Asymptotic Behavior of Solutions of Model Problems for a Coupled System

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We study the asymptotic behavior of solutions of model problems for a coupled system. By consistent use of a priori estimates and asymptotic analysis, we present here an more efficient approach which provides precise descriptions of the asymptotic behabior of solutions of this system. Our results amplify and extend earlier results of Dorr, Parter, and Shampine and their treatment of this system. Meanwhile, the stability of the steady-state solutions of the corresponding time-dependent system is discussed.

1. INTRODUCTION

Consider a Dirichlet problem for the coupled system of equations

$$
\begin{gathered}
u'' = v \qquad \text{in } (0, 1), \\
u(0, \varepsilon) = 0, \qquad u(1, \varepsilon) = 0, \\
\varepsilon v'' + f(u, u')v' - g(x, u, u')v = 0 \qquad \text{in } (0, 1), \\
v(0, \varepsilon) = v_0, \qquad v(1, \varepsilon) = v_1,
\end{gathered} \tag{1.1}
$$

for $g(x, u, u') \geqslant 0$ and $f(u, u') = h(u)$ or $h(u')$, where $h(0) = 0$. We assume that for each $\varepsilon > 0$, there exists at most one interior turning point x_0 in $(0, 1)$ such that $u(x_0, \varepsilon) = 0$. Throughout this paper we also assume that for $0 \leqslant x \leqslant 1$,

(i) $f(u, u'')$ and $g(x, u, u')$ are continuously differentiable functions of all variables and

(ii) $\partial f/\partial u$ and $\partial f/\partial u'$, do not change sign.

We are particularly interested in the case for which (i) $dh/dz > 0$ or (ii) $dh/dz < 0$ in $(0, 1)$.

System (1.1) is a simple model of the steam function-vorticity equations for two-dimensional, viscous, incompressible flow [8]. Historically this problem was considered by Dorr, Parter, and Shampine [3, 4]. They used analytical and numerical methods to study the existence of solutions of (1.1) and their asymptotic behavior as $\varepsilon \to 0^+$ for the cases $f(u, u') = u$ or u'. Howes and Shao [8] give an intuitive discussion of the system (1.1) for the case $g \equiv 0$. The purpose of this paper is to provide an alternative approach, which gives precise descriptions of the asymptotic behavior as $\varepsilon \to 0^+$ of the solutions of system (1.1) for all boundary values of v and proves the existence of the limiting solutions of system (1.1) utilizing differential inequality techniques. Our results amplify and extend the earlier results of Dorr *et al.* and our treatment via asymptotic analysis is more appealing on both intuitive and analytical grounds. We hope that our results will be useful in the study of more complicated systems of ordinary and partial differential equations, such two-dimensional versions of system (1.1), etc.

The outline of this paper is as follows. In Section 2 we state some results from singular perturbation theory and some basic techniques of differential inequalities used later. The solutions of Model I, $u'' = v$, $\varepsilon v'' + h(u)v' = 0$, Model II, $u'' = v$, $\varepsilon v'' + h(u')v' = 0$, and Model III, $u'' = v$, $\varepsilon v'' + f(u, u')\, v' - g(x, u, u')v = 0$, are discussed in Sections 3, 4, and 5, respectively. Finally, we discuss the stability of the steady-state solutions of the corresponding time-dependent system in Section 6.

2. Preliminaries

We first state some elementary results of the theory of a singularly perturbed boundary value problem. Let us consider a linear model problem

$$\varepsilon y'' + f(x)y' = 0, \qquad x \text{ in } (0, 1), k \text{ is a constant}, \tag{LP}$$
$$y(0, \varepsilon) = A, \qquad y(1, \varepsilon) = B.$$

Then as $\varepsilon \to 0^+$, we have the following classical results (cf. [5, Chap. 3; 1, Chap. 4]):

(i) $y(x, e) \to B + BL(0)$, $\quad BL(0) = O(|A - B| \exp[-kx/\varepsilon])$ if $f(x) \geqslant k > 0$ in $[0, 1]$;

(ii) $y(x, \varepsilon) \to A + BL(1)$, $\quad BL(1) = O(|B - A| \exp[-k(1-x)/\varepsilon])$ if $f(x) \leqslant -k < 0$ in $[0, 1]$;

(iii) if $\exists x_0$ in $(0, 1)$ such that $f(x_0) = 0$ with $f'(x_0) \neq 0$ (assume $A < 0 < B$), then

(a) If $f'(x_0) < 0$, then $y(x, \varepsilon) \to B + BL(0)$ if
$x_0 > 1/2$, $y(x, \varepsilon) \to A + BL(1)$ if $x_0 < 1/2$;

(b) $y(x, \varepsilon) \to (A + B)/2 + BL(0) + BL(1)$
if $f'(x_0) < 0$ in $[0, 1]$ and $x_0 = 1/2$;

(c) $y(x, \varepsilon) \to \begin{cases} A \\ B \end{cases} + IL(x_0)$ if $f'(x_0) > 0$ in $[0, 1]$.

We note that the asymptotic behavior of solutions of (LP) depends only on the signs of the coefficient $f(x)$ of the first derivative y' and the boundary values A and B. This leads us to study the sign of the coefficient $f(u, u')$ of the first derivative v' together with the boundary values of v.

A basic technique of differential inequalities is contained in the generalized Nagumo Theorem (cf. [4]). Applying the generalized Nagumo Theorem to our model problem (1.1), we only need to find bounding functions α_i and β_i $(i = 1, 2)$ such that

$$\begin{aligned} &\alpha_i \leqslant \beta_i, && i = 1, 2, \\ &\alpha_1(0, \varepsilon) \leqslant 0 \leqslant \beta_1(0, \varepsilon), && \alpha_1(1, \varepsilon) \leqslant 0 \leqslant \beta_1(1, \varepsilon), \\ &\alpha_1'' \geqslant \beta_2, && \beta_1'' \leqslant \alpha_2, \\ &\alpha_2(0, \varepsilon) \leqslant v_0 \leqslant \beta_2(0, \varepsilon), && \alpha_2(1, \varepsilon) \leqslant v_1 \leqslant \beta_2(1, \varepsilon), \\ &\varepsilon\alpha_2'' + f(u, z)\,\alpha_2' - g(x, u, z)\alpha_2 \geqslant 0, \\ &\varepsilon\beta_2'' + f(u, z)\beta_2' - g(x, u, z)\beta_2 \leqslant 0, \end{aligned} \tag{2.1}$$

for all u in $[\alpha_1, \beta_1]$ and z in $\mathbb{R}$. Then the system (1.1) has a unique solution $u = u(x, \varepsilon)$, $v = v(x, \varepsilon)$ such that

$$\alpha_1(x, \varepsilon) \leqslant u(x, \varepsilon) \leqslant \beta_1(x, \varepsilon), \qquad \alpha_2(x, \varepsilon) \leqslant v(x, \varepsilon) \leqslant \beta_2(x, \varepsilon), \tag{2.2}$$

for x in $[0, 1]$.

LEMMA. *If we choose* $\alpha_1(x, \varepsilon) = Mx(x-1)/2$, $\beta_1(x, \varepsilon) = mx(x-1)/2$, $\alpha_2(x, \varepsilon) = m$, $\beta_2(x, \varepsilon) = M$, *then the solution* $(u(x, \varepsilon), v(x, \varepsilon))$ *of* (1.1) *satisfies the inequalities*

$$\frac{Mx(x-1)}{2} \leqslant u(x, \varepsilon) \leqslant \frac{mx(x-1)}{2}, \qquad m \leqslant v(x, \varepsilon) \leqslant M, \tag{2.3}$$

for all x *in* $[0, 1]$ *and for all* $\varepsilon > 0$, *where*

$$M = \max\{v_0, v_1\}, \textit{ and } m = \min\{v_0, v_1\} \textit{ if } g(x, u, u') \equiv 0;$$

$$M = \max\{0, v_0, v_1\}, \textit{ and } m = \min\{0, v_0, v_1\} \textit{ if } g(x, u, u') > 0.$$

We divide system (1.1) into three models:

Model I, $u''=v$, $\varepsilon v''+h(u)v'=0$,

Model II, $u''=v$, $\varepsilon v''+h(u')v'=0$, and

Model III, $u''=v$, $\varepsilon v''+f(u,u')v'-g(x,u,u')v=0$.

Without lost generality, we assume $dh/du>0$ or $dh/du'>0$.

3. Model I: $u''=v$, $\varepsilon v''+h(u)v=0$

Consider the system

$$\begin{aligned} &u''=v \qquad \text{in } (0,1),\\ &u(0,\varepsilon)=u(1,\varepsilon)=0,\\ &\varepsilon v''+h(u)v'=0 \qquad \text{in } (0,1),\\ &v(0,\varepsilon)=v_0, \qquad v(1,\varepsilon)=v_1. \end{aligned} \tag{3.1}$$

THEOREM 1. *Let $(u(x,\varepsilon), v(x,\varepsilon))$ be the solution of (3.1). Then as $\varepsilon\to 0^+$, we have*

$$\frac{Mx(x-1)}{2}\leqslant u(x,\varepsilon)\leqslant\frac{mx(x-1)}{2} \qquad \text{in } [0,1];$$

(i) $v(x,\varepsilon)\to v_0+BL(1)$ *in* $[0,1]$ *if* $v_0\geqslant 0$, $v_1\geqslant 0$;

(ii) $v(x,\varepsilon)\to v_1+BL(0)$ *in* $[0,1]$ *if* $v_0\leqslant 0$, $v_1\leqslant 0$;

(iii) $v(x,\varepsilon)\to\begin{cases} v_0\\ v_1\end{cases}+IL(x_0)$ *in* $[0,1]$ *if* $v_0>0$, $v_1<0$; *and*

(iv) $v(x,\varepsilon)\to 0+BL(0)+BL(1)$ *in* $[0,1]$ if $v_0<0$, $v_1>0$,

where $M=\max\{v_0,v_1\}$, $m=\min\{v_0,v_1\}$, $BL(k)=$ *the boundary layer at* $x=k$, $k=0,1$.

Proof. We need to find bounding functions α_i and β_i $(i=1,2)$ such that

$$\alpha_i\leqslant\beta_i, \qquad i=1,2, \tag{E1}$$

$$\alpha_1(0,\varepsilon)\leqslant 0\leqslant\beta_1(0,\varepsilon), \qquad \alpha_1(1,\varepsilon)\leqslant 0\leqslant\beta_1(1,\varepsilon), \tag{BC1}$$

$$\alpha_1''\geqslant\beta_2, \qquad \beta_1''\leqslant\alpha_2, \tag{E2}$$

$$\varepsilon\alpha_2''+h(u)\alpha_2'\geqslant 0, \qquad \varepsilon\beta_2''+h(u)\beta_2'\leqslant 0, \text{ for all } u \text{ in } [\alpha_1,\beta_1] \tag{E3}$$

$$\alpha_2(0,\varepsilon)\leqslant v_0\leqslant\beta_2(0,\varepsilon), \qquad \alpha_2(1,\varepsilon)\leqslant v_1\leqslant\beta_2(1,\varepsilon). \tag{BC2}$$

Case 1. $v_0 \geqslant 0$, $v_1 \geqslant 0$. If $v_0 = v_1 = 0$, let $\alpha_i = \beta_i = 0$ $(i = 1, 2)$, then it is clear that $u(x, \varepsilon) \equiv 0$ and $v(x, \varepsilon) \equiv 0$ is the solution. If $v_0 = v_1 = p$, let $\alpha_2 = \beta_2 \equiv p$ and it follows that $v(x, \varepsilon) \equiv p$. For $0 \neq v_0 \neq v_1 \neq 0$, since $u'' = v$, $u(0, \varepsilon) = u(1, \varepsilon) = 0$, we have $u(x, \varepsilon) = v_0 x(x-1)/2$ and $u < 0$ in $(0, 1)$ and so $h(u) < 0$ in $(0, 1)$. By virtue of the linear case, the solution $v(x, \varepsilon)$ must display a boundary layer at $x = 1$.

We note that the reduced form of the v-equation $h(u)v' = 0$ implies $v =$ constant in $(0, 1)$. Therefore the constant is v_0. We set

$$\begin{aligned} \alpha_2(x, \varepsilon) &= v_0 - |v_1 - v_0| \exp\left[\frac{\lambda(1-x)}{\varepsilon}\right], \\ \beta_2(x, \varepsilon) &= v_0 + |v_1 - v_0| \exp\left[\frac{\lambda(1-x)}{\varepsilon}\right], \\ \alpha_1(x, \varepsilon) &= Mx(x-1), \qquad \beta_1(x, \varepsilon) = \frac{mx(x-1)}{4}, \end{aligned} \tag{3.2}$$

where $M = \max\{v_0, v_1\}$, $m = \min\{v_0, v_1\}$, and λ is a constant such that $0 > \lambda > h(u)$ in $[\delta, 1-\delta]$ for a given $\delta > 0$ and for $\varepsilon > 0$. Then the conditions (E1), (E2), (E3), (BC1), and (BC2) are satisfied. Therefore we have

$$\begin{aligned} & v_0 - |v_1 - v_0| \exp\left[\frac{\lambda(1-x)}{\varepsilon}\right] \\ & \qquad \leqslant v(x, \varepsilon) \leqslant v_0 + |v_1 - v_0| \exp\left[\frac{\lambda(1-x)}{\varepsilon}\right] \qquad \text{in } [\delta, 1-\delta], \end{aligned}$$

i.e., $v(x, \varepsilon) \to v_0 + BL(1)$ in $[0, 1]$ as $\varepsilon \to 0^+$.

If we look at a neighborhood of $x = 1$, then

$$u(x, \varepsilon) \sim \frac{jx(x-1)}{2} \sim j(x-1), \tag{3.3}$$

where

$$j = \begin{cases} v_0 & \text{if } v_0 > 0, \\ v_1 & \text{if } v_0 = 0. \end{cases}$$

Substituting (3.3) into the equation $\varepsilon v'' + h(u)v' = 0$, we have

$$\varepsilon v'' + h\left(\frac{j(x-1)}{2}\right) v' = 0 \qquad \text{in } (0, 1). \tag{3.4}$$

Introducing a stretched variable $\xi = (1/\varepsilon)^{1/2}\{\int h[j(x-1)/2]\, dx\}^{1/2}$ and setting $w(\xi, \varepsilon) = v(x, \varepsilon)$, then

$$w_{\xi\xi} + \xi w_\xi = 0, \qquad w(0, \varepsilon) = v_1 \tag{3.5}$$

and we have

$$w(x,\varepsilon)=v_1\left[1+\frac{2}{\pi^{1/2}}\int_0^{\xi/\sqrt{2}}\exp(-s^2)\,ds\right]$$
$$=v_1\left[1+\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right], \tag{3.6}$$

where $\operatorname{erf}[y]=(2/\pi^{1/2})\int_0^y \exp(-s^2)\,ds$ is the error function satisfying

$$\lim_{y\to-\infty}\operatorname{erf}[y]=-1.$$

Therefore, the thickness of the boundary layer at $x=1$ is of order $\varepsilon^{1/2}$.

Case 2. $v_0\leqslant 0$, $v_1\leqslant 0$. In this case, by the same arguments as in Case 1, we have $h(u)>0$ in $(0,1)$, and so the solution $v(x,\varepsilon)$ of the v-equation will display a boundary layer at $x=0$. We make the change of variables

$$y=1-x,\qquad m(y,\varepsilon)=-u(1-y,\varepsilon),\qquad n(y,\varepsilon)=-v(1-y,\varepsilon).$$

Then the system (3.1) becomes

$$m''=n,\qquad m(0,\varepsilon)=m(1,\varepsilon)=0,$$
$$\varepsilon n''+h(m)n'=0,\qquad n(0,\varepsilon)=n_0=-v_1,\qquad n(1,\varepsilon)=n_1=-v_0,$$

provided $n_0\geqslant 0$ and $m_1\geqslant 0$. By our previous results, we obtain

$$n(y,\varepsilon)\to n_0+BL(1)\qquad \text{in } [0,1] \text{ as } \varepsilon\to 0^+, \text{ i.e., } v(x,\varepsilon)\to v_1+BL(0).$$

The thickness of the boundary layer at $x=0$ is also of order $\varepsilon^{1/2}$.

Case 3. $v_0>0$, $v_1<0$. We note that $u(x,\varepsilon)$ changes sign in $(0,1)$ in this case, and so does $h(u)$. Since $u''=v$, $u(0,\varepsilon)=u(1,\varepsilon)=0$, there exists a unique x_0 in $(0,1)$ such that $u(x_0,\varepsilon)=0$ and hence $h(u(x_0),\varepsilon)=0$. Since $u<0$ and $u>0$ near $x=1$, we have $h(u)<0$ near $x=0$ and $h(u)>0$ near $x=1$. We claim that the solution $v=v(x,\varepsilon)$ of (3.1) displays an interior layer as $\varepsilon\to 0^+$ in $(0,1)$, that is,

$$\lim_{\varepsilon\to 0^+}v(x,\varepsilon)=V(x)=\begin{cases}v_0 & \text{if } 0\leqslant x\leqslant x_0-\delta,\\ v_1 & \text{if } x_0+\delta\leqslant x<1-\delta,\end{cases} \tag{3.7}$$

for x_0 in $(0,1)$, $0<\delta<1$. Since $\varepsilon v''+h(u)v'=0$ in $(0,1)$, we have

$$\varepsilon\ln\frac{|v'(1,\varepsilon)|}{|v'(0,\varepsilon)|}=-\int_0^1 h(u(s,\varepsilon))\,ds\qquad \text{in } (0,1). \tag{3.8}$$

We know that $|v'(0, \varepsilon)| = |v'(1, \varepsilon)| = O(\varepsilon^{1/2})$ as $\varepsilon \to 0^+$; it follows that

$$\lim_{\varepsilon \to 0^+} \varepsilon \ln \frac{|v'(1, \varepsilon)|}{|v'(0, \varepsilon)|} = 0 \qquad \text{in } (0, 1).$$

Hence, we have

$$\int_0^1 h(u(s, \varepsilon))\, ds \to 0 \qquad \text{in } (0, 1) \text{ as } \varepsilon \to 0^+. \tag{3.9}$$

Let $U(x) = \lim_{\varepsilon \to 0^+} u(x, \varepsilon)$, then $U'' = V$, $U(0) = U(1) = 0$. By (3.4), we obtain

$$U(x) = \begin{cases} v_0 x(x - x_0)/2, & \text{if } 0 \leqslant x \leqslant x_0, \\ -v_1(1 - x)(x - x_0)/2 & \text{if } x_0 \leqslant x \leqslant 1 \end{cases} \tag{3.10}$$

and so there exists a constant k such that $U(x) \geqslant k > 0$ for $0 < x < x_0 - \delta$ and $U(x) \leqslant -k < 0$ for $x_0 + \delta < x < 1$. It follows that $h(U(x)) \geqslant r > 0$ in $(0, x_0 - \delta)$ and $h(U(x)) \leqslant -r < 0$ in $(x_0 + \delta, 1)$, where r is a positive constant. Therefore, there is an interior layer at $x = x_0$ in $(0, 1)$. By the Dominated Convergence Theorem [9] and (3.9), it follows that as $\varepsilon \to 0^+$, $0 = \int_0^1 [h(u(s, \varepsilon)) \to 0]\, ds$, which implies that

$$\begin{aligned} 0 &= \int_0^1 h(U(s))\, ds \\ &= \int_0^{x_0} h(U(s))\, ds + \int_{x_0}^1 h(U(s))\, ds \qquad \text{in } (0, 1), \end{aligned} \tag{3.11}$$

and also

$$\begin{aligned} v(x, \varepsilon) &= V + |v_1 - v_0| \left\{1 + \operatorname{erf}\left[\frac{1}{\sqrt{2\varepsilon}} \int h\left(\frac{j}{2(x - x_0)}\right) dx\right]\right\} \\ &\qquad \text{for } 0 \leqslant x \leqslant 1, \\ v(x, \varepsilon) &= V + IL(x_0), \qquad 0 \leqslant x \leqslant 1, \end{aligned}$$

where $v(x, \varepsilon)$ is the solution of (3.1), $V = V(x)$ as in (3.7), $IL(x_0)$ means the interior layer at x_0 which is determined by (3.6).

Case 4. $v_0 < 0$, $v_1 > 0$. In this case $u(x, \varepsilon)$ also changes sign in $(0, 1)$ and there exists a unique x_0 in $(0, 1)$ such that $h(u(x_0)) = 0$. However, there is no interior layer in $(0, 1)$ because of $u(x, \varepsilon) > 0$ near $x = 0$ and $u(x, \varepsilon) < 0$ near $x = 1$. The wrong sign of u does not allow an interior layer to be displayed in $(0, 1)$. We consider the asymptotic behavior as $\varepsilon \to O^+$ of

solutions of (3.1) in the subintervals $[0, x_0]$ and $[x_0, 1]$ separately. Let $v^* = v(x_0, \varepsilon)$, then we have the systems of equations

$$\begin{aligned} &u'' = v \qquad \text{in } (0, x_0),\\ &u(0, \varepsilon) = u(x_0, \varepsilon) = 0,\\ &\varepsilon v'' + h(u)v' = 0 \qquad \text{in } (0, x_0),\\ &v(0, \varepsilon) = v_0,\ v(x_0, \varepsilon) = v^* \end{aligned} \tag{SI}$$

and

$$\begin{aligned} &u'' = v \qquad \text{in } (x_0, 1),\\ &u(x_0, \varepsilon) = u(1, \varepsilon) = 0,\\ &\varepsilon v'' + h(u)v' = 0 \qquad \text{in } (x_0, 1),\\ &v(x_0, \varepsilon) = v^*, \qquad v(1, \varepsilon) = v_1. \end{aligned} \tag{SII}$$

For $1 \gg \delta > 0$, we have $h(u) > 0$ in $[\delta, x_0 - \delta]$ and $h(u) < 0$ in $[x_0 + \delta, 1]$. Since $v_0 < 0$, $v^* = 0$, $v_1 > 0$, it follows that (SI) and (SII) are Case 2 and Case 1, respectively. Hence, by our previous results, we have that as $\varepsilon \to 0^+$,

$$v(x, \varepsilon) \to v^* + BL(0) \text{ in } [\delta, x_0 - \delta]$$

and

$$v(x, \varepsilon) \to v^* + BL(1) \text{ in } [x_0 + \delta, 1 - \delta].$$

Since $v^* = v(x_0, \varepsilon) = 0$, it follows that $v(x, \varepsilon) \to 0 + BL(0) + BL(1)$ in $[0, 1]$ as $\varepsilon \to 0^+$.

This finishes the proofs for all choices of the boundary data v_0 and v_1. All of the previous results are summarized in Fig. 1.

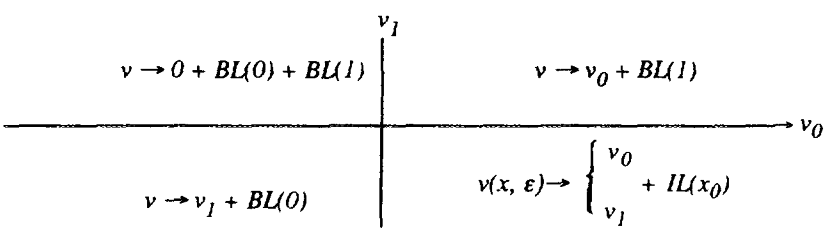

FIG. 1. Boundary value portrait for model I.

4. The Solutions of Model II: $u''=v$, $\varepsilon v''+h(u')v=0$

We consider the problem

$$\begin{aligned} u''&=v \qquad \text{in } (0,1),\\ u(0,\varepsilon)&=u(1,\varepsilon)=0,\\ \varepsilon v''+h(u')v'&=0 \qquad \text{in } (0,1),\\ v(0,\varepsilon)&=v_0, \qquad v(1,\varepsilon)=v_1. \end{aligned} \tag{4.1}$$

There is a fundamental difference between the systems (4.1) and (3.1). Because $u'(x,\varepsilon)$ always changes sign in $(0,1)$, even if $u(x,\varepsilon)$ does not change sign, then there is at least *one* interior turning point x_0 in $(0,1)$ such that $h(u'(x_0,\varepsilon))=0$. The behavior of the coefficient $h(u')$ of v' in the v-equation of (4.1) is more difficult to determine than the behavior of $h(u)$ in (3.1). We give an intuitive analysis of the asymptotic behavior as $\varepsilon\to 0^+$ of the solution of (4.1). If $v_0=v_1$, then $v(x,\varepsilon)\equiv v_0$ in $[0,1]$ and $u(x,\varepsilon)=v_0x(x-1)/2$ in $[0,1]$ [cf. Section 3]. For the other boundary values v_0 and v_1, we examine the following four cases.

Case 1. $v_0\geqslant 0$, $v_1\geqslant 0$. If $v_0>0$, $v_1>0$, then there exists some x_0 in $(0,1)$ such that $u'(x_0)=0$ and $u'(x,\varepsilon)<0$ near $x=0$, while $u'(x,\varepsilon)>0$ near $x=1$. It follows that $h(u'(x_0))=0$, $h(u')<0$ near $x=0$ and $h(u')>0$ near $x=1$. Consequently, the sign of $h(u')$ at each endpoint does not allow $v(x,\varepsilon)$ to have boundary layers. The only possible asymptotic behavior available to v is then interior layer behavior at the point x_0 in $(0,1)$. Such behavior is indeed possible. If $v_1>v_0$ then $v'>0$ in $(0,1)$ and $h(u')>0$ near $x=1$; consequently, $v''<0$ in $(x_0,1)$ by (4.3). Also we have $v''>0$ in $(0,x_0)$ since $h(u')<0$ near $x=0$ and so $v(x,\varepsilon)$ must have an interior layer in $(0,1)$. If $v_0>v_1$ then $v'<0$ in $(0,1)$, $h(u')<0$ near $x=0$, and $h(u')>0$ near $x=1$; consequently, $v''>0$ in $(0,x_0)$ and $v''<0$ in $(x_0,1)$. This also implies that $v(x,\varepsilon)$ has an interior layer in $(0,1)$. To determine the location x_0 of the interior layer, we let

$$U(x)=\lim_{\varepsilon\to 0^+} u(x,\varepsilon) \qquad \text{for} \quad x\neq x_0.$$

Then $U(x)$ must satisfy the following (five) conditions: $U(0)=U(1)=0$, $U(x_0^+)=U(x_0^-)$, and $U'(x_0^+)=U'(x_0^-)=0$. We know that $U''=v_0$ for $x\leqslant x_0$ and $U''=v_1$ for $x\geqslant x_0$; therefore, $U(x)=v_0x_2/2+ax+b$ for $x\leqslant x_0$, $U(x)=v_1x^2/2+cx+d$ for $x\geqslant x_0$, where a, b, c, and d are constants. Since $U(0)=U(1)=0$, we have that $b=0$ and $c+d=-v_1/2$, and the three remaining conditions allow us to determine a, c, and x_0. Since $U'(x_0^+)=U'(x_0^-)=0$ we find that $a=-v_0x_0$ and $c=-v_1x_0$, that is, $U(x)=$

$v_0 x(x-2x_0)/2$ for $x \leqslant x_0$, $U(x) = v_1(x-1)(x+1-2x_0)/2$ for $x \geqslant x_0$. Finally, in order that $U(x_0^+) = U(x_0^-)$, we must have $-v_0 x_0^2/2 = -v_1(1-x_0)^2/2$, that is, $x_0 = v_1^{1/2}/(v_0^{1/2} + v_1^{1/2})$.

Case 2. $v_0 \leqslant 0$, $v_1 \leqslant 0$. The reader will see immediately that this case is not the reflection of Case 1, so there is no change of variable of the type employed in showing the equivalence of Cases 1 and 2 for Model I. Since $u'(x, \varepsilon) > 0$ near $x = 0$, $u'(x_0, \varepsilon) = 0$ for some point x_0 in $(0, 1)$ and $u'(x, \varepsilon) < 0$ near $x = 1$. The coefficient $h(u')$ of v' in the v-equation behaves accordingly. Consequently, the solution $v(x, \varepsilon)$ in the v-equation cannot have an interior layer at x_0. However, the sign of $h(u')$ at the endpoints is compatible with the existence of boundary layers in $v(x, \varepsilon)$. For if $v_0 > v_1$ then $v' < 0$, since $h(u') > 0$ near $x = 0$ and $h(u') < 0$ near $x = 1$, with of course $v'' > 0$ near $x = 0$ and $v'' < 0$ near $x = 1$ by (4.3). Similarly, if $v_0 < v_1$ then $v' > 0$, and we must have $v'' < 0$ near $x = 0$ and $v'' > 0$ near $x = 1$ and near $x = 0$, $h(u') < 0$ near $x = 1$. Therefore, $v(x, \varepsilon) \to c + BL(0) + BL(1)$ as $\varepsilon \to 0^+$ with c a constant strictly between v_0 and v_1. In order to determine c and the solution $v(x, \varepsilon)$ of (4.1), we let $U(x) = \lim_{\varepsilon \to 0^+} u(x, \varepsilon)$ in $[0, 1]$ and $c^* = \lim_{\varepsilon \to 0^+} \varepsilon[v'(1, \varepsilon) - v'(0, \varepsilon)]$. Then we know that $U'' = c$, so $U(x) = cx(x-1)/2$, giving us $U'(x) = c(2x-1)/2$ in $(0, 1)$, $U'(0) = -c/2$, $U'(1) = c/2$. It follows that $U'(0) > 0$ and $U'(1) < 0$ provided $c \leqslant 0$. We also have $v(x, \varepsilon) \to c + (v_0 - c)\exp[-kx/\varepsilon] + (v_1 - c)\exp[-k(1-x)/\varepsilon]$ as $\varepsilon \to 0^+$ in $[0, 1]$, where $k = |c|/2$. From (4.3), $\varepsilon v'' = -v'h(u')$, we have $-\varepsilon \int_0^1 v''\, dx = \int_0^1 v'h(u')\, dx = vh(u')|_0^1 - \int_0^1 v\, dh(u')$, which implies that

$$-\varepsilon[v'(1, \varepsilon) - v'(0, \varepsilon)] \to v_1 h(U'(1)) - v_0 h(U'(0)) - c \int_0^1 \left(\frac{\partial h}{\partial x} + U'' \frac{\partial h}{\partial u'} \right) dx \qquad \text{as} \quad \varepsilon \to 0^+,$$

that is,

$$c^* = c\left[\frac{1}{2}(v_0 + v_1) + \int_0^1 \left(\frac{\partial h}{\partial x} + c\frac{\partial h}{\partial u'} \right) dx \right]; \tag{4.2}$$

hence, c must satisfy (4.2). If $\int_0^1 h(u')\, dx = 0$, then $v'(0, \varepsilon) = v'(1, \varepsilon)$, we have

$$\frac{-k(v_0 - c)}{\varepsilon} = \frac{k(v_1 - c)}{\varepsilon} \qquad \text{or} \qquad c = \frac{(v_0 + v_1)}{2}.$$

Case 3. $v_0 < 0$, $v_1 > 0$. We note that $u' > 0$ near both endpoints and $u'(x_0, \varepsilon) = 0$ for some x_0 in $(0, 1)$, which implies that $h(u') > 0$ near both endpoints. Hence, $v(x, \varepsilon)$ has no interior layer in $(0, 1)$, and the sign of $h(u')$ near $x = 0$ allows $v(x, \varepsilon)$ to have a boundary layer there. Meanwhile, since $v' > 0$ (from the fact that $v_1 > v_0$) in $(0, 1)$, we must have $v'' < 0$ near

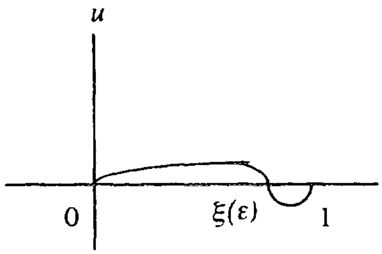

FIG. 2. Behavior of u.

$x=1$ since $h(u')>0$ there, and so $v(x,\varepsilon)$ has an "S-layer" (interior layer at $x=1$) near $x=1$. Therefore, $v(x,\varepsilon)\to c$ in $[\delta, 1-\delta^*]$ as $\varepsilon\to 0^+$, where c is a costant determined by (4.2), with $\delta>0$ and $\delta^*>0$. If $\int_0^1 h(u')\,dx=0$, then $\varepsilon\ln(|v'(1,\varepsilon)|/|v'(0,\varepsilon)|)=0$ or $|v'(1,\varepsilon)|=|v'(0,\varepsilon)|$, and so the only possible asymptotic behavior available to $v(x,\varepsilon)$ is twin layer behavior and $c=(v_0+v_1)/2$, $c\leqslant 0$. So if $v_0+v_1>0$, then c must be equal to zero. The condition $v_0+v_1>0$ says that $v_1>|v_0|$, and so there exists a unique point $\xi(\varepsilon)$ in $(0,1)$ such that $u''(x,\varepsilon)<0$ for $0\leqslant x<\xi(\varepsilon)$, $u''(x,\varepsilon)=0$, and $u''(x,\varepsilon)>0$ for $\xi(\varepsilon)<x<1$, with $\xi(\varepsilon)\to 1$ as $\varepsilon\to 0^+$; cf. Fig. 2. (This point ξ is simply the unique point in $(0,1)$ where $v(\xi(\varepsilon),\varepsilon)=0$; its existence follows from the fact that $v'(x,\varepsilon)>0$ in $[0,1]$). Therefore, in the boundary layer at $x=0$, $v''<0$, with $h(u')>0$; however, the boundary layer at $x=1$ is more complicated, in that $\xi(\varepsilon)$ is an inflection point for v. Consider $\varepsilon v''=-h(u')v'$, slightly to the left of ξ, $v''>0$ and $h(u')<0$; at $x=\xi$, $v''(\xi,\varepsilon)=0$ and $h(u'(\xi,\varepsilon))=0$; and to the right of ξ, $v''<0$ and $h(u')>0$; cf. Fig. 3.

Suppose next that $v_0+v_1<0$. We will show that $c=0$ is impossible. If $v\to 0$ in $(0,1)$ as $\varepsilon\to 0^+$, then there exists a unique point $\eta=\eta(\varepsilon)$ in $(0,1)$ such that $u''(x,\varepsilon)<0$ in $[0,\eta(\varepsilon))$, $(u''(\eta,\varepsilon))=0$ and $u''>0$ in $(\eta(\varepsilon),1]$ with $\eta(\varepsilon)\to 0$ as $\varepsilon\to 0^+$; cf. Fig. 4 (the existence of η follows from the same argument used in proving the existence of ξ). In the boundary layer at $x=0$, $v''>0$ in $[0,\eta(\varepsilon))$, $v''(\eta,\varepsilon)=0$, and $v''<0$ for x slightly to the right of η. In

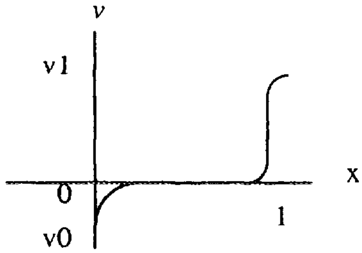

FIG. 3. $v_0+v_1>0$.

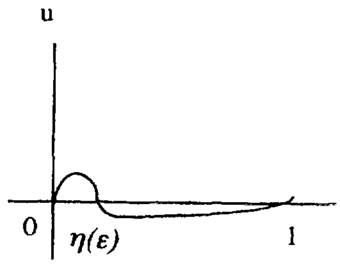

FIG. 4. Behavior of u.

turn, we see that $h(u')>0$ in $[0, \eta(\varepsilon))$, $h(u'(\eta, \varepsilon))=0$ and $h(u')<0$ for x slightly to the right of η. These relations are *incompatible* with the relation $\varepsilon v'' = -h(u')v'$. We conclude that for $v_0+v_1<0$, the limiting value c of $v(x, \varepsilon)$ must be negative, that is, $c=(v_0+v_1)/2$; cf. Fig. 5. Actually, the exact solution of the v-equation $\varepsilon v''+u'v'=0$, with $v(0, \varepsilon)=v_0$, $v(1, \varepsilon)=v_1$, is

$$v(x, \varepsilon)=v_0+(v_1-v_0)\frac{(\int_0^x \exp[-u(s, \varepsilon)/\varepsilon]\,ds)}{(\int_0^1 \exp[-u(s, \varepsilon)/\varepsilon]\,ds)}.$$

We can use Laplace's Method to obtain

$$\frac{\{\int_0^x \exp[-u(s, \varepsilon)/\varepsilon]\,ds\}}{\{\int_0^1 \exp[-u(s, \varepsilon)/\varepsilon]\,ds\}} \to \frac{1}{2} \qquad \text{as} \quad \varepsilon \to 0^+$$

in $(0, 1)$, which implies that $\lim_{\varepsilon \to 0^+} v(x, \varepsilon)=(v_0+v_1)/2+BL(0)+SL(1)$ for x in $[0, 1]$.

Case 4. $v_0>0$, $v_1<0$. Fortunately the asymptotic behavior of $v(x, \varepsilon)$ in this case is a reflection of that observed in the previous case. So there is a "Z-layer" (backwards S-layer) displayed at $x=0$. Thus $v(x, \varepsilon) \to c+BL(1)+ZL(0)$ as $\varepsilon \to 0^+$ in $[0, 1]$, where c is determined by (4.2). We

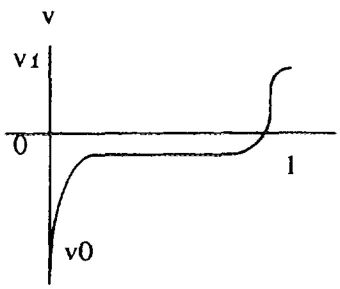

FIG. 5. $v_0+v_1<0$.

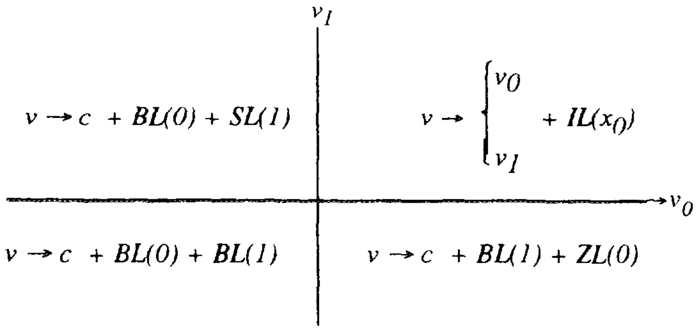

FIG. 6. Boundary value portrait for model II.

summarize our results for (4.1) in the form of a boundary value portrait in Fig. 6.

5. Solution of Model III: $u'' = v$, $\varepsilon v'' + f(u, u')\, v' - g(x, u, u')v = 0$

The problem we consider in this section is the system (1.1) in which $g(x, u, u') \geqslant g^* > 0$, for g^* a constant, that is,

$$\begin{aligned} u'' &= v \qquad \text{in } (0, 1), \\ u(0, \varepsilon) &= u(1, \varepsilon) = 0, \\ \varepsilon v'' + f(u, u')v' - g(x, u, u')v &= 0 \qquad \text{in } (0, 1), \\ v(0, \varepsilon) = v_0, &\qquad v(1, \varepsilon) = v_1, \end{aligned} \tag{5.1}$$

where $f(u, u') = h(u)$ or $h(u')$ and $g(x, u, u') \in C^1[0, 1] \times \mathbb{R}^2$. The strict positivity of the function $g(x, u, u')$ is the v-equation forces the asymptotic behavior as $\varepsilon \to 0^+$ of the solution $v(x, \varepsilon)$ to be independent of the coefficient $f(u, u')$ of v', because the reduced form of the v-equation, $f(u, u')v' + g(x, u, u')v = 0$, has only the zero solution $v \equiv 0$ in $(0, 1)$; cf. [3]. We have the following theorem.

THEOREM 2. *Let $v(x, \varepsilon)$ be the solution of* (6.1), *then as* $\varepsilon \to 0^+$,

$$v(x, \varepsilon) \to 0 + BL(0) + BL(1)$$

in $[0, 1]$ *for all boundary data* $v_0 \neq 0$ *and* $v_1 \neq 0$,

$$v(x, \varepsilon) \to 0 + BL(1) \qquad \textit{for} \quad v_0 \neq 0,\ v_1 > 0$$

and

$$v(x, \varepsilon) \to 0 + BL(0) \qquad \textit{for} \quad v_0 > 0,\ v_1 \neq 0.$$

Proof. The basic technique we use is the construction of bounding functions α_i and β_i ($i = 1, 2$) such that

$$\alpha_i \leqslant \beta_i, \qquad i = 1, 2, \tag{A1}$$

$$\alpha_1(0, \varepsilon) \leqslant 0 \leqslant \beta_1(0, \varepsilon), \qquad \alpha_1(1, \varepsilon) \leqslant 0 \leqslant \beta_1(1, \varepsilon), \tag{AB1}$$

$$\alpha_1'' \geqslant \beta_2, \qquad \beta_1'' \leqslant \alpha_2, \tag{A2}$$

$$\alpha_2(0, \varepsilon) \leqslant v_0 \leqslant \beta_2(0, \varepsilon), \qquad \alpha_2(1, \varepsilon) \leqslant v_1 \leqslant \beta_2(1, \varepsilon), \tag{AB2}$$

$$\left.\begin{aligned} &\varepsilon\alpha_2'' + f(u, z)\,\alpha_2' - g(x, u, z)\alpha_2 \geqslant 0, \\ &\varepsilon\beta_2'' + f(u, z)\,\beta_2' - g(x, u, z)\,\beta_2 \leqslant 0, \end{aligned}\right. \tag{A3}$$

for all u in $[\alpha_1, \beta_1]$ and $z \in \mathbb{R}$. Again, we divide our discussion into four cases for the different signs of the boundary values v_0, v_1 when $f(u, u') = h(u)$ or $h(u')$. We note that $v_0 = v_1 = 0$ implies $v = 0$ and $u = 0$. We now consider the cases in which v_0 and v_1 are not both zero, and without loss of generality, we assume $\partial h/\partial z \geqslant 0$ for $z = u$ or u'.

Part (I). $f(u, u') = h(u)$. *Case* 1. $v_0 \geqslant 0$, $v_1 \geqslant 0$. We note that it is sufficient to show that $\varepsilon\beta_2'' + h(\beta_1)\,\beta_2' - g^*\beta_2 \leqslant 0$ if $\alpha_2(x, \varepsilon) = 0$. We define, for $0 \leqslant x \leqslant 1$ and $\varepsilon > 0$,

$$\alpha_1(x, \varepsilon) = \tfrac{1}{2}(v_1 + v_0)(x^2 - x), \qquad \beta_1(x, \varepsilon) = 0,\ \alpha_2(x, \varepsilon) = 0,$$

and

$$\beta_2(x, \varepsilon) = v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] + v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1 - x)\right].$$

Then it follows that (A1), (A2), and (AB1) are satisfied. Since $h(0) = 0$, that is, $h(\beta_1) = 0$, it follows that

$$\begin{aligned} &\varepsilon\beta_2'' + h(\beta_1)\,\beta_2' - g^*\beta_2 \\ &\quad = \varepsilon\left(\frac{g^*}{\varepsilon}\right) v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] \\ &\qquad + \varepsilon\left(\frac{g^*}{\varepsilon}\right) v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1 - x)\right] \\ &\qquad - g^* v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] - g^* v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1 - x)\right] = 0 \end{aligned}$$

and

$$\alpha_1''(x, \varepsilon) = (v_1 + v_0) \geqslant \beta_2, \qquad \beta_1'' = 0 = \alpha_2.$$

Therefore, we see that

$$0 \leqslant v(x, \varepsilon) \leqslant v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] + v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1-x)\right] \qquad \text{in } [0, 1].$$

As $\varepsilon \to 0^+$, we have

$$v(x, \varepsilon) \to 0 + BL(0) + BL(1) \qquad \text{for} \quad v_0 > 0,\ v_1 > 0,$$
$$v(x, \varepsilon) \to 0 + BL(1) \qquad \text{for} \quad v_0 = 0,\ v_1 > 0,$$

and

$$v(x, \varepsilon) \to 0 + BL(0) \qquad \text{for} \quad v_0 > 0,\ v_1 = 0.$$

Case 2. $v_0 \leqslant 0$, $v_1 \leqslant 0$. In this case, for x in $[0, 1]$ and $\varepsilon > 0$, we define

$$\alpha_1(x, \varepsilon) = 0, \qquad \beta_1(x, \varepsilon) = \tfrac{1}{2}(v_1 + v_0)(x^2 - x),$$
$$\alpha_2(x, \varepsilon) = -|v_0| \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] - |v_1| \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1-x)\right] - \Gamma(\varepsilon), \qquad \beta_2(x, \varepsilon) = 0,$$

where $\Gamma(\varepsilon) = \varepsilon\rho/g^*$, ρ is a positive constant. By the same calculations as in Case 1, we have

$$-|v_0| \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] - |v_1| \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1-x)\right] - \Gamma(\varepsilon) \leqslant v(x, \varepsilon) \leqslant 0,$$

from which it follows that

$$v(x, \varepsilon) \to 0 + BL(0) + BL(1) \qquad \text{for} \quad v_0 < 0,\ v_1 < 0,$$
$$v(x, \varepsilon) \to 0 + BL(1) \qquad \text{for} \quad v_0 = 0,\ v_1 < 0,$$

and

$$v(x, \varepsilon) \to 0 + BL(0) \qquad \text{for} \quad v_0 < 0,\ v_1 = 0 \text{ as } \varepsilon \to 0^+ \text{ in } [0, 1].$$

Case 3. $v_0<0$, $v_1>0$. Since $u(x, \varepsilon)$ changes sign in this case, there exists a unique interior turning point x_0 in $(0, 1)$ such that $u(x_0, \varepsilon)=0$, which implies that $h(u(x_0))=0$ and $v(x_0, \varepsilon)=0$. We consider the systems

$$\begin{gathered} u''=v \qquad \text{in } (0, x_0), \\ u(0, \varepsilon)=u(x_0, \varepsilon)=0, \\ \varepsilon v''+h(u)v'-g(x, u, u')v=0 \qquad \text{in } (0, x_0), \\ v(0, \varepsilon)=v_0, \qquad v(x_0, \varepsilon)=v^* \end{gathered} \tag{T1}$$

and

$$\begin{gathered} u''=v \qquad \text{in } (x_0, 1), \\ u(x_0, \varepsilon)=u(1, \varepsilon)=0, \\ \varepsilon v''+h(u)v'-g(x, u, u')v=0 \qquad \text{in } (x_0, 1), \\ v(x_0, \varepsilon)=v^*, \qquad v(1, \varepsilon)=v_1. \end{gathered} \tag{T2}$$

Since $v^*=0$, $v_0<0$, and $v_1>0$, applying the result of Case 1 to (T2) and applying the result of Case 2 to (T1), we have $v(x, \varepsilon)\to 0+BL(0)$ as $\varepsilon\to 0^+$ in $(0, x_0)$ and $v(x, \varepsilon)\to 0+BL(1)$ as $\varepsilon\to 0^+$ in $(x_0, 1)$. Therefore, we conclude that

$$v(x, \varepsilon)\to 0+BL(0)+BL(1) \qquad \text{as} \quad \varepsilon\to 0^+ \text{ in } [0, 1].$$

Case 4. $v_0>0$, $v_1<0$. We note that this case is just the reflection of Case 3, and so we have $v(y, \varepsilon)\to 0+BL(0)+BL(1)$ as $\varepsilon\to 0^+$ in $[0, 1]$.

Part (II). $f(u, u')=h(u')$. The difference between (I) and (II) is that there is no relation between the behavior of $h(u')$ and $h(\alpha_1)$ or $h(\beta_1)$, in general. However, the reduced equation, $f(u, u')v'+g(x, u, u')v=0$, has only the trivial solution, and so the asymptotic behavior of the solution $v(x, \varepsilon)$ of the coupled system (6.1) in this case is not as complicated as in Models I and II. In fact, we only need to prove that $v(x, \varepsilon)\to 0$ as $\varepsilon\to 0^+$ for $v_0\geqslant 0$, $v_1\geqslant 0$, the three cases $v_0\leqslant 0$, $v_1\leqslant 0$; $v_0<0$, $v_1>0$; and $v_0>0$, $v_1<0$, can be reduced to the first case. It is convenient to consider the subspaces $v_0=0$, $v_1>0$; $v_0>0$, $v_1=0$; and $v_0>0$, $v_1>0$ separately.

(i) $v_0=0$, $v_1>0$. We define $\alpha_1(x, \varepsilon)=\zeta v_1(x^2-x)/2$, $\beta_1(x, \varepsilon)=0$, $\alpha_2(x, \varepsilon)=0$, and $\beta_2(x, \varepsilon)=w_R(x, \varepsilon)+\Gamma(\varepsilon)$, where $w_R(x, \varepsilon)=|v_1| \exp[-(g_\varepsilon^*)^{1/2}(1-x)]$, $\Gamma(\varepsilon)=\varepsilon\rho/g^*$, and ζ and ρ are positive constants to be determined. We note that $\alpha_1(x, \varepsilon)\leqslant 0$ in $[0, 1]$; it is clear that conditions (A1), (AB1), and (AB2) are satisfied. From Taylor's Theorem, we have

$$\begin{aligned} -h(u')\beta_2'+g^*\beta_2-\varepsilon\beta_2'' &\geqslant g^*\beta_2-|h(u')w_R'(x, \varepsilon)|-\varepsilon\beta_2'' \\ &\geqslant \varepsilon\rho-|h(u')w_R'(x, \varepsilon)| \end{aligned}$$

since $h(u')\,w'_R(x,\varepsilon)>0$ in $[1-\delta, 1]$ for some $\delta>0$. In the rest of the interval $[0, 1]$,

$$w'_R(x,\varepsilon)=|v_1|\left(\frac{g^*}{\varepsilon}\right)^{1/2}\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right],$$

and so for sufficiently small ε, we have $|h(u')\,w'_R(x,\varepsilon)|\leqslant c_1\varepsilon$ for some constant $c_1>0$. Choose $\rho\geqslant c_1$, then we see that $-h(u')\beta'_2+g^*\beta_2-\varepsilon\beta''_2\geqslant 0$. Thus (A3) holds. To prove (A2), since $\beta_1(x,\varepsilon)=\alpha_2(x,\varepsilon)=0$ and $\alpha''_1(x,\varepsilon)=\zeta v_1$, choose $\zeta v_1\geqslant |v_1|+\varepsilon\rho/g^*$, then we have $\alpha''_1\geqslant\beta_2$. Therefore, $0\leqslant v(x,\varepsilon)\leqslant |v_1|\exp[-(g^*/\varepsilon)^{1/2}(1-x)]+\Gamma(\varepsilon)$. It follows that $v(x,\varepsilon)\to 0+BL(1)$ as $\varepsilon\to 0^+$ in $[0, 1]$. If $v_0>0$, $v_1=0$. This case is the reflection of case (i). We conclude that $v(x,\varepsilon)\to 0+BL(0)$ as $\varepsilon\to 0^+$ in $[0, 1]$.

(ii) $v_0>0$, $v_1>0$. Since both boundary values v_0 andv_1 are positive, we expect that there are boundary layers at both endpoints $x=0$ and $x=1$. We define, for $0\leqslant x\leqslant 1$ and $\varepsilon>0$, $\alpha_1(x,\varepsilon)=w(v_0+v_1)(x^2-x)/2$, $\beta_1(x,\varepsilon)=0$, $\alpha_2(x,\varepsilon)=0$, and

$$\beta_2(x,\varepsilon)=w_R(x,\varepsilon)+w_L(x,\varepsilon)+\Gamma(\varepsilon),$$

where $\Gamma(\varepsilon)=\varepsilon\tau/g^*$ for τ is a positive constant to be determined, $w_R(x,\varepsilon)$ is defined in (i), and $w_L(x,\varepsilon)=|v_0|\exp[-(g^*/\varepsilon)^{1/2}x]$. We note that $h(u')\,w'_R(x,\varepsilon)>0$ in $[1-\delta, \delta]$, $h(u')\,w'_L(x,\varepsilon)>0$ in $[0, \delta]$ for some $\delta>0$, and

$$|h(u')\{w'_R(x,\varepsilon)+w'_L(x,\varepsilon)\}|\leqslant c_3\varepsilon \qquad \text{in } (\delta, 1-\delta)$$

for ε sufficiently small and some constant $c_3>0$. If we now choose $\tau\geqslant c_3$ and $w(v_0+v_1)\geqslant |v_0+v_1|+\varepsilon\tau/g^*$, then it follows that

$$\begin{aligned}-h(u')\beta'_2+g^*\beta_2-\varepsilon\beta''_2&=-h(u')(w'_R+w'_L)+g^*\beta_2-\varepsilon\beta''_2\\ &\geqslant\varepsilon\tau-|h(u')(w'_R+w'_L)|\geqslant 0 \qquad \text{in } [0, 1]\end{aligned}$$

and

$$\alpha''_1(x,\varepsilon)=w(v_0+v_1)\geqslant\beta_2.$$

Therefore, we finish the proof to have $0\leqslant v(x,\varepsilon)\leqslant\beta_2(x,\varepsilon)$ in $[0, 1]$, that is,

$$v(x,\varepsilon)\to 0+BL(0)+BL(1) \qquad \text{as } \varepsilon\to 0^+ \qquad \text{in } [0, 1].$$

We expect that this result is true for the more general system

$$u''=k(v), \qquad \text{in } [0, 1]$$
$$u(0, \varepsilon)=u(1, \varepsilon)=0$$
$$\varepsilon v''+f(u, u')v'-g(x, u, u')v=0, \qquad \text{in } [0, 1]$$
$$v(0, \varepsilon)=v_0, \qquad v(1, \varepsilon)=v_1,$$

where $k(v)$ is a continuous function and $\partial k/\partial v$ does not change sign for x in $(0, 1)$.

6. Stability of the Steady-State Solution

In the last section of this paper, we turn to examine the stability of the steady-state solution $U=U(x, \varepsilon)$, $V=V(x, \varepsilon)$ with respect to the perturbations in the initial data as $\varepsilon \to 0^+$. Consider th time-dependent version of (1.1)

$$\begin{aligned}
&u_{xx}-v=u_t, && (x, t) \text{ in } (0, 1)\times(0, \infty),\\
&u(x, 0, \varepsilon)=\phi_1(x, \varepsilon), && x \text{ in } [0, 1],\\
&u(0, t, \varepsilon)=u(1, t, \varepsilon)=0, && t \text{ in } [0, \infty),\\
&\varepsilon v_{xx}+f(u, u_x)\, v_x-g(x, u, u_x)v=v_t, && (x, t) \text{ in } (0, 1)\times(0, \infty),\\
&v(x, 0, \varepsilon)=\phi_2(x, \varepsilon), && x \text{ in } [0, 1],\\
&v(0, t, \varepsilon)=v_0, \quad v(1, t, \varepsilon)=v_1, && t \text{ in } [0, \infty),
\end{aligned} \tag{6.1}$$

where $f(u, u_x)=h(u)$ or $h(u_x)$, $\partial h/\partial y>0$ for $h=h(y)$, and $g(x, u, u_x)\geqslant 0$. We assume that $\|\phi_i\|_\infty$ $(i=1, 2)$ is bounded in $[0, 1]\times[0, \infty)$ as $\varepsilon\to 0^+$, where $\|\psi\|_\infty$ is defined as the L^∞-norm of y. In other words, (U, V) is the solution of the system of equations

$$\begin{aligned}
&U_{xx}=V && \text{in } (0, 1),\\
&U(0, \varepsilon)=U(1, \varepsilon)=0, &&\\
&\varepsilon V_{xx}+f(U, U_x)\, V_x-g(x, U, U_x)V=0 && \text{in } (0, 1),\\
&V(0, \varepsilon)=v_0, \quad V(1, \varepsilon)=v_1, &&
\end{aligned} \tag{6.2}$$

that displays boundary layer or interior layer behavior as $\varepsilon\to 0^+$. We want to examine whether the solution (U, V) of (6.2) is stable or asymptotically stable with respect to perturbatios of the initial data. We use the criteria

which Howes gave in [6, 7] to test the stability of (U, V) for all boundary values v_0 and v_1.

Considering the u-equation $u_t = u_{xx} - v$ of (6.1), we see from the maximum principle that $|u| \leqslant \max |v|$ for all (x, t) in $[0, 1] \times (0, \infty)$; in particular, $|U| \leqslant \max |V|$. Thus, it is sufficient to test the stability of the solution V of (6.2). We introduce the pertubations

$$w = w(x, t, \varepsilon) = u(x, t, \varepsilon) - U(x, \varepsilon), \qquad z = z(x, t, \varepsilon) = v(x, t, \varepsilon) - V(x, \varepsilon)$$

into (6.1) and define

$$\begin{aligned} A(x, w, w_x, \varepsilon) &= f(w + U, w_x + U_x), \\ B(x, w, w_x, z, \varepsilon) &= -g(x, w + U, w_x + U_x) z \\ &\quad + [f(w + U, w_x + U_x) - f(U, U_x)] V_x \\ &\quad + [g(x, w + U, w_x + U_x) - g(x, U, U_x)] V, \\ y_1(x, \varepsilon) &= \phi_1(x, \varepsilon) - U(x, \varepsilon), \qquad y_2(x, \varepsilon) = \phi_2(x, \varepsilon) - V(x, \varepsilon). \end{aligned}$$

Here $f(w + U, w_x + U_x) = h(w + U)$ or $h(w_x + U_x)$ and $f(U, U_x) = h(U)$ or $h(U_x)$.

The perturbations are governed then by the system

$$\begin{gathered} w_{xx} = z + w_t, \\ w(0, t, \varepsilon) = w(1, t, \varepsilon) = 0, \qquad w(x, 0, \varepsilon) = \psi_1(x, \varepsilon), \\ \varepsilon z_{xx} + A(x, w, \varepsilon) z_x + B(x, w, w_x, z, \varepsilon) = z_t, \\ z(x, 0, \varepsilon) = \psi_2(x, \varepsilon), \qquad z(0, t, \varepsilon) = z(1, t, \varepsilon) = 0 \end{gathered} \tag{6.3}$$

We now turn to testing the stability of the zero solution of (6.3). Since $-\partial B/\partial z = g(x, u, u_x) \geqslant 0$ for all (x, t) in $[0, 1] \times [0, \infty)$, it is clear that the zero solution of (6.3) is stable with respect to all initial perturbations as $\varepsilon \to 0^+$. It follows that the solution V of (6.2) is stable for all initial perturbations as $\varepsilon \to 0^+$. For given boundary values v_0 and v_1 which are not both zero, we may ask under what conditions V is asymptotically stable. To answer this question, we need to find a positive constant c such that either $|A(x, w, w_x, \varepsilon)| \geqslant c$ or $-\partial B/\partial z \geqslant c$; cf. [6, 7].

We note that $|f(w + U, \; w_x + U_x)| \geqslant k > 0$ in $\Delta \times (0, \infty)$, where $\Delta = \{[0, 1] - T\}$, $T = \{x^* \mid x^*$ in $(0, 1)$ and $f(x^*, u(x^*), u'(x^*)) = 0\}$. Therefore, if (i) $f(u, u_x) = h(u)$, $g(x, u, u_x) \equiv 0$, for some small δ, $\delta^* > 0$, we have

(1) $\Delta = [\delta, 1 - \delta]$ if $v_0 \geqslant 0$, $v_1 \geqslant 0$ or $v_0 \leqslant 0$, $v_1 \leqslant 0$;

(2) $\Delta = [\delta, x_0 - \delta^*] \cup [x_0 + \delta^*, 1 - \delta]$ if $v_0 > 0$, $v_1 < 0$ or $v_0 < 0$, $v_1 > 0$.

Then for $\|\psi_2\|_\infty \leqslant \eta$, we have $|z(x, t, \varepsilon)| \leqslant C\exp(-\sigma t)$ in $\Omega = \Delta \times (0, \infty) \times (0, \varepsilon_0]$, $0 < \varepsilon_0 \leqslant \varepsilon$, where $C = \|\psi_2\|_\infty [2e^{-\lambda} - 1]$ and $\sigma = r/(e^{-\lambda} - 1)$, for $\lambda = -r/k + O(\varepsilon)$ a negative root of the equation $\varepsilon\lambda^2 + k\lambda + r = 0$. Therefore, $z(x, t, \varepsilon)$ is asymptotically stable in Ω. That is, the solution $V(x, \varepsilon)$ of (6.2) is asymptotically stable in Δ.

(ii) $f(u, u_x) = h(u_x)$, $g(x, u, u_x) \equiv 0$. In this case, there exist x_m in $(0, 1)$, $m = 1, 2, \ldots, 6$, such that

$$h(w_x(x_n, t, \varepsilon) + U_x(x_n, \varepsilon)) = 0 \qquad \text{for} \quad (II_n) \text{ if } n = 1, 2;$$

$$h(w_x(x_m, t, \varepsilon) + U_x(x_m, \varepsilon)) = 0 \qquad \text{for} \quad (II_n) \text{ if } n = 3, 4 \text{ and } m = n \text{ or } n+1,$$

where

$$(II_1) := \{v_0 \geqslant 0, v_1 \geqslant 0\}, \qquad (II_2) := \{v_0 \leqslant 0, v_1 \leqslant 0\},$$
$$(II_3) := \{v_0 < 0, v_1 > 0\}, \qquad \text{and} \qquad (II_4) := \{v_0 > 0, v_1 < 0\}.$$

Thus there exist positive constants k_n $(n = 1, 2, 3, 4)$ such that

$$|h(w_x + U_x)| \geqslant k_n > 0 \qquad \text{in } (0, 1) \times (0, \infty) - \omega_n, n = 1, 2, 3, 4,$$

where

$$\omega_n = I_n \times (0, \infty), \qquad n = 1, 2, 3, 4,$$
$$I_n = (x_n - \delta_n, x_n + \delta_n) \qquad \text{if} \quad n = 1 \text{ or } 2,$$
$$I_n = \{(x_n - \delta_n, x_n + \delta_n) \cup (x_{n+1} - \delta_{n+1}, x_{n+1} + \delta_{n+1})\}$$
$$\text{if} \quad n = 3 \text{ or } 4,$$

for some δ_n such that $0 < \delta_n < 1$. Therefore, by the same arguments as in above, we conclude that the zero solution of (6.2) in this case is asymptotically stable with respect to all initial perturbations as $\varepsilon \to 0^+$ in $\Omega_n = \omega_n \times (0, \varepsilon_0]$ for (II_n), $n = 1, 2, 3, 4$. This is equivalent to saying that the steady-state solution V of (6.1) is asymptotically stable with respect to all initial perturbations as $\varepsilon \to 0^+$ in $\Delta_n = \{[\delta, 1 - \delta] - I_n\}$ for (II_n), $n = 1, 2, 3, 4$.

(iii) $f(u, u_x) = h(u)$ or $h(u_x)$, $g(x, u, u_x) \geqslant g^* > 0$. In this final case, since $-\partial B/\partial z = g(x, u, u_x) \geqslant g^* > 0$, the $B(x, w, w_x, z, \varepsilon)$-term dominates the stability of the solution z. We have therefore that for all initial perturbations ψ_2,

$$|z(x, t, \varepsilon)| \leqslant \|\psi_2\|_\infty \exp(-g^* t) \qquad \text{in } \Delta = (0, 1) \times (0, \infty) \times (0, \varepsilon_0].$$

It follows that the zero solution of (6.3) is asymptotically stable with respect to all initial perturbations as $\varepsilon \to 0^+$, which implies the asymptotic stability of the steady-state solution V of (6.1).

To summarize our results in this section, we conclude that the steady-state solution (U, V) of (6.1) is stable with respect to all initial perturbations as $\varepsilon \to 0^+$ in $[0, 1]$. Moreover, V is asymptotically stable for x in $[\delta, 1-\delta]$ when $g(x, u, u') \geq g^* > 0$ and V is asymptotically stable for x in $\{[0, 1] - \{x^*\}\}$ when $g(x, u, u') \equiv 0$, where x^* in $(0, 1)$ and $f(x^*, u(x^*), u'(x^*)) = 0$.

ACKNOWLEDGMENT

The author expresses the deepest gratitude to her advisor Professor F. A. Howes for his guidance and encouragement.

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