On The Construction of Mixed Orthogonal Arrays of Strength Two

Chung Yi Suen  
*Cleveland State University*, C.SUEN@csuohio.edu

Warren F. Kuhfeld  
*SAS Institute*

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1. Introduction

An orthogonal array of strength two, $L_N(s_1 \cdots s_k)$, is an $N \times k$ matrix with symbols in the $i$th column from a finite set of $s_i$ symbols $(1 \leq i \leq k)$, such that in every $N \times 2$ submatrix, all possible combinations of symbols occur equally often as a row. If among $s_1, \ldots, s_k$, there are $n_i$ that equal $\mu_i$ $(1 \leq i \leq u)$ where $\{n_i\}$ and $\{\mu_i\}$ are positive integers, $\mu_i \geq 2$, $n_1 + \cdots + n_u = k$, then we will write $L_N(\mu_1^{n_1} \cdots \mu_u^{n_u})$ for $L_N(s_1 \cdots s_k)$. When $s_1 = \cdots = s_k$ the orthogonal array is called symmetric; otherwise it is called asymmetric or mixed. An orthogonal array $L_N(s_1 \cdots s_k)$ is tight if $\sum_{i=1}^k s_i - k = N - 1$. Orthogonal arrays have extensive applications in statistical design of experiments, computer science, and cryptography.
and large orthogonal arrays, sometimes with hundreds of runs, are becoming increasingly popular among researchers modelling consumer choice (Kuhfeld, 2004). Methods for constructing mixed orthogonal arrays of strength two have been developed recently by Wang and Wu (1991), Dey and Midha (1996, 2001), Wang (1996a, b), Zhang et al. (1999), Xu (2002), and many other authors. For an excellent description of the methods of construction of orthogonal arrays, see Hedayat et al. (1999). For extensive construction methods, see Kuhfeld (2004).

Wang and Wu (1991) used the Kronecker sum of orthogonal arrays and difference schemes to construct several families of mixed orthogonal arrays. Dey and Midha (1996, 2001) extended the method of Wang and Wu (1991) to construct more families of mixed orthogonal arrays. In this paper, we modify this method to allow more flexibility. As a consequence, some new families of mixed orthogonal arrays are obtained.

2. Basic concepts and notations

Let $G$ be an additive group with $p$ elements, $0, 1, \ldots, p-1$. An $rp \times k$ matrix with entries from $G$ is called a difference scheme $D_{r, p, k, p}$, if among the differences of the corresponding elements of any two columns, each element of $G$ appears $r$ times.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times r$ and $m \times s$ matrices respectively with entries from an additive group $G$ of $p$ elements. The Kronecker sum of $A$ and $B$, denoted by $A \ast B$, is defined to be an $nm \times rs$ matrix $(b_{ij} + a)_{1 \leq i \leq n, 1 \leq j \leq r}$. Throughout, we let $0_n$ be the $n \times 1$ vector of zeros and let $1_n$ be the $n \times 1$ vector $(0, 1, \ldots, n - 1)'$. We now list some useful properties of difference schemes.

1. We can assume, without loss of generality, that the first column of a difference scheme $D_{r, p, k, p}$ is $0_{rp}$. Then every element of $G$ appears exactly $r$ times in all other columns.

2. The Kronecker sum of a difference scheme $D_{r, p, k, p}$ and an orthogonal array $L_{N}(p^s)$ is an orthogonal array $L_{rN}(p^{ks})$. The Kronecker sum of two difference schemes $D_{r_1, p_1, k_1, p}$ and $D_{r_2, p_2, k_2, p}$ is also a difference scheme $D_{r_1r_2, r_2, k_1k_2, p}$.

3. If $D_{r, p, k, p}$ exists then $k \leq rp$. $D_{r, p, r, p}$ is called a generalized Hadamard matrix. $D_{h, h, 2}$ is a Hadamard matrix of order $h$. If a Hadamard matrix of order $h$ exists, $h$ is called a Hadamard number. It is conjectured that $h$ is a Hadamard number if $h = 1, 2,$ or a multiple of 4.

4. If $p$ is a prime or a prime power then $D_{r, p, p, p}$ exists in each of the following cases: (a) $r = 2$ or 4; (b) $r$ and $p$ are powers of the same prime; (c) $r = q^m(q + 1)/p$ for all $m \geq 0$ if $q$ is a prime power and $D_{q+1, q+1, p}$ exists.

Suppose an $L_{N}(s_1^{n_1} \cdots s_u^{n_u})$ and difference schemes $D_{M, k_1, s_1}, \ldots, D_{M, k_u, s_u}$ exist. Partition the $L_{N}(s_1^{n_1} \cdots s_u^{n_u})$ as $[L_{N}(s_1^{n_1}), \ldots, L_{N}(s_u^{n_u})]$. By using Kronecker sum, Wang and Wu (1991) constructed the following mixed orthogonal array $L_{MN}(s_1^{k_1n_1} \cdots s_u^{k_un_u} M^T)$,

$$[D_{M, k_1, s_1} \ast L_{N}(s_1^{n_1}), \ldots, D_{M, k_u, s_u} \ast L_{N}(s_u^{n_u}), \tau_M \ast 0_N].$$
Theorem 1. If there exists an orthogonal array \( LN(w^m) \) constructed as follows. Arrange the rows of the \( LN(w^m) \) and obtain an \( LN(w^m) \). Furthermore, if \( a_1, a_2, a_3 \) are three 2-symbol columns such that \( a_1 + a_2 = a_3 \), then we can replace these three 2-symbol columns by a 4-symbol column. By using this procedure, Wang and Wu (1991) constructed several families of mixed orthogonal arrays.

Dey and Midha (1996) modified the construction of Wang and Wu (1991) and obtained the following result. If there exist an orthogonal array \( LN(w^m) \) and difference schemes \( DM,k_1,s_1, \ldots, DM,k_u,s_u \), then an \( LMN(s^{(k_1-1)n_1}_1, \ldots, s^{(k_u-1)n_u}_u (Mw)^4) \) can be constructed as follows. Arrange the rows of the \( LN(w^m) \) such that the first column is \( \tau_w = 0N/w \). Partition \( LN(w^m) \) as [\( \tau_w \star 0N/w \), \( LN(s^{n_1}_1) \), \( \ldots, LN(s^{n_u}_u) \)]. Then an \( LMN(s^{(k_1-1)n_1}_1, \ldots, s^{(k_u-1)n_u}_u (Mw)^4) \) can be constructed as

\[
[D_{M,k_1,s_1} \ast L_N(s^{n_1}_1), \ldots, D_{M,k_u,s_u} \ast L_N(s^{n_u}_u), \tau_{Mw} \ast 0N/w].
\]

3. Main results

We first modify the result of Wang and Wu (1991) to obtain an \( N \)-symbol column by sacrificing several columns in the construction.

**Theorem 1.** If there exists an orthogonal array \( LN(s^{n_1}_1, \ldots, s^{n_u}_u) \) and difference schemes \( DM,k_1,s_1, \ldots, DM,k_u,s_u \), then we can construct an orthogonal array \( LMN(s_k^{(k_1-1)n_1}_1, \ldots, s_k^{(k_u-1)n_u}_u) \).

**Proof.** For \( i = 1, \ldots, u \), let \( D_{M,k_i,s_i} = [0M, D_{M,k_i-1,s_i}] \). Then each of the \( s_i \) symbols appears \( M/s_i \) times in every column of \( D_{M,k_i-1,s_i} \). We can verify that

\[
[D_{M,k_1-1,s_1} \ast L_N(s^{n_1}_1), \ldots, D_{M,k_u-1,s_u} \ast L_N(s^{n_u}_u), \tau_{M} \ast 0N, 0M \ast \tau_N]
\]

is an \( LMN(s^{(k_1-1)n_1}_1, \ldots, s^{(k_u-1)n_u}_u (Mw)^4) \). \( \square \)

For examples, we obtain an \( L_{216}(18^112^16^23^6) \) by using \( L_{18}(6^13^6), D_{12,6} \), and \( D_{12,12,3} \); obtain an \( L_{216}(18^112^13^72^11) \) by using \( L_{18}(3^72^1), D_{12,12,3} \), and \( D_{12,12,2} \); and obtain an \( L_{144}(12^23^11^24^4) \) by using \( L_{12}(3^12^4), D_{12,12,3}, \) and \( D_{12,12,2} \) in Theorem 1.

The result in Theorem 1 was, in a slightly different formulation, also obtained by Dey and Midha (2001) by a slightly different method. Note that \( LMN(s^{(k_1-1)n_1}_1, \ldots, s^{(k_u-1)n_u}_u (Mw)^4) \) in Theorem 1 is tight if \( LN(s^{n_1}_1, \ldots, s^{n_u}_u) \) is tight and \( D_{M,k_1,s_1}, \ldots, D_{M,k_u,s_u} \) are generalized Hadamard matrices. Several families of tight orthogonal arrays are constructed in the following by using Theorem 1.

**Corollary 1.1.** If \( p \) is a prime power and \( D_{rp^2,rp^2,rp^2} \) exists, then we can construct a tight orthogonal array \( L_{rp^3}((rp^2)^4(p^5)^4) \).

\[
L_{rp^3}((rp^2)^4(p^5)^4) \text{ exists.}
\]
Wang and Wu (1991), the other three arrays are believed to be new. Theorem 2. If there exist orthogonal arrays \(DM, ki, si\) and \(LMN (s(ku, tmv)\) in Theorem 1. The existence of \(D_{rp^2, rp^2, p}\) is implied by the existence of \(D_{rp^2, rp^2, p}\).

\[\square\]

For \(p = 2\) and \(r = 3\) in Corollary 1.1, we have a new array \(L_{96}(121/1/4^{11}/2^{44})\). For \(p = 3\) and \(r = 2\) we obtain \(L_{486}(271/1/4^{17}/2^{53})\).

Corollary 1.2. If \(p\) is a prime power and \(D_{rp^p, p}\) exists, then we can construct a tight orthogonal array \(L_{rpn}((rp)^1(p^{n-1})^1(p^{(p-1)}(p^{n-1}-(p-1))/p))\) for all \(n \geq 3\).

\[\square\]

Example 1. We use Theorem 2 to construct many new 72-run orthogonal arrays in the following by combining two 36-run orthogonal arrays.
(a) By using \( L_{36}(3^{1}2^{27}) \), assorted \( L_{36} \), and \( D_{2,2,2} \) in Theorem 2, we obtain \( L_{72} \):
\[
\begin{align*}
L_{36}(3^{1}2^{1}3^{11}) & \rightarrow L_{72}(12^{6}3^{1}1^{27}) & \quad & L_{36}(3^{1}2^{3}3^{6}) & \rightarrow L_{72}(6^{1}3^{9}2^{27}) \\
L_{36}(3^{1}2^{3}3^{2}2^{1}) & \rightarrow L_{72}(6^{1}3^{2}2^{28}) & \quad & L_{36}(3^{1}2^{3}3^{1}2^{3}) & \rightarrow L_{72}(6^{1}3^{2}2^{30}) \\
L_{36}(3^{1}2^{3}2^{4}) & \rightarrow L_{72}(6^{1}2^{3}2^{31})
\end{align*}
\]

(b) By using \( L_{36}(2^{1}2^{3}3^{4}) \), assorted \( L_{36} \), and \( D_{2,2,2} \) in Theorem 2, we obtain \( L_{72} \):
\[
\begin{align*}
L_{36}(2^{1}18^{1}2^{1}) & \rightarrow L_{72}(18^{1}4^{1}2^{35}) & \quad & L_{36}(2^{1}9^{1}2^{12}) & \rightarrow L_{72}(9^{1}4^{1}2^{46}) \\
L_{36}(2^{1}6^{2}3^{3}) & \rightarrow L_{72}(6^{2}4^{1}3^{2}3^{4}) & \quad & L_{36}(2^{1}6^{2}3^{2}2^{2}) & \rightarrow L_{72}(6^{2}4^{1}3^{2}3^{6}) \\
L_{36}(2^{1}6^{2}3^{2}3^{1}) & \rightarrow L_{72}(6^{2}4^{1}3^{2}2^{7}) & \quad & L_{36}(2^{1}6^{2}3^{2}7) & \rightarrow L_{72}(6^{2}4^{1}3^{2}4^{1}) \\
L_{36}(2^{1}6^{2}3^{2}8) & \rightarrow L_{72}(6^{2}4^{1}3^{2}3^{4}) & \quad & L_{36}(2^{1}6^{2}3^{2}3^{4}) & \rightarrow L_{72}(6^{2}4^{1}3^{2}5^{2}3^{5}) \\
L_{36}(2^{1}6^{2}3^{2}8) & \rightarrow L_{72}(6^{2}4^{1}3^{2}3^{4}2^{3}) & \quad & L_{36}(2^{1}6^{2}3^{2}3^{2}9) & \rightarrow L_{72}(6^{2}4^{1}3^{2}3^{3}4^{3}) \\
L_{36}(2^{1}6^{1}3^{2}2^{2}) & \rightarrow L_{72}(6^{1}4^{1}3^{2}3^{6}) & \quad & L_{36}(2^{1}6^{1}3^{2}3^{6}) & \rightarrow L_{72}(6^{1}4^{1}3^{2}3^{2}3^{4}) \\
L_{36}(2^{1}6^{1}3^{2}1^{7}) & \rightarrow L_{72}(6^{1}4^{1}3^{2}3^{5}1) & \quad & L_{36}(2^{1}3^{2}3^{19}) & \rightarrow L_{72}(4^{1}3^{2}2^{3}5^{3}) \\
L_{36}(2^{1}3^{1}3^{1}2^{26}) & \rightarrow L_{72}(4^{1}3^{1}2^{3}6^{0})
\end{align*}
\]

We now construct two families of orthogonal arrays by using Theorem 2. Let \( r (\geq 3) \) be an odd number. It is known that \( L_{4r} ((2r)^{1}2^{r}) \) exists, and it is not possible to have more than two 2-symbol columns. Let \( \phi_{r} \) denote the largest possible \( m \) in an \( L_{4r} (r^{1}2^{m}) \). It is known that \( \phi_{3} = 4 \) and \( \phi_{5} = 8 \). For \( r \geq 7 \) we do not know the exact value of \( \phi_{r} \) except that \( \phi_{7} \geq 12 \), \( \phi_{9} \geq 13 \), \( \phi_{11} \geq 12 \), \( \phi_{13} \geq 12 \), and \( \phi_{r} \geq 13 \) for \( r \geq 15 \).

**Corollary 2.1.** If \( r \geq 3 \) is an odd number and \( h \) is an Hadamard number, then we can construct (a) \( L_{4r} ((2h)^{1}(2r)^{1}2^{(4r-2)(h-1)+1}) \); and (b) \( L_{4r}(2^{1}h^{1}r^{1}2^{(4r-2)(h-1)+\phi_{r}-1}) \), where \( \phi_{r} \) is the maximum number \( m \) such that \( L_{4r} (r^{1}2^{m}) \) exists.

**Proof.** \( L_{4r}(2^{1}h^{1}(2h)^{1}(2r)^{1}2^{(4r-2)(h-1)+1}) \) is obtained by using \( L_{4r}(2^{1}2^{4r-2}) \), \( L_{4r}(2^{1}(2r)^{1}2^{1}) \), and \( D_{h,h,2} \) in Theorem 2. \( L_{4r}(2^{1}h^{1}(2h)^{1}r^{1}2^{(4r-2)(h-1)+\phi_{r}-1}) \) is obtained by using \( L_{4r} (2^{1}2^{4r-2}) \), \( L_{4r}(2^{1}r^{1}2^{\phi_{r}-1}) \), and \( D_{h,h,2} \) in Theorem 2. \( \square \)

For \( h = 2 \) in Corollary 2.1(a), we obtain \( L_{8r}((2r)^{1}2^{1}4^{1}2^{4r-1}) \) which was also constructed by Agrawal and Dey (1982). For \( h = 2 \) and \( r = 3, 5, 7, 9, \) and 11 in Corollary 2.1(b), we obtain \( L_{24}(4^{1}3^{1}2^{13}) \), \( L_{40}(5^{1}4^{1}2^{25}) \), \( L_{56}(7^{1}4^{1}2^{37}) \), \( L_{72}(9^{1}4^{1}2^{46}) \), and \( L_{88}(11^{1}4^{1}2^{53}) \) respectively. The first two arrays were also obtained by Wang and Wu (1991), and the last three arrays are believed to be new. Also for \( h = 4, 8, 12 \) and \( r = 3, 5, 7, 9 \) in Corollary 2.1, we obtain new arrays \( L_{112}(14^{1}3^{1}2^{79}) \), \( L_{112}(8^{1}7^{1}2^{89}) \), \( L_{144}(24^{1}6^{1}2^{111}) \), \( L_{144}(24^{1}3^{1}2^{113}) \), \( L_{144}(18^{1}8^{1}2^{103}) \), \( L_{144}(9^{1}8^{1}1^{114}) \), \( L_{160}(16^{1}10^{1}2^{127}) \), \( L_{160}(16^{1}5^{1}2^{133}) \), \( L_{224}(16^{1}14^{1}2^{183}) \), \( L_{224}(16^{1}7^{1}2^{193}) \), \( L_{240}(24^{1}10^{1}2^{199}) \), \( L_{240}(24^{1}5^{1}2^{205}) \), \( L_{288}(18^{1}16^{1}2^{239}) \), \( L_{288}(16^{1}9^{1}2^{250}) \), \( L_{336}(24^{1}4^{1}2^{287}) \), \( L_{336}(24^{1}7^{1}2^{297}) \), \( L_{432}(24^{1}18^{1}2^{375}) \), and \( L_{432}(24^{1}9^{1}2^{386}) \).

In the following example, we obtain two new \( 96 \)-run orthogonal arrays by using Dey and Midha’s (1996) construction, and replacing several 2-symbol columns by 4-symbol columns.

**Example 2.** Partition the \( L_{8}(4^{1}2^{4}) \) as \([\tau_{2} * 0_{4}, L_{8}(4^{1}), L_{8}(2^{3})]\). By sacrificing a 2-symbol column in \( L_{8}(4^{1}2^{4}) \), Dey and Midha (1996) obtained \( L_{96}(24^{1}4^{1}2^{12}2^{36}) \) as
\[
[D_{12,12,4} * L_{8}(4^{1}), D_{12,12,2} * L_{8}(2^{3}), \tau_{24} * 0_{4}].
\]
More arrays can be obtained by replacing $\tau_{24}$ with any 24-run array $L_{24}$. Let $L_{8}(2^{3}) = [a_{1}, a_{2}, a_{3}]$ and $D_{12,12,2} = [0_{12}, b_{1}, b_{2}, B]$. If the 24-run array has at least two 2-symbol columns, we can permute the rows of $L_{24}$ such that $L_{24} = [b_{1} \ast 0_{2}, b_{2} \ast 0_{2}, L]$. Since for $i = 1, 2$ we have

$$0_{12} \ast a_{i} + b_{i} \ast a_{i} = b_{i} \ast 0_{8} = b_{i} \ast 0_{2} \ast 0_{4},$$

the three 2-symbol columns $0_{12} \ast a_{i}$, $b_{i} \ast a_{i}$, $b_{i} \ast 0_{2} \ast 0_{4}$ can be replaced by a 4-symbol column. For example, if we choose $L_{24}$ to be $L_{24}(12^{1212})$ and $L_{24}(6^{14^{124}})$, we obtain two new 96-run arrays $L_{96}(12^{14^{14}2^{42}})$ and $L_{96}(6^{14^{15}2^{41}})$, respectively.

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References


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