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A NEW PRE-LOADED MEMBRANE GEOMETRIC STIFFNESS MATRIX WITH FULL RIGID BODY CAPABILITIES

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NOTATION

a	membrane element length
b	membrane element width
$[DFC]$	directed force correction matrix
$[DFC\text{Total}]$	matrix of all corrections ($[DFC]+[MCOR]$)
E	Young's modulus (Modulus of elasticity)
h	element thickness
$[Ke]$	elastic stiffness matrix
$[Kg]$	geometric stiffness matrix
$\mathbf{K}_{ij}$	i, j element of stiffness matrix
$[M]$	mass matrix
$[MCOR]$	fixed end moment correction
N	distributed tensile load ($N_x = N$ for example, where $N_y = N_{xy} = 0$.)
N_x	distributed tensile load in x-direction
N_y	distributed tensile load in y-direction
N_{xy}	distributed shear load on membrane
α_i	element constant
β_i	element constant
ρ	mass density
v	Poisson's ratio

INTRODUCTION

Due to economic constraints associated with payload cost, space structures must consist of light-weight, and subsequently relatively flexible, components. The total stiffness of a member, in many cases, includes both the elastic stiffness, due to the material properties and member configuration, as well as geometric or initial stress stiffness due to pre-loads. Accurate prediction of the natural frequencies and mode shapes is essential for determining the adequacy of components, and for designing a controls system. The finite element method is normally used to perform this analysis. However, a phenomenon known as "grounding" or false stiffening, occurs when the geometric stiffness matrix is used. When a pre-loaded model with free/free boundary conditions is analyzed, it behaves as if it is restrained internally. Thus, it is unable to rotate as a rigid body. Instead, erroneous "pseudo-forces" develop during rigid body motion. Free vibration analysis also yields an erroneous (non-zero) natural frequency associated with rigid body rotation. Bosela *et al.* [1–3] have examined the grounding phenomenon for a pre-loaded beam, and determined that it is caused by a force imbalance during rigid body rotation, and is typical of beam geometric stiffness matrices formulated by others, even those which include higher order effects. By utilizing a directed force premise, and extending to a global model the natural mode approach used by Argyris and Symeonidis [4] for developing load correction factors for non-conservative forces, a pre-loaded beam geometric stiffness matrix with full rigid body capabilities was developed and tested [3, 5].

This paper has the following objectives:

(1) Examine the grounding phenomenon as it relates to pre-loaded membrane elements.

(2) Develop a pre-loaded membrane element with full rigid body capabilities by following the directed force approach used successfully in developing the pre-loaded beam element.

MEMBRANE ELEMENTS

Although various membrane elements are used in commercial finite element programs, most are not readily available in explicit form. One which is stated explicitly by Yang [6] is the four-node rectangular element in Fig. 1, which has either 12 or 16 degrees of freedom (DOF). The degrees of freedom numbered as 13, 14, 15 and 16, correspond to second-order twist terms such as $\partial^2 w/\partial x\,\partial y$. This element is obtained by combining the shape functions for the elementary beam element in both the x and y directions.

The geometric, or initial stress stiffness matrix $[Kg]$ for this plate element when the distributed load is in the x-direction only is

$$[K_g] = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & B_9 & B_{10} & B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ D_1 & D_2 & D_3 & D_4 & D_5 & D_6 & D_7 & D_8 & D_9 & D_{10} & D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ E_1 & E_2 & E_3 & E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} & E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ G_1 & G_2 & G_3 & G_4 & G_5 & G_6 & G_7 & G_8 & G_9 & G_{10} & G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & G_{16} \\ H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9 & H_{10} & H_{11} & H_{12} & H_{13} & H_{14} & H_{15} & H_{16} \\ I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{11} & I_{12} & I_{13} & I_{14} & I_{15} & I_{16} \\ J_1 & J_2 & J_3 & J_4 & J_5 & J_6 & J_7 & J_8 & J_9 & J_{10} & J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9 & K_{10} & K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} & L_{13} & L_{14} & L_{15} & L_{16} \\ M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 & M_8 & M_9 & M_{10} & M_{11} & M_{12} & M_{13} & M_{14} & M_{15} & M_{16} \\ N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 & N_{10} & N_{11} & N_{12} & N_{13} & N_{14} & N_{15} & N_{16} \\ P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 & P_{10} & P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 & Q_8 & Q_9 & Q_{10} & Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \end{bmatrix}, \tag{1}$$

$$A_1 = B_2 = C_3 = D_4 = \frac{78N_x b}{175a} \qquad A_2 = B_1 = C_4 = D_3 = -\frac{78N_x b}{175a}$$

$$A_4 = B_3 = C_2 = D_1 = \frac{27N_x b}{175a} \qquad A_3 = B_4 = C_1 = D_2 = -\frac{27N_x b}{175a}$$

$$A_5 = A_6 = D_7 = D_8 = \frac{13N_x b}{350} \qquad B_5 = B_6 = C_7 = C_8 = -\frac{13N_x b}{350}$$

$$A_7 = A_8 = D_5 = D_6 = \frac{9N_x b}{700} \qquad B_7 = B_8 = C_5 = C_6 = -\frac{9N_x b}{700}$$

$$A_9 = B_{10} = C_{12} = D_{11} = \frac{11N_x b^2}{175a} \qquad A_{10} = B_9 = C_{11} = D_{12} = -\frac{11N_x b^2}{175a}$$

$$A_{11} = B_{12} = C_{10} = D_9 = \frac{13N_x b^2}{350a} \qquad A_{12} = B_{11} = C_9 = D_{10} = -\frac{13N_x b^2}{350a}$$

$$A_{13} = A_{14} = C_{15} = C_{16} = \frac{11N_x b^2}{2100} \qquad B_{13} = B_{14} = D_{15} = D_{16} = -\frac{11N_x b^2}{2100}$$

$$B_{15} = B_{16} = D_{13} = D_{14} = \frac{13N_x b^2}{4200} \qquad A_{15} = A_{16} = C_{13} = C_{14} = -\frac{13N_x b^2}{4200}$$

$$E_1 = F_1 = G_4 = H_4 = \frac{13N_x b}{350} \qquad E_2 = F_2 = G_3 = H_3 = -\frac{13N_x b}{350}$$

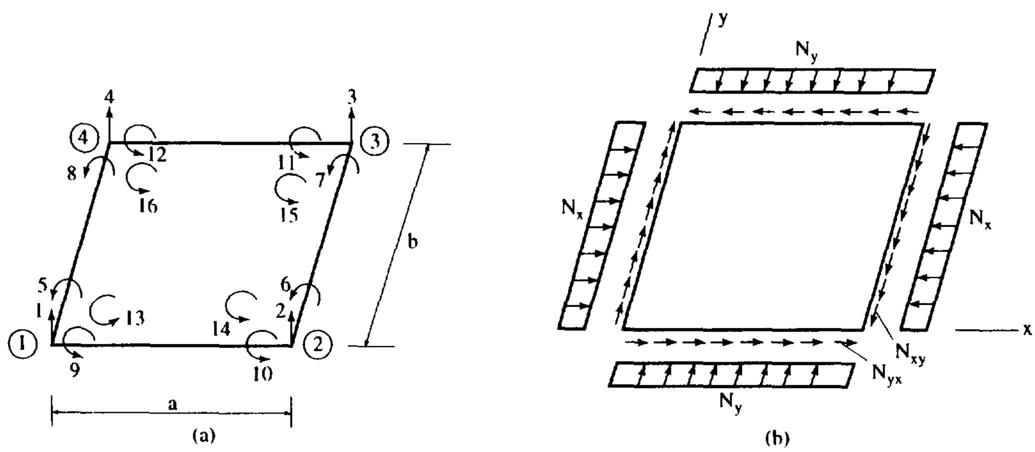

Fig. 1.(a) Four-node rectangular plate element with 16 DOF. (b) In-plane normal and shear forces on quadrilateral membrane element.

$$E_4 = F_4 = G_1 = H_1 = \frac{9N_x b}{700} \qquad E_3 = F_3 = G_2 = H_2 = -\frac{9N_x b}{700}$$

$$E_5 = F_6 = G_7 = H_8 = \frac{26N_x ab}{525} \qquad E_6 = F_5 = G_8 = H_7 = -\frac{13N_x ab}{1050}$$

$$E_7 = F_8 = G_5 = H_6 = -\frac{3N_x ab}{700} \qquad E_8 = F_7 = G_6 = H_5 = \frac{3N_x ab}{175}$$

$$E_9 = F_9 = G_{11} = H_{11} = \frac{11N_x b^2}{2100} \qquad E_{10} = F_{10} = G_{12} = H_{12} = -\frac{11N_x b^2}{2100}$$

$$E_{11} = F_{11} = G_9 = H_9 = \frac{13N_x b^2}{4200} \qquad E_{12} = F_{12} = G_{10} = H_{10} = -\frac{13N_x b^2}{4200}$$

$$E_{13} = F_{14} = \frac{11N_x ab^2}{1575} \qquad G_{15} = H_{16} = -\frac{11N_x ab^2}{1575}$$

$$G_{16} = H_{15} = \frac{11N_x ab^2}{6300} \qquad E_{14} = F_{13} = -\frac{11N_x ab^2}{6300}$$

$$E_{15} = F_{16} = \frac{13N_x ab^2}{12600} \qquad G_{13} = H_{14} = -\frac{13N_x ab^2}{12600}$$

$$G_{14} = H_{13} = \frac{13N_x ab^2}{3150} \qquad E_{16} = F_{15} = -\frac{13N_x ab^2}{3150}$$

$$I_1 = J_2 = K_4 = L_3 = \frac{11N_x b^2}{175a} \qquad I_2 = J_1 = K_3 = L_4 = -\frac{11N_x b^2}{175a}$$

$$I_4 = J_3 = K_1 = L_2 = \frac{13N_x b^2}{350a} \qquad I_3 = J_4 = K_2 = L_1 = -\frac{13N_x b^2}{350a}$$

$$I_5 = I_6 = K_7 = K_8 = \frac{11N_x b^2}{2100} \qquad J_5 = J_6 = L_7 = L_8 = -\frac{11N_x b^2}{2100}$$

$$I_7 = I_8 = K_5 = K_6 = \frac{13N_x b^2}{4200} \qquad J_7 = J_8 = L_5 = L_6 = -\frac{13N_x b^2}{4200}$$

$$I_9 = J_{10} = K_{11} = L_{12} = \frac{2N_x b^3}{175a} \qquad I_{10} = K_{12} = L_{11} = J_9 = -\frac{2N_x b^3}{175a}$$

$$I_{11} = J_{12} = K_9 = L_{10} = \frac{3N_x b^3}{350a} \qquad I_{12} = J_{11} = K_{10} = L_9 = -\frac{3N_x b^3}{350a}$$

$$I_{13} = I_{14} = L_{15} = L_{16} = M_9 = N_9 = Q_{12} = P_{12} = \frac{N_x b^3}{1050}$$

$$J_{13} = J_{14} = K_{15} = K_{16} = M_{10} = P_{11} = Q_{11} = -\frac{N_x b^3}{1050}$$

$$J_{15} = J_{16} = K_{13} = K_{14} = M_{11} = N_{11} = P_{10} = Q_{10} = \frac{N_x b^3}{1400}$$

$$I_{15} = I_{16} = L_{13} = L_{14} = M_{12} = N_{12} = P_9 = Q_9 = -\frac{N_x b^3}{1400}$$

$$M_1 = N_1 = P_3 = Q_3 = \frac{11N_x b^2}{2100} \qquad M_2 = N_2 = P_4 = Q_4 = -\frac{11N_x b^2}{2100}$$

$$M_4 = N_4 = P_2 = Q_2 = \frac{13N_x b^2}{4200} \qquad M_3 = N_3 = P_1 = Q_1 = -\frac{13N_x b^2}{4200}$$

$$M_5 = N_6 = \frac{11N_x ab^2}{1575} \qquad M_6 = N_5 = -\frac{11N_x ab^2}{6300}$$

$$M_8 = N_7 = \frac{13N_x ab^2}{3150} \qquad M_7 = N_8 = -\frac{13N_x ab^2}{12600}$$

$$M_{13} = N_{14} = P_{15} = Q_{16} = \frac{2N_x ab^3}{1575} \qquad M_{14} = N_{13} = P_{16} = Q_{15} = -\frac{N_x ab^3}{3150}$$

$$M_{15} = N_{16} = P_{13} = Q_{14} = \frac{N_x ab^3}{4200} \qquad M_{16} = N_{15} = P_{14} = Q_{13} = -\frac{N_x ab^3}{1050}$$

$$P_5 = Q_6 = \frac{13N_x ab^2}{12600} \quad N_{10} = -\frac{N_x b^3}{1050} \quad P_6 = Q_5 = -\frac{13N_x ab^2}{3150}$$

$$P_8 = Q_7 = \frac{11N_x ab^2}{6300} \quad P_7 = Q_8 = -\frac{11N_x ab^2}{1575}. \tag{2}$$

In order to solve for the natural frequencies, the corresponding four-node quadrilateral mass matrix developed by Yang was also used. The coefficients of the mass matrix are determined by using the equation

$$\mathbf{M}_{ij} = \rho h \int_0^b \int_0^a f_i(x, y) f_j(x, y) \, \partial x \, \partial y. \tag{3}$$

The mass matrix is as follows:

$$[M] = \rho abh \begin{bmatrix}
\frac{169}{1225} &&&&&&&&&&&&&&& \\
\frac{117}{2450} & \frac{129}{1225} &&&&&&&&&&&&&& \\
\frac{81}{4900} & \frac{117}{2450} & \frac{169}{1225} &&&&&&&&&&&&& \\
\frac{117}{2450} & \frac{81}{4900} & \frac{117}{2450} & \frac{169}{1225} &&&&&&&&&&&& \\
\frac{143a}{7350} & \frac{169a}{14700} & \frac{39a}{9800} & \frac{33a}{4900} & \frac{13a^2}{3675} &&&&&& \text{SYMMETRIC} &&&&& \\
-\frac{169a}{14700} & -\frac{143a}{7350} & -\frac{33a}{4900} & -\frac{39a}{9800} & -\frac{13a^2}{4900} & \frac{13a^2}{3675} &&&&&&&&&& \\
-\frac{39a}{9800} & -\frac{33a}{4900} & -\frac{143a}{7350} & -\frac{169a}{14700} & -\frac{9a^2}{9800} & \frac{3a^2}{2450} & \frac{13a^2}{3675} &&&&&&&&& \\
\frac{33a}{4900} & \frac{39a}{9800} & \frac{169a}{14700} & \frac{143a}{7350} & \frac{3a^3}{2450} & -\frac{9a^3}{9800} & -\frac{13a^3}{4900} & \frac{13a^3}{3675} &&&&&&&& \\
\frac{143b}{7350} & \frac{33b}{4900} & \frac{39b}{9800} & \frac{169b}{14700} & \frac{121ab}{44100} & -\frac{143ab}{88200} & -\frac{169ab}{176400} & \frac{143ab}{88200} & \frac{13b^2}{3675} &&&&&&& \\
\frac{36b}{4900} & \frac{143b}{7350} & \frac{169b}{14700} & \frac{39b}{9800} & \frac{143ab}{88200} & -\frac{121ab}{44100} & -\frac{143ab}{88200} & \frac{169ab}{176400} & \frac{3b^2}{2450} & \frac{13b^2}{3675} &&&&&& \\
-\frac{39b}{9800} & -\frac{169b}{14700} & -\frac{143b}{7350} & -\frac{33b}{4900} & -\frac{169ab}{176400} & \frac{143ab}{8820} & \frac{121ab}{44100} & -\frac{143ab}{88200} & -\frac{9b}{9800} & -\frac{13b^2}{4900} & \frac{13b^2}{3675} &&&&& \\
-\frac{169b}{14700} & -\frac{39b}{9800} & -\frac{33b}{4900} & -\frac{143b}{7350} & -\frac{143ab}{88200} & \frac{169ab}{176400} & \frac{143ab}{88200} & -\frac{121ab}{44100} & -\frac{13b^3}{4900} & -\frac{9b^3}{7800} & \frac{3b^3}{2450} & \frac{13b^3}{3675} &&&& \\
\frac{121ab}{44100} & \frac{143ab}{88200} & \frac{169ab}{176400} & \frac{143ab}{88200} & \frac{11a^2b}{22050} & -\frac{11a^2b}{29400} & -\frac{13a^2b}{58800} & \frac{13a^2b}{44100} & \frac{11ab^2}{22050} & \frac{13ab^2}{44100} & -\frac{13ab^2}{58800} & -\frac{11ab^2}{29400} & \frac{a^2b^2}{11025} &&& \\
-\frac{143ab}{88200} & -\frac{121ab}{44100} & -\frac{143ab}{88200} & -\frac{169ab}{176400} & -\frac{11a^2b}{29400} & \frac{11a^2b}{22050} & \frac{13a^2b}{44100} & -\frac{13a^2b}{58800} & -\frac{13ab^2}{44100} & -\frac{11ab^2}{22050} & \frac{11ab^2}{29400} & \frac{13ab^2}{58800} & -\frac{a^2b^2}{14700} & \frac{a^2b^2}{11025} && \\
\frac{169ab}{176400} & \frac{143ab}{88200} & \frac{121ab}{44100} & \frac{143ab}{88200} & \frac{13a^2b}{58800} & -\frac{13a^2b}{44100} & -\frac{11a^2b}{22050} & \frac{11a^2b}{29400} & \frac{13ab^2}{58800} & \frac{11ab^2}{29400} & -\frac{11ab^2}{22050} & -\frac{13ab^2}{44100} & \frac{a^2b^2}{19600} & -\frac{a^2b^2}{14700} & \frac{a^2b^2}{11025} & \\
-\frac{143ab}{88200} & -\frac{169ab}{176400} & -\frac{143ab}{88200} & -\frac{121ab}{44100} & -\frac{13a^2b}{44100} & \frac{13a^2b}{58800} & \frac{11a^2b}{29400} & -\frac{11a^2b}{22050} & -\frac{11ab^2}{29400} & -\frac{13ab^2}{58800} & \frac{13ab^2}{44100} & \frac{11ab^2}{22050} & -\frac{a^2b^2}{14700} & \frac{a^2b^2}{19600} & -\frac{a^2b^2}{14700} & \frac{a^2b^2}{11025}
\end{bmatrix} \quad (4)$$

GROUNDING OF MEMBRANE ELEMENTS

Suppose one considers a free/free model with two 12 DOF elements, and a uniform tensile load applied in the x-direction only, as shown in Fig. 2. That element should possess five rigid body modes: (1) translation in the y-direction; (2) rigid body rotation about the x-axis; (3) rotation of the edge line between nodes 1, 2 and 5; (4) rotation of the edge lines between nodes 4, 3 and 6; (5) rigid body rotation about the y-axis. It should be noted that (2) and (4) only occur because for the example problem, there is no tension in the y-direction, and we are neglecting elastic stiffness effects.

In matrix form, assembling $[Kg]$ and multiplying a matrix of these rigid body modes yields

$$[Kg]\cdot\begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & \frac{2}{b} & 1 & 0 & 0 \\ 0 & \frac{2}{b} & 1 & 0 & 0 \\ 0 & \frac{2}{b} & 0 & 1 & 0 \\ 0 & \frac{2}{b} & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & 0 & 0 & \frac{1}{a} \\ 0 & \frac{2}{b} & 1 & 0 & 0 \\ 0 & \frac{2}{b} & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{Nb}{2a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{Nb}{2a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{Nb^2}{12a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{Nb^2}{12a} \\ 0 & 0 & 0 & 0 & \frac{Nb}{2a} \\ 0 & 0 & 0 & 0 & \frac{Nb}{2a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{Nb^2}{12a} \\ 0 & 0 & 0 & 0 & -\frac{Nb^2}{12a} \end{bmatrix}. \quad (5)$$

Examining the right hand side of eqn (5) reveals that rigid body modes 1–4 are present in the global $[Kg]$. However, mode #5, which corresponds to rigid body rotation about the y-axis, was not present. Instead, pseudo-forces developed in degree of freedoms 1, 4, 9, 12, 13, 14, 17 and 18. The pseudo-forces in degrees of freedom 1, 4, 13 and 14 correspond to the forces required to keep the rotated model in equilibrium, which is consistent with the behavior of a pre-loaded beam element [2]. Hence, the procedure used successfully for the pre-loaded beam element [3] will be adapted.

The pseudo-moments developed in degrees of freedom 9, 12, 17 and 18 did not occur with rotation of the pre-loaded beam element. It should be noted, however, that the form resembles a fixed-end moment caused by a distributed load.

DIRECTED-FORCE APPROACH

Consider Argyris's methodology [4] for the directed force problem. Let $\phi \cong (u_4 - u_{18})/2a$ or

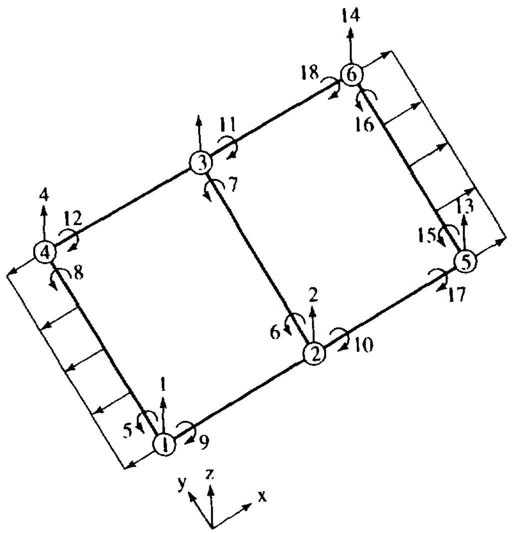

Fig. 2. Two element 18 DOF.

$(u_1 - u_{17})/2a$. If one neglects the change in the axial component during a small rotation, one obtains $N(b/2)\cos\phi \cong Nb/2$. The load vector for this force becomes

$$\mathbf{R}^{\mathrm{DFC}} = [(Nb/2)\sin(u_{17} - u_1/2a)\ \ 0\ \ 0\ \ (Nb/2)\sin \times (u_{18} - u_4/2a) 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ -(Nb/2)\sin(u_{17} - u_1/2a) \ -(Nb/2)\sin(u_{18} - u_4/2a)\ \ 0\ 0\ 0\ 0\ 0\ 0]. \quad (6)$$

Taking

$$\frac{\partial \mathbf{R}_i^{\mathrm{DFC}}}{\partial u_i}$$

and assuming small angles yields

$$[K^{\mathrm{DFC}}] = \begin{bmatrix}
-\frac{bN}{4a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{bN}{4a} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{bN}{4a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{bN}{4a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{bN}{4a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{bN}{4a} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{bN}{4a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{bN}{4a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (7)$$

Combining $[Kg]+[K^{\mathrm{DFC}}]$ and multiplying by the rigid body mode matrix yields

$$[[Kg]+[K^{\mathrm{DFC}}]\cdot[\text{Rigid Body Modes}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{Nb^2}{12a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{Nb^2}{12a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{Nb^2}{12a} \\ 0 & 0 & 0 & 0 & -\frac{Nb^2}{12a} \end{bmatrix} \tag{8}$$

Note that the pseudo-force terms associated with degrees of freedom 1, 4, 13 and 14 have disappeared, but the pseudo-moment terms associated with degrees of freedom 9, 12, 17 and 18 remain.

Once again, consider a rotation ϕ about the y-axis, and let $\phi=(u_{13}+u_{14}-u_1-u_4)/4a$. We have considered the equivalent concentrated vertical loads at the nodes due to the rotation in the determination of $[K^{\mathrm{DFC}}]$. However, this vertical component is really based on a distributed load. In order to replace it with concentrated loads at the node points, we must also include fixed-end moments. The load vector for the fixed end moments caused by the vertical distributed loads would be

$$\mathbf{R}_{\mathrm{FEM}} = \Big[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{Nb^2}{12}\times\sin\phi \quad 0 \quad 0 \quad -\frac{Nb^2}{12}\sin\phi \quad 0 \quad 0 \quad 0 \quad 0 \quad -\frac{Nb^2}{12}\sin\phi \quad \frac{nB^2}{12}\sin\phi\Big]. \tag{9}$$

Taking $\dfrac{\partial R_i^{\mathrm{FEM}}}{\partial v_i}$, assuming small angles and enforcing symmetry yields

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{Nb^2}{48a} & 0 & 0 & \frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & \frac{Nb^2}{48a} & -\frac{Nb^2}{48a} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{Nb^2}{48a} & 0 & 0 & \frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & \frac{Nb^2}{48a} & -\frac{Nb^2}{48a} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{Nb^2}{48a} & 0 & 0 & -\frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Nb^2}{48a} & \frac{Nb^2}{48a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{Nb^2}{48a} & 0 & 0 & \frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{Nb^2}{48a} & -\frac{Nb^2}{48a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Nb^2}{48a} & 0 & 0 & -\frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & -\frac{Nb^2}{48a} & \frac{Nb^2}{48a} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Nb^2}{48a} & 0 & 0 & -\frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & -\frac{Nb^2}{48a} & \frac{Nb^2}{48a} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{Nb^2}{48a} & 0 & 0 & \frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{Nb^2}{48a} & -\frac{Nb^2}{48a} & 0 & 0 & 0 & 0 \\
-\frac{Nb^2}{48a} & 0 & 0 & -\frac{Nb^2}{48a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Nb^2}{48a} & \frac{Nb^2}{48a} & 0 & 0 & 0 & 0
\end{bmatrix}
\quad (10)
$$

At this point,

$$
[[Kg] - [K^{DFC}] + [K^{FEM}]] \cdot [\text{Rigid Body Modes}] =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
. \quad (11)
$$

Thus, the all pseudo-forces and pseudo-moments have been eliminated.

COMPARISON OF FREQUENCIES

In order to determine the effect of the correction factors, a comparison of the first 12 natural frequencies of vibration was performed for the two element model. The results are included in Table 1.

Table 1. Comparison of frequencies of vibration for two-element free/free model

Frequency	Yang's [*Kg*] 16 DOF	Yang's [*Kg*] 24 DOF	Corrected 16 DOF	Corrected 24 DOF
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	103.2	0	0	0.0152
6	103.2	0	103.2	103.2
7	118.0	0	118.0	103.2
8	118.0	0	206.4	174.7
9	206.4	0	206.4	206.4
10	206.4	0	206.4	206.4
11	206.4	0	206.4	206.4
12	206.4	0	268.3	206.4

As the table indicates, the missing rigid body mode (mode #5, frequency = 0), was obtained when the total directed force correction factors were added to Yang's [*Kg*].

SUMMARY

Based upon this investigation, the following conclusions can be made:

(1) Pre-loaded membrane elements exhibit the same "grounding" problems associated with pre-loaded beam elements. In other words, an internal grounding, or false stiffening, occurs when they are subjected to a rigid body rotation, which generates erroneous pseudo-forces.

(2) In addition to the pseudo-forces, erroneous pseudo-moments also develop during rigid body rotation.

(3) An erroneous non-zero frequency develops during rigid body rotation, instead of the required zero frequency associated with a rigid body mode.

(4) Using a directed force premise, and extending the Argyris approach for developing correction factors for non-conservative forces to the global level, the pseudo-forces were eliminated, but the pseudo-moments still remained.

(5) By considering the fixed-end moments associated with the distributed load, and similarly developing correction factors, the pseudo-moments were eliminated, and the missing zero frequency associated with rigid body rotation were obtained.

(6) Future work should include comparison of results for larger models and with test results.

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