x, \varepsilon) < 0$ for x near 1; whence, $u''(= v)$ is positive near $x = 0$ and negative near $x = 1$, and so $u(x, \varepsilon)$ is negative near 0 and positive near 1. Thus the sign of u does not allow boundary layer behavior in v at either endpoint. Finally, we can eliminate the limits $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = v_0$ in $[0, 1)$ and $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = v_1$ in $(0, 1]$, on the grounds that again the sign of $u(x, \varepsilon)$ near $x = 1$ or $x = 0$ does not permit boundary layer behavior (cf. Section 2).

The only type of asymptotic behavior in v that we have not excluded is then interior (shock) layer behavior, wherein

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = \begin{cases} v_0, 0 \le x \le x_0 - \delta, \\ v_1, x_0 + \delta \le x \le 1, \end{cases}$$

for some point x_0 in $(0, 1)$. To see why such behavior is permissible, let us examine the sign of $u(x, \varepsilon)$ for $x < x_0$ and $x > x_0$. For $x < x_0\, v(x, \varepsilon) \sim v_0 > 0$ and so $u'' \sim v_0, 0 < x < x_0$, with

$u(0, \varepsilon) = 0$ and $u(x_0, \varepsilon) \sim 0$, that is, $u(x, \varepsilon) \sim v_0 x(x - x_0)/2 \le 0$ as $\varepsilon \to 0^+$, while for $x > x_0$ $v(x, \varepsilon) \sim v_1 < 0$ and so $u(x, \varepsilon) \sim -v_1(1 - x)(x - x_0)/2 \ge 0$ as $\varepsilon \to 0^+$. Consequently $u(x, \varepsilon)$ passes through zero at $x = x_0$ like a positive multiple of $x - x_0$, making an interior layer in v at x_0 possible (cf. Section 2). It only remains to determine the location x_0 of the layer. Again we make use of the relation (3.6). Since $v'(0, \varepsilon)$ and $v'(1, \varepsilon)$ are of the same size as $\varepsilon \to 0^+$ (there are no boundary layers!), we know that $\lim_{\varepsilon \to 0^+} \varepsilon \ln(|v'(1, \varepsilon)|/|v'(0, \varepsilon)|) = 0$, and so (3.6) implies that

$$\lim_{\varepsilon \to 0^+} \int_0^1 u(s, \varepsilon)\, ds = 0. \tag{3.7}$$

But we just saw that as $\varepsilon \to 0^+$ $u(x, \varepsilon) \sim v_0 x(x - x_0)/2$ in $[0, x_0]$ and $u(x, \varepsilon) \sim -v_1(1 - x) \times (x - x_0)/2$ in $[x_0, 1]$, and so the relation (3.7), in conjunction with the dominated convergence theorem (cf. [9; Chapter 1]), implies that

$$\int_0^{x_0} [\lim_{\varepsilon \to 0^+} u(s, \varepsilon)]\, dx + \int_{x_0}^1 [\lim_{\varepsilon \to 0^+} u(s, \varepsilon)]\, ds = 0,$$

that is,

$$(v_0/2) \int_0^{x_0} s(s - x_0)\, ds = -(v_1/2) \int_{x_0}^1 (s - 1)(s - x_0)\, ds$$

or $v_0 x_0^3 = -v_1(1 - x_0)^3$. Therefore, the interior layer is located at

$$x_0 := (-v_1)^{1/3}/[(-v_1)^{1/3} + v_0^{1/3}].$$

We can now proceed as in case 1 and describe this layer in terms of an error function centered at this value of x_0.

Case 4. $v_0 < 0$, $v_1 > 0$. Taking a cue from the previous case we can eliminate all possible types of limiting behavior for the solution $v(x, \varepsilon)$, except the correct one. First of all, we note that interior layer behavior is not possible here. To see this, suppose that for some x_1 in $(0, 1)$

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = \begin{cases} v_0, & 0 \le x \le x_1 - \delta, \\ v_1, & x_1 + \delta \le x \le 1. \end{cases}$$

Then, as in case 3, we have $u(x, \varepsilon) \sim -v_0 x(x_1 - x)/2 \ge 0$ in $[0, x_1]$ and $u(x, \varepsilon) \sim -v_1(1 - x)(x - x_1)/2 \le 0$ in $[x_1, 1]$ as $\varepsilon \to 0^+$, that is, $u(x, \varepsilon)$ passes through zero at x_1 like a positive multiple of $-(x-x_1)$. Thus the signs of u near x_1 are incompatible with the existence of an interior layer there (cf. Section 2). It is also not possible that $v(x, \varepsilon) \to v_0$ in $[0, 1)$ or $v(x, \varepsilon) \to v_1$ in $(0, 1]$ as $\varepsilon \to 0^+$. To see this suppose, for example, that the former relation obtains. Then $v'(0, \varepsilon) \sim 0$, $v'(0, \varepsilon) \ge 0$ and $v'(1, \varepsilon) = O(\varepsilon^{-1/2})$ as $\varepsilon \to 0^+$; consequently, $\lim_{\varepsilon \to 0^+} [\varepsilon \ln v'(1, \varepsilon) - \varepsilon \ln v'(0, \varepsilon)] = \lim_{\varepsilon \to 0^+} (-\varepsilon \ln v'(0, \varepsilon)) > 0$. However, $\lim_{\varepsilon \to 0^+} (-\int_0^1 u(s, \varepsilon))\, ds = -\int_0^1 (v_0/2)(s^2 - s)\, ds = v_0/12 < 0$, in contradiction of the relation (3.6). A similar argument eliminates the latter relation as well. Finally we can eliminate the possibility that $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = c$ in $(0, 1)$, for c a nonzero constant in the interval (v_0, v_1). For should this limit be obtained, then $v'(0, \varepsilon) = O(\varepsilon^{-1/2})$ and $v'(1, \varepsilon) = O(\varepsilon^{-1/2})$ as $\varepsilon \to 0^+$ (there are boundary layers of width $O(\varepsilon^{1/2})$ at the endpoints!), and so $\lim_{\varepsilon \to 0^+} \varepsilon \ln(v'(1, \varepsilon)/v'(0, \varepsilon)) = 0$. However, we see that $\lim_{\varepsilon \to 0^+} (-\int_0^1 u(s, \varepsilon)\, ds) = -\int_0^1 (c/2)(s^2 - s)\, ds = c/12 \neq 0$, in contradiction of (3.6).

This leaves us with the only possible type of limiting behavior that has not been excluded, namely

$$\lim_{\varepsilon \to 0+} v(x, \varepsilon) = 0 \qquad \text{for } x \text{ in } [\delta, 1 - \delta].$$

To see why this limit is permissible, note that $v'(0, \varepsilon)$ and $v'(1, \varepsilon)$ are the same size as $\varepsilon \to 0^+$; consequently, $\lim_{\varepsilon \to 0^+} \varepsilon \ln (v'(1, \varepsilon)/v'(0, \varepsilon)) = 0$ as before. But now $u(x, \varepsilon) \sim 0$ in $[0, 1]$ as $\varepsilon \to 0^+$, since $u'' = v \sim 0$ in $(0, 1)$, and so $\lim_{\varepsilon \to 0^+} (-\int_0^1 u(s, \varepsilon)\, ds) = 0$, in conformity with the relation (3.6). The sign of $u(x, \varepsilon)$ near $x = 0$ and $x = 1$ is also compatible with the existence of boundary layers in v there.

We can summarize our results for (E1) in the form of a boundary value portrait (cf. Fig. 1), where the abbreviation $BL(x)[IL(x)]$ denotes the term "boundary layer at x" ["interior layer at x"].

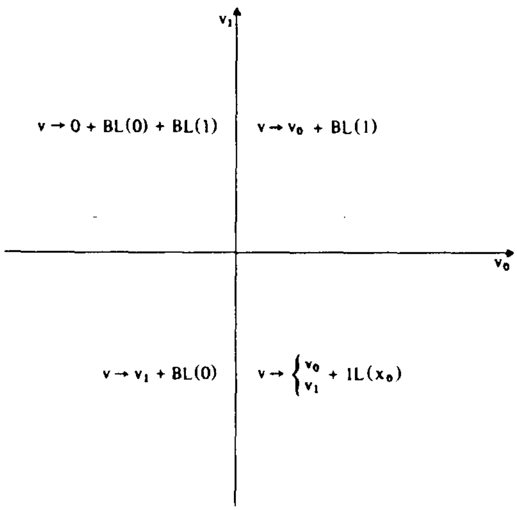

Fig. 1.

4. THE SECOND CASE

We take as our second problem

$$\text{(E2)} \qquad \begin{aligned} u'' &= v, \qquad u(0, \varepsilon) = u(1, \varepsilon) = 0, \\ \varepsilon v'' + u'v' &= 0, \qquad v(0, \varepsilon) = v_0, \qquad v(1, \varepsilon) = v_1, \end{aligned}$$

in order to illustrate the effect of a driving term like $u'(x, \varepsilon)$ on the behavior of solutions of the perturbed equation for v. It was first studied by Parter *et al.* [5, 6] for all choices of the boundary values v_0, v_1. Fortunately there is an immediate observation that simplifies dramatically our consideration of the different types of asymptotic behavior available to the function v; namely, for all values of $\varepsilon > 0$

$$v'(0, \varepsilon) = v'(1, \varepsilon). \tag{4.1}$$

If $v_0 = v_1$ then this result is trivial, while if $v_0 \neq v_1$ then $v'(x, \varepsilon) > 0$, $[v'(x, \varepsilon) < 0]$ in $[0, 1]$ if $v_0 < v_1 [v_0 > v_1]$. Therefore $\varepsilon v''/v' = -u'$, that is, $\varepsilon \ln(v'(1, \varepsilon)/v'(0, \varepsilon)) = -\int_0^1 u'(s, \varepsilon)\, ds = u(0, \varepsilon) - u(1, \varepsilon) = 0$, leading directly to (4.1). An important consequence of this observation is the fact that if the solution $v(x, \varepsilon)$ displays boundary layer behavior it must have boundary layers at *both* endpoints. The discussion of (E2) now proceeds by examining the following four cases.

Case 1. $v_0 \geq 0$, $v_1 \geq 0$. As in case 1 of Section 3 we utilize the *a priori* estimate on u provided by lemma 2.1

$$M(x^2 - x)/2 \leq u(x, \varepsilon) \leq m(x^2 - x)/2, \qquad 0 \leq x \leq 1, \tag{4.2}$$

where $M := \max\{v_0, v_1\}$ and $m := \min\{v_0, v_1\}$. Thus if $v_0 = v_1 = 0$ then $u \equiv 0$, $v \equiv 0$ is the solution of (E2). The two subcases $v_0 > 0$, $v_1 > 0$ and $v_0 = 0$ or $v_1 = 0$ will be considered separately. First of all, if $v_0 > 0$ and $v_1 > 0$ then $u(x, \varepsilon) < 0$ in $(0, 1)$ *and* $u'(x, \varepsilon) < 0$ near $x = 0$, while $u'(x, \varepsilon) > 0$ near $x = 1$. Consequently the sign of $u'(x, \varepsilon)$ at the endpoints does *not* allow v to have boundary layers. But $v'(0, \varepsilon) = v'(1, \varepsilon)$ by virtue of (4.1) and so if $v_0 \neq v_1$, the only possible asymptotic behavior available to v is interior (shock) layer behavior at some point x_0 in $(0, 1)$, that is,

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = \begin{cases} v_0, 0 \leq x \leq x_0 - \delta, \\ v_1, x_0 + \delta \leq x \leq 1. \end{cases}$$

Such behavior is indeed possible by virtue of the fact that if $u'' \sim v_0$ for $x < x_0$ and $u'' \sim v_1$ for $x > x_0$ then $u'(x, \varepsilon) \sim v_0(x - x_0)$ for $x < x_0$ and $u'(x, \varepsilon) \sim v_1(x - x_0)$ for $x > x_0$, that is, in a neighborhood of x_0, the v-equation reduces to the equation $\varepsilon v'' + k(x - x_0)v' = 0$, for k a positive constant. It remains only to determine the location x_0 of the interior layer. To this end, we note that if $u(x, \varepsilon) \sim U(x)$ as $\varepsilon \to 0^+$ for $x \neq x_0$, then $U(x)$ must satisfy the following (five) conditions: $U(0) = 0$, $U(1) = 0$, $U(x_0^+) = U(x_0^-)$ and $U'(x_0^+) = U'(x_0^-) = 0$. We do know that $U'' = v_0$ for $x \leq x_0$ and $U'' = v_1$ for $x \geq x_0$; therefore, for $x \leq x_0\, U(x) = v_0 x^2/2 + c_1 x + c_2$ and for $x \geq x_0\, U(x) = v_1 x^2/2 + d_1 x + d_2$, where c_1, c_2, d_1 and d_2 are constants. Since $U(0) = U(1) = 0$, we have that $c_2 = 0$ and $d_1 + d_2 = -v_1/2$, and the three remaining conditions allow us to determine c_1, d_1 and x_0. Since $U'(x_0^-) = U'(x_0^+) = 0$ we find that $c_1 = -v_0 x_0$ and $d_1 = -v_1 x_0$, that is, $U(x) = v_0 x(x - 2x_0)/2$ for $x \leq x_0$ and $U(x) = v_1(x - 1)(x + 1 - 2x_0)/2$ for $x \geq x_0$. Finally, in order that $U(x_0^-) = U(x_0^+)$ we must have $-v_0 x_0^2/2 = -v_1(1 - x_0)^2/2$, that is,

$$x_0 := v_1^{1/2}/(v_0^{1/2} + v_1^{1/2}) = 1/(1 + (v_0/v_1)^{1/2}). \tag{4.3}$$

Suppose now that either $v_0 = 0$ or $v_1 = 0$; let us say that $v_0 = 0$, since the discussion of the case $v_1 = 0$ proceeds analogously. The behavior of the solution of the v-equation in this case is a limiting form of the behavior just observed. To wit, if in formula (4.3) we set v_0 equal to zero then x_0 is equal to 1; whence, there is an "interior" layer at $x = 1$! In reality there is a boundary layer at $x = 1$ (with $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = v_0 = 0$ in $[0, 1 - \delta]$) whose structure is described by an error function, since $v'(0, \varepsilon) = v'(1, \varepsilon)$ must of course obtain (cf. (3.5)). In [7] we coined

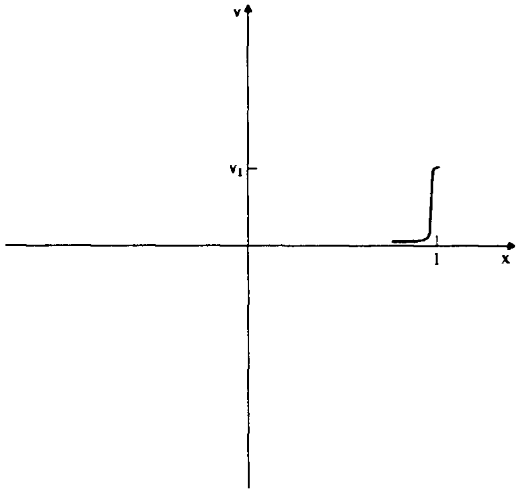

Fig. 2.

the term "S-layer" to describe this situation; cf. Fig. 2. In the remaining case $v_1 = 0$ there is a "Z-layer" (backwards S-layer) at $x = 0$, with $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = v_1 = 0$ in $[\delta, 1]$.

Case 2. $v_0 \le 0$, $v_1 \le 0$. The reader will see immediately that this case is not the reflection of case 1, in the sense that there is no change of variable of the type employed in showing the equivalence of cases 1 and 2 for problem (E1). The present cases 1 and 2 are fundamentally different from each other. It is convenient to consider first the subcase $v_0 < 0$, $v_1 < 0$ and to delay considering the subcase $v_0 = 0$ or $v_1 = 0$ until later.

If $v_0 < 0$ and $v_1 < 0$ then from the estimate (4.2)

$$M(x^2 - x)/2 \le u(x, \varepsilon) \le m(x^2 - x)/2, \qquad 0 \le x \le 1,$$

for $M := \max\{v_0, v_1\}$ and $m := \min\{v_0, v_1\}$, we see that $u(x, \varepsilon) > 0$ in $(0, 1)$, $u'(x, \varepsilon) > 0$ near $x = 0$, $u'(x, \varepsilon) < 0$ near $x = 1$, and $u'(x, \varepsilon)$ passes through zero at some point x_0 in $(0, 1)$ like a positive multiple of $-(x - x_0)$. Consequently, the solution of the v-equation cannot exhibit interior layer behavior at x_0; however, the sign of $u'(x, \varepsilon)$ at the endpoints is compatible with the existence of boundary layers in v there. It follows that the only limiting behavior available to v is boundary layer behavior at each endpoint, that is, $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = c$, $\delta \le x \le 1 - \delta$, for c a constant strictly between v_0 and v_1. In order to determine c we proceed as follows. We know that $u'' = v \sim c$ as $\varepsilon \to 0^+$, and so $u(x, \varepsilon) \sim c(x^2 - x)/2$, giving us $u'(x, \varepsilon) \sim c(2x - 1)/2$. In particular, $u'(0, \varepsilon) = -c/2$ and $u'(1, \varepsilon) = c/2$, and so we see that for x in $[0, 1]$ as $\varepsilon \to 0^+$

$$v(x, \varepsilon) \sim c + (v_0 - c)\exp[-kx/\varepsilon] + (v_1 - c)\exp[-k(1 - x)/\varepsilon], \tag{4.4}$$

where $k := |c|/2$. Therefore $v'(0, \varepsilon) = v'(1, \varepsilon)$ if and only if $-k(v_0 - c)/\varepsilon = k(v_1 - c)/\varepsilon$ or $c := (v_0 + v_1)/2$. We conclude that for x in $[0, 1]$

$$\begin{aligned} v(x, \varepsilon) \sim (v_0 + v_1)/2 &+ (v_0 - v_1)/2\exp[-kx/\varepsilon] \\ &+ (v_1 - v_0)/2\exp[-k(1 - x)/\varepsilon], \end{aligned} \tag{4.5}$$

where $k := |v_0 + v_1|/4$, showing that the thickness of each boundary layer is of order ε.

If now $v_0 = 0$ or $v_1 = 0$ then it is easy to see that the appropriate limiting form of (4.5) obtains, that is, $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = v_1/2[v_0/2]$ if $v_0 = 0[v_1 = 0]$.

Case 3. $v_0 < 0$, $v_1 > 0$. By arguing as in case 2 we see immediately that the only possible asymptotic behavior available to v is twin boundary layer behavior, that is,

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = c \qquad \text{for } x \text{ in } [\delta, 1 - \delta], \tag{4.6}$$

where c is a constant in the interval (v_0, v_1).

Let us note, to begin with, that the constant c in (4.6) must be nonpositive. For $u'' = v - c$, and so $u(x, \varepsilon) \sim c(x^2 - x)/2$ as $\varepsilon \to 0^+$; whence, the signs of $u'(x, \varepsilon)$ required for boundary layer behavior at the endpoints $u'(0, \varepsilon) \sim -c/2 \geq 0$ and $u'(1, \varepsilon) \sim c/2 \leq 0$ obtain provided $c \leq 0$. The discussion now divides into three subcases. Suppose first that $v_0 + v_1 > 0$; we will show that $c < 0$ is impossible. If $c < 0$ then $u'(0, \varepsilon) \sim -c/2 > 0$ and $u'(1, \varepsilon) = c/2 < 0$, and it follows that the relation (4.4) obtains for x in $[0, 1]$. By the same argument that led to (4.5) we see that c must equal $(v_0 + v_1)/2 > 0$. It follows then that c must in fact be zero if $v_0 + v_1 > 0$, that is,

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = 0 \qquad \text{for } x \text{ in } [\delta, 1 - \delta]. \tag{4.7}$$

Since this is a somewhat nonintuitive result we check that it is self-consistent. The condition $v_0 + v_1 > 0$ says that $v_1 > |v_0|$, and so in view of (4.7), there exists a unique point $\xi = \xi(\varepsilon)$ in $(0, 1)$ such that $u''(x, \varepsilon) < 0$ for $0 \leq x < \xi(\varepsilon)$, $u''(\xi, \varepsilon) = 0$ and $u''(x, \varepsilon) > 0$ for $\xi(\varepsilon) < x \leq 1$, with $\lim_{\varepsilon \to 0^+} \xi(\varepsilon) = 1$. (This point ξ is simply the unique point in $(0, 1)$ where $v(\xi(\varepsilon), \varepsilon) = 0$; its existence follows from the fact that $v'(x, \varepsilon) > 0$ in $[0, 1]$.) Therefore, in the boundary layer at $x = 0$, $v''(x, \varepsilon) < 0$, with of course $u'(0, \varepsilon) > 0$; however, the boundary layer at $x = 1$ is more complicated, in that $\xi(\varepsilon)$ is an inflection point for v. To see that this is consistent with the behavior of u', rewrite the v-equation as

$$\varepsilon v'' = -u'v'. \tag{4.8}$$

Then slightly to the left of ξ, $v''(x, \varepsilon) > 0$ and $u'(x, \varepsilon) < 0$; at $x = \xi v''(x, \varepsilon) = 0$ and $u'(x, \varepsilon) = 0$; and to the right of ξ, $v''(x, \varepsilon) < 0$ and $u'(x, \varepsilon) > 0$. Thus this behavior *is* compatible with the equation (4.8).

Suppose next that $v_0 + v_1 < 0$. We will show that $c = 0$ is now impossible. If $v \to 0$ in $(0, 1)$ as $\varepsilon \to 0^+$ then in view of the fact that $|v_0| > v_1$ there exists a unique point $\eta = \eta(\varepsilon)$ in $(0, 1)$ such that $u''(x, \varepsilon) < 0$ for $0 \leq \eta(\varepsilon)$, $u''(\eta, \varepsilon) = 0$ and $u''(x, \varepsilon) > 0$ for $\eta(\varepsilon) < x \leq 1$, with $\lim_{\varepsilon \to 0^+} \eta(\varepsilon) = 0$. (The existence of η follows from the same argument that gave us the existence of ξ.) In the boundary layer at $x = 0$ $v''(x, \varepsilon) > 0$ for $0 \leq x < \eta(\varepsilon)$, $v''(\eta, \varepsilon) = 0$ and $v''(x, \varepsilon) < 0$ for x slightly to the right of η. In turn, we see that $u'(x, \varepsilon) > 0$ for $0 \leq x < \eta(\varepsilon)$, $u'(\eta, \varepsilon) = 0$ and $u'(x, \varepsilon) < 0$ for x slightly to the right of η. But these relations are now *incompatible* with the relation (4.8). For instance, we know that $v'' > 0$ for $0 \leq x < \eta$; however, (4.8) tells us that $v'' = -\varepsilon^{-1}u'v' < 0$ there. We conclude that for $v_0 + v_1 < 0$ the limiting value c of $v(x, \varepsilon)$ must be negative; in fact, c must equal $(v_0 + v_1)/2$, that is,

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = (v_0 + v_1)/2 \qquad \text{for } x \text{ in } [\delta, 1 - \delta]. \tag{4.9}$$

This follows directly from (4.4) by repeating the argument given there.

Finally, for the remaining case $v_0 + v_1 = 0$, we see with little difficulty that

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = 0 \qquad \text{for } x \text{ in } [\delta, 1 - \delta].$$

Case 4. $v_0 > 0$, $v_1 < 0$. Fortunately the asymptotic behavior of $v(x, \varepsilon)$ in this case is the same as that described in the previous case. To see this simply introduce the changes of variable $r := 1 - x$, $\hat{u}(r, \varepsilon) := u(1 - r, \varepsilon)$ and $\hat{v}(r, \varepsilon) := v(1 - r, \varepsilon)$ into (E2), and observe that for x in $[\delta, 1 - \delta]$

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = (v_0 + v_1)/2 \qquad \text{if} \quad v_0 + v_1 < 0$$

and

$$\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = 0 \qquad \text{if} \quad v_0 + v_1 \geq 0,$$

as before.

We summarize our results for (E2) in the form of a boundary value portrait (cf. Fig. 3).

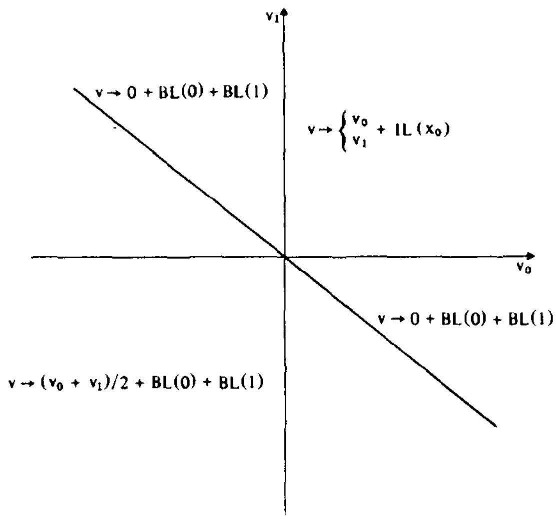

Fig. 3.

REFERENCES

1. Batchelor G. K., *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge (1967).
2. Bernfeld S. R. & Lakshmikantham V., *An Introduction to Nonlinear Boundary Value Problems*. Academic Press, New York (1974).
3. Chang K. W. & Howes F. A., *Nonlinear Singular Perturbation Phenomena: Theory and Applications*. Springer, New York (1984).
4. Dorr F. W., Some examples of singular perturbation problems with turning points, *SIAM J. math. Analysis* **1**, 141–146 (1970).

5. DORR F. W. & PARTER S. V., Singular perturbations of nonlinear boundary value problems with turning points, *J. math. Analysis Applic.* **29**, 273–293 (1970).
6. DORR F. W., PARTER S. V. & SHAMPINE L. F., Applications of the maximum principle to singular perturbation problems, *SIAM Rev.* **15**, 43–88 (1973).
7. HOWES F. A., Boundary-interior layer interactions in nonlinear singular perturbation theory, *Mem. Am. math. Soc.* **203**, (1978).
8. O'MALLEY R. E. JR., *Introduction to Singular Perturbations*. Academic Press, New York (1974).
9. RUDIN W., *Real and Complex Analysis*. McGraw-Hill, New York (1974).
10. VANDYKE M., *Perturbation Methods in Fluid Mechanics*. Parabolic Press, Stanford (1975).