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Refined Approximations of the Solutions of a Coupled System with Turning Points

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We present in this paper the asymptotic behavior of solutions of a boundary value problem for a coupled system of differential equations $u''=v$, $\varepsilon v''+f(u,u')v'-g(x,u,u')v=0$ on a compact interval I, where $f(u,u')$ has turning points in I. We provide upper and lower solutions, $\beta(x,\varepsilon)$ and $\alpha(x,\varepsilon)$, respectively, which bound solutions, exhibiting boundary layer and interior layer behavior, for which $\lim_{\varepsilon\to 0^+}\{\beta(x,\varepsilon)-\alpha(x,\varepsilon)\}=0$ uniformly on I.

1. INTRODUCTION

Consider the Dirichlet problem for the coupled system of differential equations

$$\begin{aligned} &u''=v \qquad \text{in } (0,1),\\ &u(0,\varepsilon)=0, \qquad u(1,\varepsilon)=0,\\ &\varepsilon v''+f(u,u')v'-g(x,u,u')v=0 \qquad \text{in } (0,1),\\ &v(0,\varepsilon)=v_0, \qquad v(1,\varepsilon)=v_1, \end{aligned} \tag{1.1}$$

for $g(x,u,u')\geqslant 0$ and $f(u,u')=h(u)$ or $h(u')$, where $h(0)=0$, and ε is small positive parameter. We assume that for each $\varepsilon>0$, there exists at most one interior turning point x_0 in $(0,1)$ such that $u(x_0,\varepsilon)=0$. We also assume that $\partial f/\partial u$ and $\partial f/\partial u'$ do not change sign, and $f(u,u')$ and $g(x,u,u')$ are of class $C^{(1)}[0,1]$. System (1.1) was studied by Dorr and Parter [3, 4] and recently by Howes and Shao [8] and Shao [12] who extended and amplified their results. This system is a simple model of the streamfunctionvorticity equations governing the steady-state, two-dimensional, viscous, incompressible flow as the Reynolds number $\mathrm{Re} \to \infty$. That is,

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = -\omega$$

$$\frac{1}{\mathrm{Re}} \nabla^2\omega + \psi_x\omega_y - \psi_y\omega_x = 0$$

$$(x, y) \text{ in } G,\ \psi,\ \omega \text{ prescribed on the boundary of } G, \qquad (1.2)$$

where ψ is the streamfunction of the flow, that is, $\psi_y = u$, $-\psi_x = v$, for $\underline{u} = (u, v, 0)$ the velocity field of the flow, and G is a bounded, open connected subset of $\mathbb{R}^2$. The purpose of this paper is to provide refined approximations to the solutions of (1.1), which describe the limiting solutions, boundary layer solutions, interior layer solutions, as well as the S-layer or Z-layer solutions as $\varepsilon \to 0^+$ which exhibits the thickness of the boundary layers. Our refinements are in the spirit of Kirschvink [9] as refinements of Howes [1, 5] for similar problems in the absence of turning points.

The contents of the various sections are as follows. In Section 2 we state some preliminaries which include the limiting solutions of system (1.1) (cf. [8, 12]). In Sections 3–5 we discuss Model I: $u'' = v$, $\varepsilon v'' + h(u)v' = 0$; Model II: $u'' = v$, $\varepsilon v'' + h(u')v' = 0$; Model III: $u'' = v$, $\varepsilon v'' + h(u)v' - g(x, u, u')v = 0$, $g \geq g^* > 0$; and Model IV: $u'' = v$, $\varepsilon v'' + h(u')v' - g(x, u, u')v = 0$, $g \geq g^* > 0$.

2. Preliminaries

For convenience and simplification, we consider system (1.1) for the following models:

Model I: $u'' = v$, $\varepsilon v'' + h(u)v' = 0$;

Model II: $u'' = v$, $\varepsilon v'' + h(u')v' = 0$;

Model III: $u'' = v$, $\varepsilon v'' + h(u)v' - g(x, u, u')v = 0$, $g \geq g^* > 0$;

Model IV: $u'' = v$, $\varepsilon v'' + h(u')v' - g(x, u, u')v = 0$, $g \geq g^* > 0$.

The limiting behavior of solutions in of these models can be summarized in Fig. 1–3.

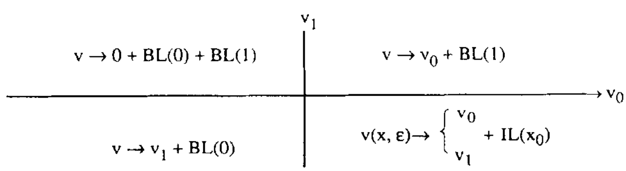

FIG. 1. Boundary value portrait for Model I.

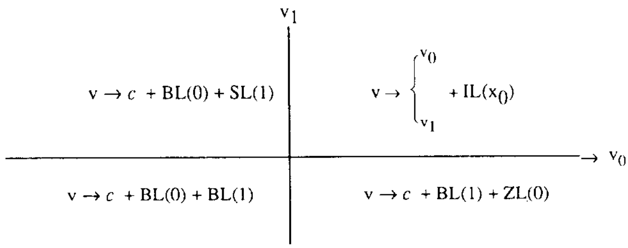

FIG. 2. Boundary value portrait for Model II.

Our treatment is through bounding functions and the generalized Nagumo's Theorem (cf. [12, 13]). Clearly, Nagumo's condition is satisfied, so we only need to exhibit bounding functions α_i and β_i ($i=1, 2$) such that

$$\begin{aligned}&\alpha_i \leqslant \beta_i, \qquad i=1,2,\\ &\alpha_1(0,\varepsilon) \leqslant 0 \leqslant \beta_1(0,\varepsilon), \qquad \alpha_1(1,\varepsilon) \leqslant 0 \leqslant \beta_1(1,\varepsilon),\\ &\alpha_1'' \geqslant \beta_2, \qquad \beta_1'' \leqslant \alpha_2,\\ &\alpha_2(0,\varepsilon) \leqslant v_0 \leqslant \beta_2(0,\varepsilon), \qquad \alpha_2(1,\varepsilon) \leqslant v_1 \leqslant \beta_2(1,\varepsilon),\\ &\varepsilon\alpha_2'' + f(u,z)\alpha_2' - g(x,u,z)\alpha_2 \geqslant 0,\\ &\varepsilon\beta_2'' + f(u,z)\beta_2' - g(x,u,z)\beta_2 \leqslant 0,\end{aligned} \tag{2.1}$$

for all u in $[\alpha_1, \beta_1]$ and z in $\mathbb{R}$. Then the system (1.1) has a (unique) solution ($u=u(x,\varepsilon)$, $v=v(x,\varepsilon)$) such that

$$\begin{aligned}&\alpha_1(x,\varepsilon) \leqslant u(x,\varepsilon) \leqslant \beta_1(x,\varepsilon),\\ &\alpha_2(x,\varepsilon) \leqslant v(x,\varepsilon) \leqslant \beta_2(x,\varepsilon),\end{aligned} \tag{2.2}$$

for x in [0, 1] (the existence and uniqueness of the solution was proved by Dorr and Parter [3]). We divide our discussion into four cases according to the different signs of the boundary values v_0 and v_1, that is, (1) $v_0 \geqslant 0$,

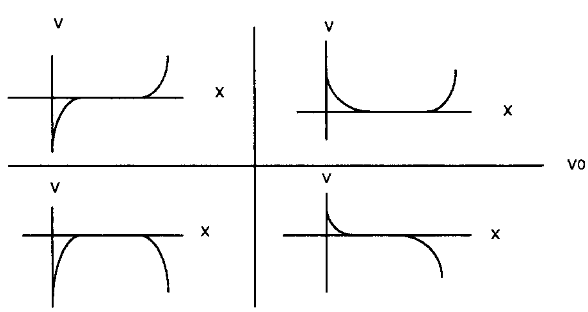

FIG. 3. Boundary value portrait of Model III and Model IV.

$v_1 \geqslant 0$; (2) $v_0 \leqslant 0$, $v_1 \leqslant 0$; (3) $v_0 < 0$, $v_1 > 0$; and (4) $v_0 > 0$, $v_1 < 0$, for each model.

3. Model I: $u'' = v$, $\varepsilon v'' + h(u)v' = 0$

Consider the following system:

$$\begin{aligned} &u'' = v \qquad \text{in } (0, 1), \\ &u(0, \varepsilon) = u(1, \varepsilon) = 0, \\ &\varepsilon v'' + h(u)v' = 0 \qquad \text{in } (0, 1), \\ &v(0, \varepsilon) = v_0, \qquad v(1, \varepsilon) = v_1. \end{aligned} \tag{3.1}$$

We have the following theorem.

Theorem 1. *If $(u(x, \varepsilon), v(x, \varepsilon))$ is the solution of (3.1), then*

(i) $$v(x, \varepsilon) = v_0 + (v_1 - v_0) \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} (1 - x) \right] + O(\varepsilon^{1/2})$$

for $v_0 \geqslant 0, v_1 \geqslant 0$;

(ii) $$v(x, \varepsilon) = v_1 + (v_1 - v_0) \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} x \right] + O(\varepsilon^{1/2})$$

for $v_0 \leqslant 0, v_1 \leqslant 0$;

(iii) $$v(x, \varepsilon) = \begin{cases} v_0 + w_L(x - O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } 0 \leqslant x \leqslant x_0, \\ v_1 + w_R(x + O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } x_0 \leqslant x \leqslant 1, \end{cases}$$

for $v_0 > 0, v_1 < 0$;

(iv) $$v(x, \varepsilon) = w_L^*(x, \varepsilon) + w_R^*(x, \varepsilon) + O(\varepsilon^{1/2})$$

for $v_0 < 0, v_1 > 0$.

where $w_L(x, \varepsilon) = v_0 \exp[-(k/\varepsilon)^{1/2} (x_0 - x)]$ *and* $w_R(x, \varepsilon) = v_1 \exp[-(k/\varepsilon)^{1/2} (x - x_0)]$, *for all x in* $[0, 1]$ *and each* $\varepsilon > 0$.

Proof. We know that the $u(x, \varepsilon)$ depends on $v(x, \varepsilon)$ and bounding functions α_i and β_i $(i = 1, 2)$ must satisfy the following conditions:

(A1) $\alpha_i \leqslant \beta_i\, i = 1, 2$,

(A2) $\alpha_1'' \geqslant \beta_2$, $\beta_1'' \leqslant \alpha_2$,

(A3) $\varepsilon\alpha_2'' + h(u)\alpha_2' \geqslant 0$, $\varepsilon\beta_2'' + h(u)\beta_2' \leqslant 0$, for all u in $[\alpha_1, \beta_1]$

(B1) $\alpha_1(0, \varepsilon) \leqslant 0 \leqslant \beta_1(0, \varepsilon)$, $\alpha_1(1, \varepsilon) \leqslant 0 \leqslant \beta_1(1, \varepsilon)$,

(B2) $\alpha_2(0, \varepsilon) \leqslant v_0 \leqslant \beta_2(0, \varepsilon)$, $\alpha_2(1, \varepsilon) \leqslant v_1 \leqslant \beta_2(1, \varepsilon)$.

Without lost generality, we assume $dh/du > 0$, and we divide the discussion into the four cases indicated above.

Case 1. $v_0 \geqslant 0, v_1 \geqslant 0$.

Since $(v_0, v_1) = (0, 0)$ implies $(u(x, \varepsilon), v(x, \varepsilon)) \equiv (0, 0)$, we assume now that $(v_0, v_1) \neq (0, 0)$. The u-equation $u'' = v$ with the boundary conditions $u(0, \varepsilon) = u(1, \varepsilon) = 0$ gives $u \leqslant 0$ in $[0, 1]$, $u < 0$ in $[\delta, 1-\delta]$ for each $1 > \delta > 0$ and so $h(u) \leqslant 0$ in $[0, 1]$, $h(u) < 0$ in $[\delta, 1-\delta]$. By virtue of the linear case, the solution $v(x, \varepsilon)$ should display a boundary layer at $x = 1$ and thus the outer solution of v is v_0 in this case. We define the bounding functions α_i and β_i $(i = 1, 2)$:

$$\alpha_1(x, \varepsilon) = (M + c^*)\, x(x-1), \qquad \beta_1(x, \varepsilon) = \frac{mx(x-1)}{4};$$

$$\alpha_2(x, \varepsilon) = v_0 + w_1(x, \varepsilon), \qquad \beta_2(x, \varepsilon) = v_0 + w_2(x, \varepsilon),$$

where $M = \max\{v_0, v_1\}$, $m = \min\{v_0, v_1, (1/2)\,|v_1 - v_0|\}$, $w_i = w_i(x, \varepsilon)$ $(i = 1, 2)$ are the unique solutions of the problems

$$\varepsilon w_1'' - k w_1 = c \exp\left[-\sigma\left(\frac{k}{\varepsilon}\right)^{1/2}\right], \qquad x \text{ in } (0, 1),$$

$$w_1(0, \varepsilon) = (v_1 - v_0), \qquad \lim_{\varepsilon \to 0^+} w_1(x, \varepsilon) = 0$$

and

$$\varepsilon w_2'' - k w_2 = -c \exp\left[-\sigma\left(\frac{k}{\varepsilon}\right)^{1/2}\right], \qquad x \text{ in } (0, 1),$$

$$w_2(0, \varepsilon) = (v_1 - v_0), \qquad \lim_{\varepsilon \to 0^+} w_2(x, \varepsilon) = 0,$$

i.e.

$$w_1 = w(x, \varepsilon) - c^* \Gamma(x, \varepsilon), \qquad w_2 = w(x, \varepsilon) + c^* \Gamma(x, \varepsilon),$$

where

$$w(x, \varepsilon) = |v_1 - v_0| \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right], \qquad c^* = \frac{kc}{1-\sigma^2},$$

$$\Gamma(x, \varepsilon) = \exp\left[-\sigma\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right] - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right],$$

k, c^*, and σ are positive constants such that

$$k\colon h(u) < -k < 0 \text{ in } [\delta, 1-\delta] \text{ for small } \delta > 0 \text{ and } 0 < k < -h(u)\left(\frac{k}{\varepsilon}\right)^{1/2};$$

$$\sigma\colon 0 < \sigma < 1, \left\{\sigma \exp\left[(1-\sigma)\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\} > 1$$

$$\text{and } \left\{\sigma^2 \exp\left[(1-\sigma)\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\} < 1;$$

$$c^*\colon c^*\left\{\exp\left[-\sigma\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right] - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\}$$

$$> w,\ |w(x, \varepsilon) - c^*\Gamma(x, \varepsilon)| < O(\varepsilon^{1/2}).$$

It is clear that α_i and β_i $(i = 1, 2)$ satisfy conditions (A1), (B1), and (B2), but we need to show that (A2) and (A3) are true. Since

$$\alpha_1''(x, \varepsilon) = 2M + 2c^* \geqslant v_0 + (v_1 - v_0)\exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right] + c^*\Gamma(x, \varepsilon)$$

and

$$\beta_1''(x, \varepsilon) = \frac{m}{2} \leqslant v_0 + (v_1 - v_0)\exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right] - c^*\Gamma(x, \varepsilon)$$

for all x in $[0, 1]$, we have $\alpha_1'' \geqslant \beta_2$ and $\beta_1'' \leqslant \alpha_2$. Furthermore,

$$\begin{aligned}
&\varepsilon\alpha_2'' + h(u)\alpha_2' \\
&\quad = \varepsilon w_1'' + h(u) w_1' \\
&\quad = k\left[w + c^*\left\{\exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right] - \sigma^2 \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\}\right] \\
&\qquad + h(u)\left(\frac{k}{\varepsilon}\right)^{1/2}\left[w + c^*\left\{\exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right.\right. \\
&\qquad \left.\left. - \sigma \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\}\right] \geqslant 0,
\end{aligned}$$

$$\begin{aligned}
&\varepsilon\beta_2'' + h(u)\beta_2' \\
&\quad = \varepsilon w_2'' + h(u) w_2' \\
&\quad = k\left[w + c^*\left\{\sigma^2 \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right] - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\}\right] \\
&\qquad + h(u)\left(\frac{k}{\varepsilon}\right)^{1/2}\left[w + c^*\left\{\exp\left[-\sigma\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right.\right. \\
&\qquad \left.\left. - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\}\right] \leqslant 0
\end{aligned}$$

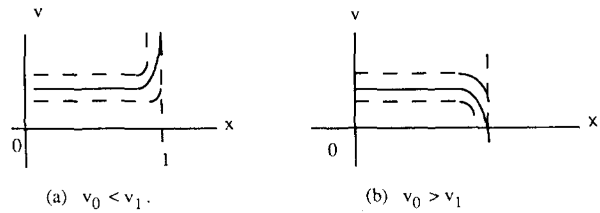

FIG. 4. $v_0 \geqslant 0, v_1 \geqslant 0.$

by $w > 0$, $\Gamma(x, \varepsilon) > 0$. Thus from our definitions of the constants, we have

$$\alpha_1(x, \varepsilon) \leqslant u(x, \varepsilon) \leqslant \beta_1(x, \varepsilon)$$

and

$$v(x, \varepsilon) = v_0 + (v_1 - v_0) \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} (1-x)\right] + O(\varepsilon^{1/2}),$$

for all x in $[0, 1]$ and each $\varepsilon > 0$, where $|v_1 - v_0| \exp[-(k/\varepsilon)^{1/2}(1-x)]$ is the boundary layer solution at $x = 1$. The thickness of the boundary layer is of order $\varepsilon^{1/2}$. The refined approximation of the solution is shown by the narrow region in Fig. 4.

Case 2. $v_0 \leqslant 0, v_1 \leqslant 0.$

This case is handled by reflection. Making the change of variables $y = 1 - x$, $\mathbf{m}(y, \varepsilon) = -u(1-y, \varepsilon)$, $\mathbf{n}(y, \varepsilon) = -v(1-y, \varepsilon)$, the system (3.1) becomes

$$\mathbf{m}'' = \mathbf{n}$$

$$\mathbf{m}(0, \varepsilon) = \mathbf{m}(1, \varepsilon) = 0,$$

$$\varepsilon\mathbf{n}'' + h(\mathbf{m})\mathbf{n}' = 0,$$

$$\mathbf{n}(0, \varepsilon) = \mathbf{n}_0 = -v_1, \mathbf{n}(1, \varepsilon) = \mathbf{n}_1 = -v_0,$$

provided $v_0 \geqslant 0$ and $v_1 \geqslant 0$, which is Case 1.

Case 3. $v_0 > 0, v_1 < 0.$

We note that $u(x, \varepsilon)$ changes sign in $(0, 1)$ in this case, and so does $h(u)$. Since $u'' = v$, $u(0, \varepsilon) = u(1, \varepsilon) = 0$, there exists a unique x_0 in $(0, 1)$ such that $u(x_0, \varepsilon) = 0$, $u'(x_0, \varepsilon) \neq 0$, and hence $h(u(x_0, \varepsilon)) = 0$ and $v(x_0, \varepsilon) = 0$. Since $h(u) \leqslant 0$ in $[0, x_0]$ and $h(u) \geqslant 0$ in $[x_0, 1]$; $J[x] = \int_{v_1}^{v_0} -h(u(x_0, \varepsilon))\,ds = [-h(u(x_0, \varepsilon)](v_0 - v_1) = 0$ iff $x = x_0$, there must be an interior layer at

$x = x_0$ as $\varepsilon \to 0^+$ in $[0, 1]$. We approach this case by considering the following submodel problems

$$\begin{aligned} &u'' = v \qquad \text{in } (0, x_0), \\ &u(0, \varepsilon) = u(x_0, \varepsilon) = 0, \\ &\varepsilon v'' + h(u)v' = 0 \qquad \text{in } (0, x_0), \\ &v(0, \varepsilon) = v_0, \qquad v(x_0, \varepsilon) = 0 \end{aligned} \tag{V$_\text{I}$1}$$

and

$$\begin{aligned} &u'' = v \qquad \text{in } (x_0, 1), \\ &u(x_0, \varepsilon) = u(1, \varepsilon) = 0, \\ &\varepsilon v'' + h(u)v' = 0 \qquad \text{in } (x_0, 1), \\ &v(x_0, \varepsilon) = 0, \qquad v(1, \varepsilon) = v_1 \end{aligned} \tag{V$_\text{I}$2}$$

separately. The conditions $h(u) \leqslant 0$ in $[0, x_0]$ and $h(u) \geqslant 0$ in $[x_0, 1]$ imply that there are boundary layers at $x = x_0$ in $(\mathrm{V_I}1)$ and $(\mathrm{V_I}1)$ respectively. Choosing $\delta = O(\varepsilon)$ it follows from Cases 1 and 2 that $v_L(x, \varepsilon) = v_0 + w_L(x, \varepsilon) + O(\varepsilon^{1/2})$ is the solution of system $(\mathrm{V_I}1)$ and $v_R(x, \varepsilon) = v_0 + w_R(x, \varepsilon) + O(\varepsilon^{1/2})$ is the solution of system $(\mathrm{V_I}2)$, where w_L and w_R are the unique solutions of the problems

$$\begin{aligned} &\varepsilon w_L'' - k w_L = 0 \qquad x \text{ in } (0, x_0), \\ &w_L(x_0, \varepsilon) = v_0, \qquad \lim_{\varepsilon \to 0^+} w_L(x, \varepsilon) = 0, \end{aligned}$$

and

$$\begin{aligned} &\varepsilon w_R'' - k w_R = 0 \qquad x \text{ in } (x_0, 1), \\ &w_R(x_0, \varepsilon) = v_1, \qquad \lim_{\varepsilon \to 0^+} w_R(x, \varepsilon) = 0, \end{aligned}$$

respectively; i.e., $w_L(x, \varepsilon) = v_0 \exp[-(k/\varepsilon)^{1/2}(x_0 - x)]$ and $w_R(x, \varepsilon) = v_1 \exp[-(k/\varepsilon)^{1/2}(x - x_0)]$. Therefore, the solution $v(x, \varepsilon)$ of Model I in this case has the form

$$v(x, \varepsilon) = \begin{cases} v_0 + w_L(x - O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } \ 0 \leqslant x \leqslant x_0, \\ v_1 + w_R(x + O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } \ x_0 \leqslant x \leqslant 1, \end{cases}$$

for each $\varepsilon > 0$. The location x_0 of the interior layer satisfies the integral equation

$$0 = \int_0^1 h(U(s))\, ds = \int_0^{x_0} h(U(s))\, ds + \int_{x_0}^1 h(U(s))\, ds,$$

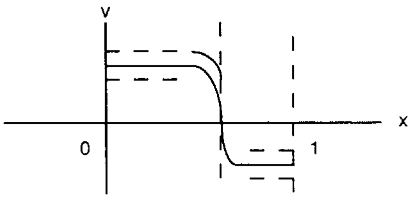

FIG. 5. $v_0 > 0$, $v_1 < 0$.

where

$$U(x) = \begin{cases} v_0 x(x - x_0)/2 & \text{for} \quad 0 < x \leqslant x_0 \\ -v_1(1-x)(x-x_0)/2 & \text{for} \quad x_0 < x < 1, \end{cases}$$

$(U(x) = \lim_{\varepsilon \to 0^+} u(x, \varepsilon))$. If $h(u) = u$, x_0 is given explicitly by $x_0 = (-v_1)^{1/3}/((-v_1)^{1/3} + (v_0)^{1/3})$. The asymptotic solution $v(x, \varepsilon)$ is shown in Fig. 5.

Case 4. $v_0 < 0, v_1 > 0$.

In this case $u(x, \varepsilon)$ also changes sign in (0, 1). However, there is no interior layer in (0, 1) because $u(x, \varepsilon) > 0$ near $x = 0$ and $u(x, \varepsilon) < 0$ near $x = 1$, which implies the same behavior for $h(u)$. It follows that there exists a unique x_0 in (0, 1) such that $h(u(x_0)) = 0$, $h(u) \leqslant 0$ in $[0, x_0]$ and $h(u) \geqslant 0$ in $[x_0, 1]$. The signs of the coefficient $h(u)$ of v' allow a boundary layer at both endpoints $x = 0$ and $x = 1$. We consider again the system $(V_I 1)$ and $(V_I 2)$ in Case 3 but with the opposite signs of the coefficient $h(u)$ of the v' term. We can conclude from Case 1 and Case 2 that

$$v(x, \varepsilon) = w_L^*(x, \varepsilon) + w_R^*(x, \varepsilon) + O(\varepsilon^{1/2})$$

for all x in [0, 1] and for each $\varepsilon > 0$, where

$$w_L^*(x, \varepsilon) = v_0 \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} x \right]$$

and

$$w_R^*(x, \varepsilon) = v_1 \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} (1 - x) \right].$$

4. The Solution of Model II: $u''=v$, $\varepsilon v''+h(u')v'=0$

We consider the problem

$$
\begin{aligned}
&u''=v \qquad \text{in } (0,1),\\
&u(0,\varepsilon)=u(1,\varepsilon)=0,\\
&\varepsilon v''+h(u')v'=0 \qquad \text{in } (0,1),\\
&v(0,\varepsilon)=v_0, \qquad v(1,\varepsilon)=v_1.
\end{aligned} \tag{4.1}
$$

There is a fundamental difference between the systems (4.1) and (3.1) because $u'(x,\varepsilon)$ always changes sign in $(0,1)$, even if $u(x,\varepsilon)$ does not change sign. Thus there is at least *one* interior turning point x_0 in $(0,1)$ such that $h(u'(x_0,\varepsilon))=0$. Therefore, we have the following theorem.

Theorem 2. *Let* $(u(x,\varepsilon), v(x,\varepsilon))$ *be the solution of* (4.1), *then*

(i) $v(x,\varepsilon)=\begin{cases} v_0+w_L(x-O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if } 0\leqslant x\leqslant x_0,\\ v_1+w_R(x+O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if } x_0\leqslant x\leqslant 1,\end{cases}$

for $v_0\geqslant 0, v_1\geqslant 0$;

(ii) $v(x,\varepsilon)=\mathbf{c}+w_1(x,\varepsilon)+w_2(x,\varepsilon)+O(\varepsilon^{1/2})$, *for* $v_0\leqslant 0, v_1\leqslant 0$;

(iii) $v(x,\varepsilon)=\mathbf{c}+SL(1)+BL(0)$ *for* $v_0<0, v_1>0$.

(iv) $v(x,\varepsilon)=\mathbf{c}+ZL(0)+BL(1)$ *for* $v_0>0, v_1<0$,

where $w_L(x,\varepsilon)=v_0\exp[-(k/\varepsilon)^{1/2}(x_0-x)]$, $w_R(x,\varepsilon)=v_1\exp[-(k/\varepsilon)^{1/2}(x-x_0)]$, $\mathbf{c}=\lim_{\varepsilon\to 0^+} v(x,\varepsilon)$, $w_1(x,\varepsilon)=(v_0-\mathbf{c})\exp[-(k/\varepsilon)^{1/2}x]$, $w_2(x,\varepsilon)=(v_1-\mathbf{c})\exp[-(k/\varepsilon)^{1/2}(1-x)]$, *$k$ is defined as in Theorem* 1, *for* (iii) *and* (iv) $\mathbf{c}=0$ *if* $v_0+v_1>0$, $c=(v_0+v_1)/2$ *if* $v_0+v_1<0$,

$$SL(1)=\begin{cases} c+w_{BR}(x-O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if } \xi(\varepsilon)\leqslant x\leqslant x_1,\\ v_1+w_S(x+O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if } x_1\leqslant x\leqslant 1;\end{cases}$$

$$ZL(0)=\begin{cases} v_0+w_S^*(x-O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if } 0\leqslant x\leqslant x_2,\\ c+w_{BL}^*(x+O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if } x_2\leqslant x\leqslant \eta(\varepsilon).\end{cases}$$

Proof. We examine the appropriate four cases reflecting the different signs of the boundary values v_0 and v_1.

Case 1. $v_0\geqslant 0, v_1\geqslant 0$.

Because $v_0>0$, $v_1>0$, $u''=v$, and $u(0,\varepsilon)=u(1,\varepsilon)=0$ imply $u(x,\varepsilon)<0$ in $(0,1)$. This implies that there exists some x_0 in $(0,1)$ such that $u'(x_0)=0$ and $u'(x,\varepsilon)<0$ near $x=0$, while $u'(x,\varepsilon)>0$ near $x=1$. It follows that $h(u'(x_0))=0$, $h(u')\leqslant 0$ in $[0,x_0]$ and $h(u')\geqslant 0$ in $[x_0,1]$. Consequently, the sign of $h(u')$ at each endpoint does not allow $v(x, \varepsilon)$ to have boundary layers. The only possible asymptotic behavior available to v is then an interior layer behavior at the point x_0 in $[0, 1]$. By the same arguments in Case 3 of Model I, we have

$$v(x, \varepsilon) = \begin{cases} v_0 + w_L(x - O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } 0 \leqslant x \leqslant x_0, \\ v_1 + w_R(x + O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } x_0 \leqslant x \leqslant 1, \end{cases}$$

where $w_L(x, \varepsilon) = v_0 \exp[-(k/\varepsilon)^{1/2}(x_0 - x)]$ and $w_R(x, \varepsilon) = v_1 \exp[-(k/\varepsilon)^{1/2}(x - x_0)]$. To determine the location x_0 of the interior layer, we let $U(x) = \lim_{\varepsilon \to 0^+} u(x, \varepsilon)$ for $x \neq x_0$. Then $U(x)$ must satisfy the following (five) conditions: $U(0) = U(1) = 0$, $U(x_0^+) = U(x_0^-)$ and $U'(x_0^+) = U'(x_0^-) = 0$ (cf. [8]). Therefore we have

$$U(x) = \frac{v_0 x(x - 2x_0)}{2} \qquad \text{for} \quad x \leqslant x_0,$$

$$U(x) = \frac{v_1(x-1)(x+1-2x_0)}{2} \qquad \text{for} \quad x \geqslant x_0.$$

Finally, in order that $U(x_0^+) = U(x_0^-)$, we must have $-v_0 x_0^2/2 = -v_1(1 - x_0)^2/2$; that is,

$$x_0 = \frac{v_1^{1/2}}{v_0^{1/2} + v_1^{1/2}} \tag{4.2}$$

(cf. [8]).

If $v_0 = 0$, then $x_0 = 1$, and $v(x, \varepsilon)$ has an interior layer at $x = 1$ called an "S-layer". If $v_1 = 0$, then $x_0 = 0$, and there is a "Z-layer" (backwards S-layer) at $x = 0$.

Case 2. $v_0 \leqslant 0, v_1 \leqslant 0$.

This case is not a reflection of Case 1 as in Model I. If $v_0 \leqslant 0$ and $v_1 \leqslant 0$, then $u(x, \varepsilon) \geqslant 0$ in $(0, 1)$, $u'(x, \varepsilon) \geqslant 0$ in $[0, x_0]$ and $u'(x, \varepsilon) \leqslant 0$ in $[x_0, 1]$ for some point x_0 in $(0, 1)$. The coefficient $h(u')$ of v' in the v-equation behaves similarly. Consequently, the solution $v(x, \varepsilon)$ in the v-equation cannot have an interior layer at x_0. However, the sign of $h(u')$ at the endpoints is compatible with the existence of boundary layers in $v(x, \varepsilon)$. Hence, $v(x, \varepsilon) = \mathbf{c} + w_1(x, \varepsilon) + w_2(x, \varepsilon) + O(\varepsilon^{1/2})$, for all x in $[0, 1]$ and for each $\varepsilon > 0$, where $\mathbf{c}$ is a constant such that $\mathbf{c} = \lim_{\varepsilon \to 0^+} v(x, \varepsilon)$,

$$w_1(x, \varepsilon) = (v_0 - \mathbf{c}) \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} x\right],$$

$$w_2(x, \varepsilon) = (v_1 - \mathbf{c}) \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} (1 - x)\right].$$

If $\int_0^1 h(u')\, dx = 0$, then $\mathbf{c} = (v_0 + v_1)/2$; cf. [8].

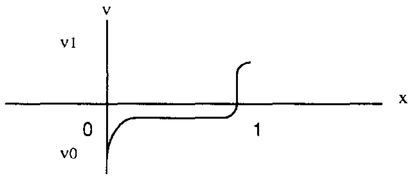

FIG. 6. $v_0 < 0, v_1 > 0$.

Case 3. $v_0 < 0, v_1 > 0$.

We note that $u > 0$ near $x = 0$ and $u < 0$ near $x = 1$ in this case. It follows that $u' > 0$ near both endpoints and $u'(x_0, \varepsilon) = 0$ for some x_0 in $(0, 1)$, which implies that $h(u') > 0$ near both endpoints. Hence, $v(x, \varepsilon)$ has no interior layer in $(0, 1)$, and the sign of $h(u')$ near $x = 0$ allows $v(x, \varepsilon)$ to have a boundary layer at $x = 0$ and $v(x, \varepsilon)$ has an S-layer near $x = 1$ (cf. [12]); that is, $v(x, \varepsilon) = \mathbf{c} + BL(0) + SL(1)$ as shown in Fig. 6 (where $\lim_{\varepsilon \to 0^+} v(x, \varepsilon) = \mathbf{c} \leqslant 0$, $\mathbf{c} = 0$ if $v_0 + v_1 > 0$, and $\mathbf{c} = (v_0 + v_1)/2$ if $v_0 + v_1 < 0$; these results can be found in [8, 12]). The condition $v_0 + v_1 > 0$ says that $v_1 > |v_0|$, and so there exists a unique point $\xi(\varepsilon)$ in $(0, 1)$ such that $u''(x, \varepsilon) < 0$ for $0 \leqslant x < \xi(\varepsilon)$, $u''(x, \varepsilon) = 0$ and $u''(x, \varepsilon) > 0$ for $\xi(\varepsilon) < x < 1$, with $\xi(\varepsilon) \to 1$ as $\varepsilon \to 0^+$; cf. Fig. 7. Slightly to the left of $\xi, v'' > 0$ and $h(u') < 0$; at $x = \xi$, $v''(\xi, \varepsilon) = 0$ and $h(u'(\xi, \varepsilon)) = 0$; and to the right of $\xi, h(u') > 0$ near $x = 1$. Suppose $v_0 + v_1 < 0$, then there exists a unique point $\eta = \eta(\varepsilon)$ in $(0, 1)$ such that $u''(x, \varepsilon) < 0$ in $[0, \eta(\varepsilon))$, $u''(\eta, \varepsilon)) = 0$ and $u'' > 0$ in $(\eta(\varepsilon), 1]$ with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$; cf. Fig. 8. We can determine the thickness of the boundary layer in the following manner. In the boundary layer at $x = 0$, $v'' > 0$ in $[0, \eta(\varepsilon))$, $v''(\eta, \varepsilon) = 0$ and $v'' < 0$ for x slightly to the right of η. In turn, we see that $h(u') > 0$ in $[0, \eta(\varepsilon))$, $h(u'(\eta, \varepsilon)) = 0$ and $h(u') < 0$ for x slightly to the right of η. We conclude that the limiting value $\mathbf{c}$ of $v(x, \varepsilon)$ must be nonpositive. We consider the problems

$$u'' = v, \qquad u(0, \varepsilon) = 0 = u(\xi(\varepsilon), \varepsilon), \qquad x \text{ in } (0, \xi(\varepsilon)), \tag{VB}$$

$$\varepsilon v'' + h(u')v' = 0, \qquad v(\xi(\varepsilon), \varepsilon) = v^*, \qquad v(0, \varepsilon) = v_0, \qquad x \text{ in } (0, \xi(\varepsilon)),$$

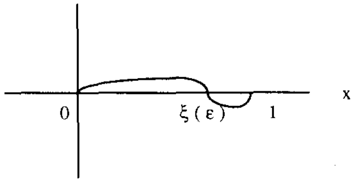

FIGURE 7

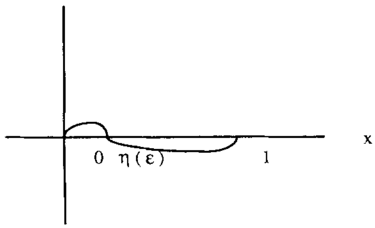

FIGURE 8

and

$$u'' = v, \qquad u(\xi(\varepsilon), \varepsilon) = 0 = u(1, \varepsilon) \qquad x \text{ in } (0, \xi(\varepsilon)), \tag{VI}$$
$$\varepsilon v'' + h(u')v' = 0, \qquad v(\xi(\varepsilon), \varepsilon) = v^*, \qquad v(1, \varepsilon) = v_1, \qquad x \text{ in } (0, \xi(\varepsilon)).$$

From the results of Case 1 and Case 2, we have $v(x, \varepsilon) = \mathbf{c} + SL(1) + BL(0)$; i.e.,

$$v(x, \varepsilon) = \begin{cases} \mathbf{c} + w_{BL}(x, \varepsilon) + w_{BR}(x - O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } 0 \leqslant x \leqslant x_1, \\ v_1 + w_S(x + O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } x_1 \leqslant x \leqslant 1, \end{cases} \tag{4.3}$$

where x_1 in $(\xi(\varepsilon), 1)$ such that $h(u'(x_1, \varepsilon)) = 0$, and where w_{BL}, w_{BR}, and w_S are the unique solutions of the problems

$$\varepsilon w''_{BL} - h(u'(0, \varepsilon))\, w_{BL} = 0, \qquad x \text{ in } (0, \xi(\varepsilon)),$$
$$w_{BL}(0, \varepsilon) = v_0 - \mathbf{c}, \qquad \lim_{\varepsilon \to 0^+} w_{BL}(x, \varepsilon) = 0,$$
$$\varepsilon w''_{BR} - h(u'(\xi(\varepsilon), \varepsilon)\, w_{BR} = 0 \qquad x \text{ in } (\xi(\varepsilon), x_1),$$
$$w_{BR}(\xi(\varepsilon), \varepsilon) = \mathbf{c}, \qquad \lim_{\varepsilon \to 0^+} w_{BR}(x, \varepsilon) = 0,$$

and

$$\varepsilon w''_S - h(u'(x_1 + \delta, \varepsilon)\, w_S = 0 \qquad x \text{ in } (x_1, 1),$$
$$w_S(x_1, \varepsilon) = v_1, \qquad \lim_{\varepsilon \to 0^+} w_S(x, \varepsilon) = 0,$$

where $\mathbf{c} = 0$ if $v_0 + v_1 > 0$, $\mathbf{c} = (v_0 + v_1)/2$ if $v_0 + v_1 < 0$ and $\delta = O(\varepsilon)$. The S-layer solution is

$$SL(1) = \begin{cases} \mathbf{c} + w_{BR}(x - O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } \xi(\varepsilon) \leqslant x \leqslant x_1, \\ v_1 + w_S(x + O(\varepsilon), \varepsilon) + O(\varepsilon^{1/2}) & \text{if } x_1 \leqslant x \leqslant 1. \end{cases}$$

Case 4. $v_0>0, v_1<0$.

This case is a reflection of that obseved in the previous case with a "Z-layer" at $x=0$. Making the changes of variables

$$r=1-x, \qquad \hat{u}(r,\varepsilon)=u(1-r,\varepsilon), \qquad \hat{v}(r,\varepsilon)=v(1-r,\varepsilon),$$
$$\hat{u}(0,\varepsilon)=\hat{u}(1,\varepsilon)=0,$$
$$\hat{v}(0,\varepsilon)=\hat{v}_0=-v_1, \qquad \hat{v}(1,\varepsilon)=\hat{v}_1=-v_0,$$

by the same arguments in Case 3 we have $v(x,\varepsilon)=\mathbf{c}+ZL(0)+BL(1)$; i.e.,

$$v(x,\varepsilon)=\begin{cases} v_0+w_S^*(x-O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if} \quad 0\leqslant x\leqslant x_2, \\ \mathbf{c}+w_{BL}^*(x+O(\varepsilon),\varepsilon)+w_{BR}^*(x,\varepsilon)+O(\varepsilon^{1/2}) & \text{if} \quad x_2\leqslant x\leqslant 1, \end{cases} \tag{4.4}$$

where w_{BL}^*, w_{BR}^*, and w_S^* are the unique solutions of the problems

$$\varepsilon w_{BL}^{*\prime\prime}-h(u'(\eta(\varepsilon),\varepsilon))\,w_{BL}^*=0, \qquad x \text{ in } (0,\eta(\varepsilon)),$$
$$w_{BL}^*(\eta(\varepsilon),\varepsilon)=\mathbf{c}, \qquad \lim_{\varepsilon\to 0^+} w_{BL}^*(x,\varepsilon)=0,$$
$$\varepsilon w_{BR}^{*\prime\prime}-h(u'(1,\varepsilon))\,w_{BR}^*=0 \qquad x \text{ in } (\eta(\varepsilon),1),$$
$$w_{BR}^*(1,\varepsilon)=v_1-\mathbf{c}, \qquad \lim_{\varepsilon\to 0^+} w_{BR}^*(x,\varepsilon)=0,$$

and

$$\varepsilon w_S^{*\prime\prime}-h(u'(x_2-\delta,\varepsilon)\,w_S^*=0 \qquad x \text{ in } (0,x_2),$$
$$w_S^*(x_2,\varepsilon)=v_0, \qquad \lim_{\varepsilon\to 0^+} w_S^*(x,\varepsilon)=0,$$

where $\lim_{\varepsilon\to 0^+} v(x,\varepsilon)=\mathbf{c}$ in $[\delta^*, 1-\delta]$, $\mathbf{c}=0$ if $v_0+v_1>0$, $\mathbf{c}=(v_0+v_1)/2$ if $v_0+v_1<0$ and if $\int_0^1 h(u')\,dx=0$, $\delta^*, \delta=O(\varepsilon)$. The Z-layer solution is

$$ZL(0)=\begin{cases} v_0+w_S^*(x-O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if} \quad 0\leqslant x\leqslant x_2, \\ \mathbf{c}+w_{BL}^*(x+O(\varepsilon),\varepsilon)+O(\varepsilon^{1/2}) & \text{if} \quad x_2\leqslant x\leqslant \eta(\varepsilon). \end{cases}$$

5. Solutions of Model III:
$u''=v,\ \varepsilon v''+h(u)v'-g(x,u,u')v=0,\ g\geqslant g^*>0$
and Model IV:
$u''=v,\ \varepsilon v''+h(u')v'-g(x,u,u')v=0,\ g\geqslant g^*>0$

We consider in this section the system (1.1) in which $g\geqslant g^*>0$, for g^* a constant,

$$
\begin{aligned}
&u'' = v \qquad \text{in } (0, 1),\\
&u(0, \varepsilon) = u(1, \varepsilon) = 0,\\
&\varepsilon v'' + f(u, u')v' - g(x, u, u')v = 0 \qquad \text{in } (0, 1),\\
&v(0, \varepsilon) = v_0, \qquad v(1, \varepsilon) = v_1,
\end{aligned}
\tag{5.1}
$$

where $f(u, u') = h(u)$ or $h(u')$, $dh/dz > 0$ or $dh/dz < 0$, and $g(x, u, u') \in C^1[0, 1] \times \mathbb{R}^2$. The condition $g(x, u, u') \geqslant g^* > 0$ forces the reduced form of the v-equation of (5.1) to have only the zero solution $v \equiv 0$ in (0, 1) for all the boundary values v_0 and v_1; cf. [3, 12]. The asymptotic behavior of solution $v(x, \varepsilon)$ cannot have an interior layer in the interval (0, 1). Therefore, we have the following theorem.

THEOREM 3. *If* $(u(x, \varepsilon), v(x, \varepsilon))$ *is the solution of* (5.1), *then*

$$
v(x, \varepsilon) = v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x \right] + v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1 - x) \right] + O(\varepsilon^{1/2}),
$$

for all x *in* $[0, 1]$ *and each* $\varepsilon > 0$.

Proof. We will construct the bounding functions α_i and β_i $(i = 1, 2)$ such that

(A1) $\alpha_i \leqslant \beta_i$, $i = 1, 2$,

(B1) $\alpha_1(0, \varepsilon) \leqslant 0 \leqslant \beta_1(0, \varepsilon)$, $\alpha_1(1, \varepsilon) \leqslant 0 \leqslant \beta_1(1, \varepsilon)$,

(A2) $\alpha_1'' \geqslant \beta_2$, $\beta_1'' \leqslant \alpha_2$,

(B2) $\alpha_2(0, \varepsilon) \leqslant v_0 \leqslant \beta_2(0, \varepsilon)$, $\alpha_2(1, \varepsilon) \leqslant v_1 \leqslant \beta_2(1, \varepsilon)$,

(A3) $\varepsilon\alpha_2'' + f(u, z)\alpha_2' - g(x, u, z)\alpha_2 \geqslant 0$, $\varepsilon\beta_2'' + f(u, z)\beta_2' - g(x, u, z)\beta_2 \leqslant 0$, for all u in $[\alpha_1, \beta_1]$ and $z \in \mathbb{R}$.

Again, we divide our discussion into three cases for the different signs of the boundary values v_0, v_1 for each of the following models:

$$
\begin{aligned}
&u'' = v \qquad \text{in } (0, 1),\\
&u(0, \varepsilon) = u(1, \varepsilon) = 0,\\
&\varepsilon v'' + h(u)v' - g(x, u, u')v = 0 \qquad \text{in } (0, 1),\\
&v(0, \varepsilon) = v_0, \qquad v(1, \varepsilon) = v_1,
\end{aligned}
\tag{III}
$$

and

$$\begin{aligned} &u''=v \qquad \text{in } (0,1),\\ &u(0,\varepsilon)=u(1,\varepsilon)=0,\\ &\varepsilon v''+h(u')v'-g(x,u,u')v=0 \qquad \text{in } (0,1),\\ &v(0,\varepsilon)=v_0, \qquad v(1,\varepsilon)=v_1. \end{aligned} \tag{IV}$$

Without lost generality, we assume that $dh/dz>0$.

(I) Model III.

Case 1. $v_0 \geqslant 0$, $v_1 \geqslant 0$.

Since $u(x,\varepsilon)\leqslant 0$ in $[0,1]$ for $v_0\geqslant 0$ and $v_1\geqslant 0$ and $\lim_{\varepsilon\to 0^+} v(x,\varepsilon)=0$, we define, for $0\leqslant x\leqslant 1$ and $\varepsilon>0$,

$$\alpha_1(x,\varepsilon)=\frac{1}{2}(v_1+v_0)(x^2-x), \qquad \beta_1(x,\varepsilon)=0,$$

$$\begin{aligned}\alpha_2(x,\varepsilon)=v_0\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}x\right]+v_1\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right]\\ -c_1^*\Gamma_1(x,\varepsilon)-c_2^*\Gamma_2(x,\varepsilon)\end{aligned}$$

and

$$\beta_2(x,\varepsilon)=v_0\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}x\right]+v_1\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right],$$

where c_1^* and c_2^* are positive constants to be determined, $\Gamma_1(x,\varepsilon)=\exp[-\sigma_1(k/\varepsilon)^{1/2}x]-\exp[-(k/\varepsilon)^{1/2}x]$, $\Gamma_2(x,\varepsilon)=\exp[-\sigma_2(k/\varepsilon)^{1/2}(1-x)]-\exp[-(k/\varepsilon)^{1/2}(1-x)]$, σ_i $(i=1,2)$ are constants such that $0<\sigma_i<1$, with $\{1/\sigma_1^2\exp[(\sigma_1-1)(k/\varepsilon)^{1/2}x]\}>1$ and $\{\sigma_2\exp[(1-\sigma)(k/\varepsilon)^{1/2}(1-x)]\}>0$, k: $h(u)<-k<0$ in $[\delta,1-\delta]$ for small $\delta>0$, and $0<k<-h(u)(k/\varepsilon)^{1/2}$. Then it follows that (A1), (B1), and (B2) are satisfied. Since $h(0)=0$, that is, $h(\beta_1)=0$, it follows that

$$\begin{aligned}&\varepsilon\beta_2''+h(\beta_1)\beta_2'-g(x,u,u')\beta_2\\ &\quad\leqslant g^*v_0\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}x\right]+g^*v_1\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right]\\ &\qquad -g^*v_0\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}x\right]-g^*v_1\exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right]=0;\end{aligned}$$

$$\alpha_1''(x,\varepsilon)=(v_1+v_0)\geqslant\beta_2; \qquad \beta_1''=0=\alpha_2;$$

$$
\begin{aligned}
&\varepsilon\alpha_2'' + h(u)\alpha_2' - g(x,u,u')\alpha_2 \\
&\quad \geqslant g^* v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] + g^* v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right] \\
&\qquad - \varepsilon(c_1^* \Gamma_1'' - c_2^* \Gamma_2'') \\
&\qquad + h(u)\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}\right]\left\{v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right]\right. \\
&\qquad \left. - c_1^*\left[-\sigma_1 \exp\left[-\sigma_1\left(\frac{k}{\varepsilon}\right)^{1/2} x\right] - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} x\right]\right]\right\} \\
&\qquad + h(u)\left[\left(\frac{g^*}{\varepsilon}\right)^{1/2}\right]\left\{v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right]\right. \\
&\qquad \left. - c_2^*\left[\sigma_2 \exp\left[-\sigma_2\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right] - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right]\right\} \\
&\qquad - g v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] \\
&\qquad - g v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right] + g(c_1^*\Gamma_1 + c_2^*\Gamma_2) \\
&\quad = (g^* - g)\, v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] \\
&\qquad + (g^* - g)\, v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right] - g^* c_1^* \left\{\sigma_1^2 \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} x\right]\right. \\
&\qquad \left. - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} x\right]\right\} - g^* c_2^* \left\{\sigma_2^2 \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right. \\
&\qquad \left. - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right\} \\
&\qquad + h(u)\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}\right]\left\{v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right]\right. \\
&\qquad \left. - c_1^*\left[-\sigma_1 \exp\left[-\sigma_1\left(\frac{k}{\varepsilon}\right)^{1/2} x\right] - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2} x\right]\right]\right\} \\
&\qquad + h(u)\left[\left(\frac{g^*}{\varepsilon}\right)^{1/2}\right]\left\{v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right]\right. \\
&\qquad - c_2^*\left[\sigma_2 \exp\left[\sigma_2\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right. \\
&\qquad \left.\left. - \exp\left[-\left(\frac{k}{\varepsilon}\right)^{1/2}(1-x)\right]\right]\right\} \geqslant 0,
\end{aligned}
$$

we choose g^* such that $(g-g^*)$ is small, c_1^* small enough such that $v_0 \exp[-(g^*/\varepsilon)^{1/2} x] > c_1^* \Gamma_1$, c_2^* large enough such that $v_1 \exp[-(g^*/\varepsilon)^{1/2}(1-x)] < c_2^* \Gamma_2$, and $|v_1 \exp[-(g^*/\varepsilon)^{1/2}(1-x)] - c_2^* \Gamma_2| < v_0 \exp[-(g^*/\varepsilon)^{1/2} x] - c_1^* \Gamma_1$. Therefore, (A2) and (A3) are true, and it follows that

$$v(x, \varepsilon) = v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x\right] + v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2}(1-x)\right] + O(\varepsilon^{1/2})$$

in $[0, 1]$ for each $\varepsilon > 0$.

Case 2. $v_0 \leqslant 0$, $v_1 \leqslant 0$.

This case is again handled by reflection. Making the change of variables as in Case 2 of Model I, we have

$$\begin{aligned} &\mathbf{m}'' = \mathbf{n} \\ &\mathbf{m}(0, \varepsilon) = \mathbf{m}(1, \varepsilon) = 0, \\ &\varepsilon \mathbf{n}'' + h(\mathbf{m})\mathbf{n}' = 0, \\ &\mathbf{n}(0, \varepsilon) = \mathbf{n}_0 = -v_1, \ \mathbf{n}(1, \varepsilon) = \mathbf{n}_1 = -v_0, \end{aligned}$$

provided $v_0 \geqslant 0$ and $v_1 \geqslant 0$. This the Case 1 of Model III.

Case 3. $v_0 v_1 < 0$.

In a similar manner, since $u(x, \varepsilon)$ changes sign in either case, there exists a unique interior turning point x_0 in $(0, 1)$ such that $u(x_0, \varepsilon) = 0$, which implies that $h(u(x_0, \varepsilon)) = 0$ and $v(x_0, \varepsilon) = 0$. We consider the system

$$\begin{aligned} &u'' = v \qquad \text{in } (0, x_0), \\ &u(0, \varepsilon) = u(x_0, \varepsilon) = 0, \\ &\varepsilon v'' + h(u)v' - g(x, u, u')v = 0 \qquad \text{in } (0, x_0), \\ &v(0, \varepsilon) = v_0, \qquad v(x_0, \varepsilon) = v^* \end{aligned} \tag{T1}$$

and

$$\begin{aligned} &u'' = v \qquad \text{in } (x_0, 1), \\ &u(x_0, \varepsilon) = u(1, \varepsilon) = 0, \\ &\varepsilon v'' + h(u)v' - g(x, u, u')v = 0 \qquad \text{in } (x_0, 1), \\ &v(x_0, \varepsilon) = v^*, \qquad v(1, \varepsilon) = v_1. \end{aligned} \tag{T2}$$

Since $v^* = v(x_0, \varepsilon) = 0$ and outer solutions of systems (T1) and (T2) are zero, it follows that there are boundary layers at $x = 0$ and $x = 1$ respectively from the result of Case 1 and Case 2. Therefore, we have the same form of the result as in Case 1 and Case 2.

(II). *Model* IV.

Since outer solution of system (IV) is the trivial solution, the asymptotic behavior of the solution $v(x, \varepsilon)$ of the coupled system (5.1) in this case is not as complicated as in Models I and II. In fact, $u'' = v$, $v_0 \geqslant 0$ and $v_1 \geqslant 0$ imply $u(x, \varepsilon) \leqslant 0$. It follows that $u' < 0$ near $x = 0$ and $u' > 0$ near $x = 1$, and thus $h(u') < 0$ in $(0, x_0^*)$, $h(u'(x_0^*, \varepsilon)) = 0$ and $h(u') > 0[x_0^*, 1]$ for some x_0^* in (0, 1). By the same arguments as in Case 3 of Model (III), we have

$$v(x, \varepsilon) = v_0 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} x \right] + v_1 \exp\left[-\left(\frac{g^*}{\varepsilon}\right)^{1/2} (1 - x) \right] + O(\varepsilon^{1/2}).$$

For $v_0 \leqslant 0$, $v_1 \leqslant 0$, this case is a reflection of the case: $v_0 \geqslant 0$, $v_1 \geqslant 0$. If we consider the asymptotic behavior as $\varepsilon \to 0^+$ of the solution $v(x, \varepsilon)$ of (5.1) in the subintervals $[0, x_0]$ and $[x_0, 1]$ separately for $v_0 v_1 < 0$, the previous results can be applied in order to obtain the stated result.

6. Remark

From our constructions of the bounding functions α_i and β_i $(i = 1, 2)$ which provide refined approximations to the solutions of system (1.1), we see that similar results can be obtained for more general second-order singularly perturbed scalar and vector problems of the form $\varepsilon y'' = f(t, y) y' + g(t, y)$, $y(a) = A$, $y(b) = B$ when $f(t, y)$ has zeros in the interval $[a, b]$.

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